Europ. J. Combinatorics (2001) **22**, 865–877
doi:10.1006/eujc.2001.0505
Available online at http://www.idealibrary.com on IDEAL®

On Generalizations of Matching-covered Graphs

ZOLTÁN SZIGETI

Structural results for extensions of matching-covered graphs are presented in this paper.

1. INTRODUCTION

An ear-decomposition of a graph G is a sequence $(G_0, G_1, \dots, G_k)$ of subgraphs such that G_0 is a vertex, $G_k = G$ and each G_{i+1} is obtained from G_i by adding an ear that is a path whose end vertices belong to G_i but the inner vertices do not. It is well known that a graph has an ear-decomposition if and only if it is 2-edge-connected. We remark that each circuit can be the starting ear of an ear-decomposition. It is quite simple to see that the number of ears in each ear-decomposition of G is $m - n + 1$, where n and m denote the number of vertices and edges of G, respectively. However, the number of even ears may differ in distinct ear-decompositions of G. (The length of an ear is the number of edges contained in it.) We focus our attention on ear-decompositions (called optimal) that have minimum number $\varphi(G)$ of even ears. Frank showed in [2] how an optimal ear-decomposition can be constructed in polynomial time for any 2-edge-connected graph.

Lovász [4] observed that a graph G is factor-critical if and only if $\varphi(G) = 0$. Lovász and Plummer [5] proved that for matching-covered graphs $\varphi(G) = 1$. However, this latter one is not a characterization. To see an example, let H be the simple graph obtained from the circuit on four vertices by adding an edge e. Then $\varphi(H) = 1$ but H is not matching-covered. Let us call an edge e of a graph G φ-extreme if e may lie on an even ear of an optimal ear-decomposition of G. Note that in the example above e is not φ-extreme. This observation leads to the following characterization of matching-covered graphs. We call a graph G φ-covered if each edge of G is φ-extreme. For more definitions see Section 2.

CLAIM 1. *G is matching-covered if and only if $\varphi(G) = 1$ and G is φ-covered. In other words, G is matching-covered if and only if G/e is factor-critical for each edge e of G.*

The reader is encouraged to prove Claim 1 as a warm-up. In the light of Claim 1, φ-covered graphs can be considered as a natural generalization of matching-covered graphs. We propose the investigation of φ-covered graphs in this paper. By Claim 1, we have another way to generalize matching-covered graphs, namely we may consider graphs with $\varphi(G) = 1$. This possibility will also be exploited in this paper.

By combining the results of Lovász and Plummer [5] and Little [3], it follows that for any two edges of a matching-covered graph G there exists an optimal ear-decomposition of G such that the first ear P is even and P contains these two edges. This result can be extended to 2-vertex-connected φ-covered graphs. Note that to demonstrate this result we had to use some properties of the ear matroid. The ear matroid of a graph was introduced in [6, 7]. It will be shown that two edges of G belong to the same block of the ear matroid if and only if these two edges may lie on the starting even ear of an optimal ear-decomposition. To argue the above mentioned result we shall give a simple description of the blocks of the ear matroid. Hopefully, this result is of interest in its own right and can be considered as one of the main results of this paper.

0195–6698/01/060865 + 13 $35.00/0

By the aforementioned characterization of factor-critical graphs it follows that for an optimal ear-decomposition $(G_0, \ldots, G_k)$ of a factor-critical graph G, each subgraph G_i in this sequence is also factor-critical. This useful property does not hold for matching-covered graphs. As an example, consider the complete graph K_4 on four vertices. K_4 is matching-covered but, since $K_4 - e$ is not matching-covered for an arbitrary edge e of K_4, K_4 has no optimal ear-decomposition such that all the subgraphs in the sequence are matching-covered. To have a similar property for matching-covered graphs Lovász and Plummer [5] suggested the notion of graded ear-decomposition. Briefly this means that they allowed the addition of more ears simultaneously. With this more general notion, we can achieve our aim. It is easy to see that each matching-covered graph has a graded ear-decomposition in such a way that the first ear is even, all the other ears are of odd length and each subgraph in the sequence is matching-covered. What is much more interesting (and of course a little bit more complicated) is, as Lovász and Plummer [5] demonstrated, that we can do this by adding at most two ears in each step. This is the so-called Two Ear Theorem, and for a very short and simple proof we refer to a note of the present author [8]. We shall show, as a main result of the paper, that the Two Ear Theorem can be extended to φ-covered graphs. This theorem characterizes φ-covered graphs by means of ear-decomposition. Another constructive characterization will also be given for φ-covered graphs.

Along the way we shall also prove some structural results on the graph defined by the φ-extreme edges. The power of this approach has been utilized in [9] to provide a simple graph theoretic proof for the Tight Cut Lemma on bricks due to Edmonds, Lovász and Pulleyblank [1]. We shall also provide a new proof for the Cathedral Theorem on saturated graphs due to Lovász and Plummer [5]. In fact, an analogous construction, the Cathedral Construction for saturated graphs, can be deduced from our results for almost critical graphs.

The organization of this paper is as follows. In Section 2 we give all the definitions we need. Section 3 contains earlier results and some new simple observations that will be used in this paper. In Section 4 we shall apply our results to almost critical graphs to provide a new proof for the Cathedral Theorem on saturated graphs. In Section 5 we prove our main lemma that provides a constructive characterization for φ-covered graphs. We investigate in Section 6 the graph defined by the φ-extreme edges and give some information about the structure of this graph. Section 7, which is devoted to the ear matroid, yields a simple description of the blocks of this matroid. In Section 8 we extend results on matching-covered graphs to φ-covered graphs.

We remark that all the results here can be found in the two IPCO papers [6, 10].

2. Definitions and Notation

A connected component K of a graph G is called *odd* (*even*) if $|V(K)|$ is odd (even). For $X \subseteq V(G)$, $\boldsymbol{c_o(G - X)}$ denotes the number of odd components in $G - X$, while $\boldsymbol{C_X}$ will denote the union of the even components of $G - X$. We shall use the notation $\boldsymbol{C(G)}$ defined in the Gallai–Edmonds Decomposition Theorem [5].

Let G be a graph with a perfect matching. An edge of G is *allowed* if it lies in some perfect matching of G. $\boldsymbol{N(G)}$ denotes the subgraph of G induced by the allowed edges of G. G is *matching-covered* if it is connected and each edge of G is allowed, that is, $G = N(G)$. G is called *elementary* if $N(G)$ is connected. In particular, every matching-covered graph is elementary. A vertex set $X \subseteq V(G)$ is called a *barrier* if $c_o(G-X) = |X|$. If G is elementary, then let $\boldsymbol{\mathcal{P}(G)}$ be defined as the set of all maximal barriers of G. G is said to be *saturated* if for each pair u, v of non-adjacent vertices of G, $G - u - v$ has a perfect matching. It is equivalent to saying that the addition of the edge uv to G creates a new perfect matching of $G + uv$.

CATHEDRAL CONSTRUCTION. *Let G_0 be a saturated elementary graph and to each class $S \in \mathcal{P}(G)$ assign an already constructed saturated graph G_S or the empty set. For each $S \in \mathcal{P}(G)$ join every vertex of S to every vertex of G_S.*

A subgraph H of a graph G is called *nice* if $G - V(H)$ has a perfect matching. A graph G is *factor-critical* if for each vertex $v \in V(G)$, $G - v$ possesses a perfect matching.

For a graph H with a perfect matching, a non-empty barrier X of H is said to be a *strong barrier* if $H - X$ has no even components, each odd component of $H - X$ is factor-critical and the bipartite graph obtained from H by deleting the edges spanned by X and by contracting each factor-critical component of $H - X$ to a single vertex is matching-covered. Let $G = (V, E)$ be a graph and assume that the subgraph H of G induced by $U \subseteq V$ has a strong barrier X. Then H is said to be a *strong subgraph* of G with strong barrier X if X separates $U - X$ and $V - U$ in G or if $U = V$.

An *ear-decomposition* of a graph G is a sequence $(G_0, G_1, \dots, G_k)$ of subgraphs such that G_0 is a vertex, $G_k = G$ and each G_{i+1} is obtained from G_i by adding an *ear* P_{i+1} that is a path whose end vertices belong to G_i but the inner vertices do not. We shall also use the following notation for an ear-decomposition: $G = P_1 + P_2 + \dots + P_k$. Note that we allow closed ears, for example the *starting ear* P_1 is always a circuit. The *length* of an ear is the number of edges contained in it. A sequence $(G_0, G_1, \dots, G_m)$ of subgraphs of G is a *2-graded ear-decomposition* of G if G_0 is a vertex, G_1 is an even circuit, $G_m = G$, for $1 \le i \le m-1$, G_{i+1} is matching-covered, G_{i+1} is obtained from G_i by adding at most two disjoint odd paths which are openly disjoint from G_i but their end-vertices belong to G_i.

Let G be an arbitrary graph. If $X \subseteq V(G)$, then the subgraph of G induced by X is denoted by $\boldsymbol{G[X]}$. The graph obtained from G by contracting an edge set F of G will be denoted by $\boldsymbol{G/F}$. By the subdivision of an edge set F we mean the operation which subdivides each edge $f \in F$ by a new vertex, and it will be denoted by $\boldsymbol{G \times F}$.

We say that an edge set of a graph G is *critical making* if its contraction leaves a factor-critical graph. For a 2-edge-connected graph G, $\boldsymbol{\varphi(G)}$ is defined to be the minimum number of even ears in an ear-decomposition of G. An ear-decomposition is said to be *optimal* if it has exactly $\varphi(G)$ even ears. We call a graph *almost critical* if $\varphi(G) = 1$. A circuit C of G is called *good* if G has an optimal ear-decomposition such that the first ear is C. We say that an edge e of G is *φ-extreme* if e may lie on an even ear of an optimal ear-decomposition of G, in other words, $\varphi(G/e) = \varphi(G) - 1$. More generally, an edge set F of G is called *φ-extreme* if $\varphi(G/F) = \varphi(G) - |F|$. G is called *φ-covered* if each edge of G is φ-extreme. We denote by $D(G)$ the graph on $V(G)$ whose edges are exactly the φ-extreme edges of G.

The *ear matroid* $\boldsymbol{\mathcal{M}(G)}$ of a graph G was introduced in [7]. Its bases are exactly the maximum φ-extreme edge sets, or equivalently, the minimum critical making edge sets. The set of bases of $\mathcal{M}(G)$ will be denoted by $\boldsymbol{\mathcal{B}(G)}$.

The blocks of a matroid $\mathcal{N}$ are defined by an equivalence relation. For two elements e and f of $\mathcal{N}$, $e \sim f$ if there exists a circuit in the matroid containing them, or equivalently, if there exists a base B containing e such that $B - e + f$ is a base again. This is an equivalence relation and the *blocks* of $\mathcal{N}$ are the equivalence classes of $\sim$. The *blocks* of a graph G are defined to be the blocks of the circuit matroid of G, in other words the maximal 2-vertex-connected subgraphs of G.

We finish this section by giving some examples for φ-covered graphs: the complete bipartite graph $K_{2,n}$ ($n \ge 2$) is φ-covered and $\varphi(K_{2,n}) = n - 1$, a graph G whose blocks are matching-covered is φ-covered and $\varphi(G)$ is the number of blocks of G. A procedure that generates all the φ-covered graphs is presented in Section 5.

3. Preliminaries

In this section we list the results we will need in this paper.

THEOREM 1 (TUTTE [11]). *A graph G has a perfect matching if and only if for every $X \subseteq V(G)$, $c_0(G - X) \leq |X|$.*

THEOREM 2 (LOVÁSZ [4]).
(a) A graph G is factor-critical if and only if $\varphi(G) = 0$.
(b) For a factor-critical graph G, a circuit C of G is nice if and only if C is good.

THEOREM 3 (LOVÁSZ [4]). *Let H be a connected subgraph of a graph G.*
(a) If H and G/H are factor-critical, then G is factor-critical.
(b) If H is nice in the factor-critical graph G, then G/H is factor-critical.

THEOREM 4 (LOVÁSZ AND PLUMMER [5]). *Let G be an elementary graph. Then $\mathcal{P}(G)$ is a partition of $V(G)$. For every pair $x, y \in V(G)$, x and y belong to different classes of $\mathcal{P}(G)$ if and only if $G - x - y$ has a perfect matching.*

THEOREM 5 (CATHEDRAL THEOREM [5]). *If G is any saturated graph then it can be built up using the Cathedral Construction starting with a saturated elementary graph G_0. The graph G_0 may be uniquely described as the subgraph of G induced by those vertices of G which, for each $x \in V(G)$, do not lie in $C(G - x)$.*

THEOREM 6. *Let G be a matching-covered graph. Then*

(a) (*Little* [3]). *Any two edges of G belong to a nice circuit.*
(b) (*Lovász and Plummer* [5]). $\varphi(G) = 1$.
(c) (*Lovász and Plummer* [5]). *A circuit C of G is nice if and only if C is good and even.*
(d) Consequently, *any two edges of G belong to a good even circuit.*

THEOREM 7 (LOVÁSZ AND PLUMMER [5]). *Let G be a matching-covered graph. Then:*

(a) If $\{e_1, \dots, e_k\}$ is a set of non-edges of G such that $G + \{e_1, \dots, e_k\}$ is matching-covered, then there exist $i \leq j$ such that $G + e_i + e_j$ is matching-covered.
(b) G has a 2-graded ear-decomposition.
(c) Any two edges of G belong to the starting ear of a 2-graded ear-decomposition.

THEOREM 8.
(a) (Frank [2]). *$\varphi(G)$ equals the minimum size of a critical making edge set.*
(b) (Lemma 1.1 in [7]). *For any forest F of G, $\varphi(G/F) = \varphi(G \times F)$.*

THEOREM 9 ([7]). *The φ-extreme edge sets of a graph G form the independent sets of a matroid $\mathcal{M}(G)$. The bases $\mathcal{B}(G)$ of $\mathcal{M}(G)$ are exactly the minimum critical making edge sets.*

THEOREM 10 (CLAIM 7 IN [9]). *If $G - X$ has at least $|X|$ factor-critical components for a vertex set $X \neq \emptyset$, then there exists a strong subgraph H of G with strong barrier $Y \subseteq X$ such that all the components of $H - Y$ are among the factor-critical components of $G - X$.*

THEOREM 11 (FRANK [2]). *Let G be a 2-edge-connected graph. Then:*

(a) Every edge e of G belongs to a good circuit of G.

(b) *An edge e of G belongs to a good even circuit of G if and only if e is φ-extreme in G.*

THEOREM 12 (FRANK [2]). *Let G be a 2-edge-connected graph. Then:*

(a) *G has a strong subgraph if and only if it is not factor-critical.*
(b) *Let H be a strong subgraph of G. Then $\varphi(H) = 1$ and $\varphi(G/H) = \varphi(G) - 1$.*
(c) *G is almost critical if and only if G has a perfect matching and G contains no two disjoint strong subgraphs.*

THEOREM 13 ([9] (SEE ALSO IN [6])). *Let G be an almost critical graph. Then:*

(a) *$E(D(G)) = E(B(G))$, where $B(G)$ is one of the connected components of $N(G)$.*
(b) *$E(D(G)) = E(D(H))$ for every strong subgraph H of G.*
(c) *$V(B(G)) = \bigcap\{V(H) : H$ is a strong subgraph in $G\}$.*

In the rest of this section we give some simple observations on almost critical graphs.

LEMMA 1. *Suppose that H has a strong barrier X. Then:*

(a) (Frank [2]). *Each edge leaving X is φ-extreme in H, X contains no φ-extreme edge of H.*
(b) *If C is a good even circuit of H containing two vertices u and v from X then the two parts D_1 and D_2 of C between u and v are of even length.*

PROOF. (b) Let $H' := H + uv$. Then, by Theorem 12(b), $\varphi(H) = \varphi(H') = 1$. C is a good even circuit of H so there exists an optimal ear-decomposition $P_1 + \cdots + P_k$ of H such that the unique even ear is $P_1 = C$. Suppose that D_1 and D_2 are of odd length. Then $(D_1 + uv) + D_2 + P_2 + \cdots + P_k$ is an optimal ear-decomposition of H' and the unique even ear $(D_1 + uv)$ contains uv so uv is a φ-extreme edge of H' by Theorem 11(b). However, X is a strong barrier of H' containing uv, that is, uv is not a φ-extreme edge of H' by Lemma 1(a). This contradiction proves (b). □

LEMMA 2. *Let G be an almost critical graph.*

(a) *Then $B(G)$ is matching-covered and $G/B(G)$ is factor-critical.*
(b) *Any two φ-extreme edges of G belong to a good even circuit.*
(c) *Let $G_0 := G[V(B(G))]$. Any connected component of $G - V(G_0)$ has neighbours in exactly one maximal barrier of G_0.*

PROOF. (a) Since each connected component of $N(G)$ is matching-covered, so is $B(G)$ by Theorem 13(a). Let $e \in E(D(G))$. Then, by Theorem 2(a), G/e is factor-critical. By Theorem 13(a), $e \in E(B(G))$ and $G/e - V(B(G)/e) = G - V(B(G))$ has a perfect matching. Then, by Theorem 3(b), $G/B(G) = (G/e)/(B(G)/e)$ is factor-critical.

(b) Let $e, f \in E(D(G))$. Then, by Theorem 13(a), e and f belong to $B(G)$ which is matching-covered by Lemma 2(a). By Theorem 6(b) and (d), $B(G)$ has an ear-decomposition $P_1 + \cdots + P_k$ such that the unique even ear P_1 contains e and f. By Lemma 2(a), $G/B(G)$ is factor-critical, thus, by Theorem 2(a), $\varphi(G/B(G)) = 0$. By Theorem 11(a), $G/B(G)$ has an optimal ear-decomposition $P_1' + \cdots + P_l'$ such that P_1' contains the vertex of $G/B(G)$ corresponding to $B(G)$. Then $P_1 + \cdots + P_k + P_1' + \cdots + P_l'$ is an ear-decomposition of G such that the unique even ear P_1 contains e and f, and we are done.

(c) Note that G_0 is elementary by Lemma 2(a). Suppose there exists a connected component P of $G - V(G_0)$ that has neighbours in at least two maximal barriers of G_0, say S_1 is one of them. Let G' be the graph obtained from G by contracting S_1 and $V(G_0) - S_1$ into vertices v_1 and v_2 and deleting the edges between v_1 and v_2. Then $|V(G')|$ is even and the existence of P implies that G' is connected. We show that G' has a perfect matching. Otherwise, by Theorem 1, there is a set $X \neq \emptyset$ such that $c_o(G' - X) \geq |X| + 2$. Let us denote by G'' the graph obtained from G' by identifying v_1 and v_2 and let X' be the smallest vertex set in G'' that contains X. Then $c_o(G'' - X') \geq c_o(G' - X) - 1 \geq |X| + 1 > |X'|$ thus G'' is not factor-critical. However, $G'' = G/B(G)$ and it is factor-critical by Lemma 2(a). This contradiction shows that G' has a perfect matching M_1. The edge set M_1 is a matching in G that covers all the vertices in $V(G) - V(G_0)$ and two vertices $s_1 \in S_1$ and $s_2 \in S_2$, where S_2 is a maximal barrier of G_0 different from S_1. By Theorem 4, $G_0 - s_1 - s_2$ has a perfect matching M_2. Then $M_1 \cup M_2$ is a perfect matching of G that contains two edges leaving $V(B(G))$, a contradiction by Theorem 13(a). □

4. SATURATED GRAPHS

In this section we derive from our results on almost critical graphs the Cathedral Theorem 5 for saturated graphs, a result of Lovász and Plummer [5]. To be able to apply our results we need the following lemma.

LEMMA 3. *Let G be a saturated graph. Then:*

(a) For a barrier X of G, C_X is saturated and for all $x \in X$ and $y \in X \cup C_X$, $xy \in E(G)$.
(b) Every strong subgraph H of G is saturated.
(c) G is almost critical.
(d) $G_0 := G[V(B(G))]$ is (elementary and) saturated.
(e) Every maximal barrier of G_0 is a barrier of G.

PROOF. (a) Is immediate by definition.

(b) Let H be a strong subgraph of G with strong barrier X. Let $u, v \in V(H)$ be such that $uv \notin E(H)$. Then $uv \notin E(G)$. Since G is saturated, $G - u - v$ contains a perfect matching M. Since X is a barrier in G, $M' := M \cap E(H)$ is a matching of $H - u - v$ that is either perfect or covers all the vertices of $H - u - v$ except exactly two vertices x and y in X. In the latter case, by Lemma 3(a), $M' + xy$ is a perfect matching of $H - u - v$. Then, by definition, H is saturated.

(c) Let us suppose that G is not almost critical. By definition, G has a perfect matching so, by Theorem 12(c), G contains two vertex disjoint strong subgraphs H_1 and H_2 with strong barriers X_1 and X_2. Let $x \in X_1$ and $y \in V(H_2) - X_2$. Then $y \in C_{X_1}$ and, by Lemma 3(a), $xy \in E(G)$. Then X_2 does not separate $V(H_2) - X_2$ and $V(G) - V(H_2)$, hence, by definition, H_2 is not a strong subgraph of G, a contradiction.

(d) G_0 is elementary by Lemma 2(a). We prove that G_0 is saturated by induction on $|V(G)|$. For $|V(G)| = 2, 4$ it is trivial. First suppose that for each strong subgraph H of G, $V(H) = V(G)$. Then, by Theorem 13(c), $V(B(G)) = V(G)$, that is, G_0 is saturated. Secondly, suppose that there exists a strong subgraph H of G such that $|V(H)| < |V(G)|$. By Lemma 3(b), H is saturated, thus, by induction, $H_0 := H[V(B(H))]$ is saturated. By Lemma 3(c), G is almost critical, so by Theorem 13(b), $E(D(G)) = E(D(H))$, that is, $G_0 = H_0$ is saturated.

(e) Let $S \in \mathcal{P}(G_0)$ and let us denote by $F_1, \ldots, F_{|S|}$ the odd components of $G_0 - S$. By Lemma 2(c), every connected component H_i of $G - V(G_0)$ has neighbours in exactly one maximal barrier S_i of G_0. By Lemma 3(d), G_0 is saturated so, by Lemma 3(a), $G_0[S_i]$ is a complete graph, thus H_i has neighbours either only in S or in one of the F_j's. Moreover, H_i has a perfect matching by Theorem 13(a). It follows that the components F_j of $G_0 - S$ correspond to odd components of $G - S$, hence S is a barrier of G. □

PROOF OF THEOREM 5. We have to show that if G is a saturated graph then it can be built up by the Cathedral Construction. By Lemma 3(c), G is almost critical. Let $G_0 := G[V(B(G))]$. Then, by Lemma 3(d), G_0 is elementary and saturated. Let $S \in \mathcal{P}(G_0)$. Then, by Lemma 3(e), S is a barrier of G, so by Lemma 3(a), C_S is saturated and every vertex of S is adjacent to every vertex of C_S. Let H_i be an arbitrary connected component of $G - V(G_0)$. By Lemma 2(c), H_i has neighbours in one of the maximal barriers of G_0 (say S_i). Since C_{S_i} is saturated, $H_i = C_{S_i}$. Finally, it easy to see that the last claim in Theorem 5 is equivalent to Theorem 13(b). □

5. DECOMPOSITION

The main tool underlying the results to be proved in the following sections is given in the following lemma. It generalizes Theorem 13(b).

THEOREM 14. *For a strong subgraph H of G, $E(D(G)) = E(D(H)) \cup E(D(G/H))$.*

PROOF. First, let $e \in E(D(H)) \cup E(D(G/H))$. By Theorem 9, the set $e \cap E(D(H))$ $(e \cap E(D(G/H)))$ can be extended to a base $B_1 \in \mathcal{B}(H)$ $(B_2 \in \mathcal{B}(G/H))$. Let $B := B_1 \cup B_2$. Then $e \in B$ and, by Theorems 3(a) and 12(b), $B \in \mathcal{B}(G)$ so, by Theorem 9, $e \in E(D(G))$.

Secondly, let $e \in E(D(G))$. Let us denote by X the strong barrier of H that separates $V(H) - X$ and $V(G) - V(H)$.

LEMMA 4. *There is a base $B_e \in \mathcal{B}(G)$ such that $e \in B_e$ and $|B_e \cap E(H)| = 1$.*

PROOF. If at least one of the two end vertices of e is contained in one of the components of $H - X$, then let us denote this component by K, otherwise let K be an arbitrary component of $H - X$. Let f be a φ-extreme edge in H which connects K to X, such an edge exists by Lemma 1(a). Let $B' \in \mathcal{B}(G/H)$ and let $B_f := B' \cup f$. By Theorems 3(a) and 12(b), $B_f \in \mathcal{B}(G)$ with $f \in B_f$ and $|B_f \cap E(H)| = 1$. The edge e is φ-extreme in G thus, by Theorem 9, it can be extended to a base $B_e \in \mathcal{B}(G)$ using elements in B_f. We still have to show that $|B_e \cap E(H)| = 1$. By construction, $|B_e \cap E(H)| \le 2$. Let us denote by X' (by V') the smallest vertex set in G/B_e that contains X $(V(H))$ and let $H' := (G/B_e)[V']$. G/B_e is factor-critical because $B_e \in \mathcal{B}(G)$, whence, by Theorems 12(a) and 10, $c_o(H' - X') < |X'|$. Then, by construction, $|X| - 1 = c_o(H - X) - 1 \le c_o(H' - X') \le |X'| - 1 \le |X| - 1$. Thus $c_o(H' - X') = c_o(H - X) - 1$ and $|X'| = |X|$. It follows that $|B_e \cap E(H)| = 1$. □

Let $D_e = B_e - E(H)$. Let $G' := G/D_e$. Then, by Theorems 9 and 2(a) and Lemma 4, $\varphi(G') = 1$. We claim that H remains a strong subgraph in G'. Otherwise, $|X|$ decreases and then the corresponding set X' violates the Tutte's condition in G', a contradiction by Theorem 12(c).

First suppose that $e \in E(H)$. Then, by Theorem 13(b), $e \in E(D(H))$. Now suppose that $e \in E(G/H)$. By Theorem 12(b), G'/H is factor-critical. Since $(G/H)/D_e = G'/H$ and $|D_e| = \varphi(G) - 1 = \varphi(G/H)$, $e \in D_e \in \mathcal{B}(G/H)$; that is, by Theorem 9, $e \in E(D(G/H))$. □

By Theorem 12(a), a connected graph G can be decomposed (by contracting strong subgraphs) into $\varphi(G)$ almost critical graphs and a factor-critical graph; that is, any connected graph G can be constructed by starting from a factor-critical graph and by applying $\varphi(G)$ times the inverse operation of contraction of a strong subgraph.

By Theorem 14, a 2-edge-connected graph G is $\varphi(G)$-covered if and only if G can be decomposed (by contracting strong subgraphs) into $\varphi(G)$ matching-covered graphs and a single vertex, in other words, a 2-edge-connected graph G is $\varphi(G)$-covered if and only if G can be constructed by starting from a vertex and by applying $\varphi(G)$ times the inverse operation of contraction of a matching-covered strong subgraph. This way we can construct as many examples of $\varphi(G)$-covered graphs as we want.

6. φ-EXTREME EDGES

The following result generalizes Lemma 2(a) and gives some information about the structure of $D(G)$ for an arbitrary 2-edge-connected graph G.

THEOREM 15. *Let us denote by $G_1, \ldots, G_k$ the blocks of $D(G)$. Then:*

(a) The graph $S(G) := ((G/G_1)/\ldots)/G_k$ is factor-critical.
(b) $\varphi(G) = \sum_1^k \varphi(G_i)$.
(c) $\varphi(G/G_i) = \varphi(G) - \varphi(G_i)$ $(i = 1, \ldots, k)$.
(d) G_i is φ-covered $(i = 1, \ldots, k)$.

PROOF. We prove by induction on $\varphi(G)$. For $\varphi(G) = 1$, Theorem 13(a) and Lemma 2(a) imply (a),(b), (c) and (d).

Now suppose that $\varphi(G) \geq 2$. Let H be a strong subgraph of G with strong barrier X. Then, by Theorem 14, $D(G)$ contains $D(H)$ and hence, by Theorem 13(a) it contains $B(H)$. By Theorem 12(b), H is almost critical so, by Lemma 2(a), $B(H)$ is matching-covered. Thus $B(H)$ is 2-vertex-connected, and hence, by Theorem 14, it is included in some G_i, say G_1. We remark that $E(D(H)) = E(G_1) \cap E(H)$ by Theorem 14 and $X \subset V(B(H))$ by Lemma 1(a). Consider the graph $G' := G/B(H)$. Then the vertex v of G' that corresponds to $B(H)$ separates $V(H/B(H)) - v$ and $V(G') - V(H/B(H))$. Moreover, $H/B(H)$ is factor-critical by Lemma 2(a), so $\varphi(G') = \varphi(G/H)$ and $E(D(G/H)) = E(D(G'))$. By Theorem 14, $E(D(G)) - E(D(H)) = E(D(G/H))$, so $E(D(G')) = E(D(G)) - E(D(H))$. Thus the blocks $G'_1, \ldots, G'_l$ of $D(G')$ are exactly the blocks of $G_1/B(H)$ and $G_2, \ldots, G_k$. By Theorem 12(b), $\varphi(G') = \varphi(G/H) = \varphi(G) - 1$, thus, by the induction hypothesis, the theorem is true for G'.

LEMMA 5. *$B(H)$ is a strong subgraph of G_1.*

PROOF. $B(H)$ is nice in H by Theorem 13(a) so the factor-critical components of $H - X$ correspond to odd components of $B(H) - X$. Thus X is a barrier of $B(H)$. Let Y be a maximal barrier of $B(H)$ including X. Then, since $B(H)$ is matching-covered by Lemma 2(a), Y is a strong barrier of $B(H)$. Since X separates $H - X$ and $G - V(H)$ in G, Y separates $B(H) - Y$ and $G_1 - V(B(H))$. It follows that $B(H)$ is a strong subgraph of G_1 with strong barrier Y. □

(a) Since $S(G) = S(G')$ (in the second case we contracted G_1 in two steps, namely first $B(H)$ and then the blocks of $G_1/B(H)$), the statement follows from the induction hypothesis.
(b) By Lemma 5 and Theorem 12(b), $\varphi(G_1/B(H)) = \varphi(G_1) - 1$. By induction, $\varphi(G') = \sum_1^l \varphi(G'_i)$. Then $\varphi(G) = \varphi(G') + 1 = \sum_1^l \varphi(G'_i) + 1 = (\varphi(G_1) - 1) + \sum_2^k \varphi(G_i) + 1 = \sum_1^k \varphi(G_i)$.
(c) By Theorem 3(a), $\varphi(G) \leq \varphi(G/G_i) + \varphi(G_i)$ and $\varphi(G/G_i) \leq \varphi(((G/G_1)/\ldots)/G_k) + \sum_2^k \varphi(G_j)$. By adding these two inequalities, and using that $\varphi(((G/G_1)/\ldots)/G_k) = 0$ by (a) and Theorem 2(a), and $\sum_1^k \varphi(G_j) = \varphi(G)$ by (b), we have $\varphi(G) \leq \sum_1^k \varphi(G_j) = \varphi(G)$. Thus equality holds everywhere, hence $\varphi(G) = \varphi(G/G_i) + \varphi(G_i)$, as we claimed.
(d) For $i \geq 2$ the statement follows from the induction hypothesis. For G_1 it follows from the induction hypothesis and from Theorem 14. □

7. The Blocks of the Ear Matroid

In this section we present a simple description of the blocks of the ear matroid $\mathcal{M}(G)$ for an arbitrary 2-edge-connected graph G. The close relation between the circuits of the ear matroid $\mathcal{M}(G)$ and the good even circuits of G is presented in the following lemma.

LEMMA 6. *Two edges e and f of a 2-edge-connected graph G belong to a good even circuit of G if and only if e and f are in the same block of the ear matroid $\mathcal{M}(G)$.*

PROOF. If e and f belong to the starting even ear P_1 of an optimal ear-decomposition then choosing one edge from each even ear (let e be chosen from P_1) we obtain a set F for which $|F| = \varphi(G)$, G/F and $G/(F-e+f)$ are factor-critical by Theorem 2(a), thus, by Theorem 9, F and $F-e+f$ are in $\mathcal{B}(G)$, that is, e and f belong to the same block of $\mathcal{M}(G)$.

Alternatively, let $F \in \mathcal{B}(G)$ containing e such that $F-e+f \in \mathcal{B}(G)$. Let $G' := G \times (F-e)$. Since F is a minimal critical making edge set, it is a forest and $\varphi(G/(F-e)) = 1$. Then, by Theorem 8(b), $\varphi(G') = 1$. Moreover, e and f are φ-extreme in G'. By Lemma 2(b), there exists an optimal ear-decomposition of G' such that the starting ear contains e and f and it is the unique even ear. Obviously, this ear-decomposition provides the desired ear-decomposition of G. □

It is natural to investigate graphs whose ear matroid is loopless. Note that, by definition, these are exactly the φ-covered graphs. The blocks of the ear matroid $\mathcal{M}(G)$ of a φ-covered graph can easily be described.

THEOREM 16. *Let G be a 2-vertex-connected φ-covered graph. Then the ear matroid $\mathcal{M}(G)$ has one block.*

PROOF. We prove the theorem by induction on $\varphi(G)$. If $\varphi(G) = 1$, then G is matching-covered by Claim 1, and then, by Theorem 6(d) and Lemma 6, the theorem is true. In the rest of the proof we suppose that $\varphi(G) \geq 2$. Let H be a strong subgraph of G with strong barrier X. By Theorem 14, H and G/H are φ-covered and, by Theorem 12(b), $\varphi(H) = 1$ and $\varphi(G/H) = \varphi(G) - 1$. Let G_1 be an arbitrary block of G/H.

(i) Let e_1 and e_2 be two arbitrary edges of H. Let $B \in \mathcal{B}(G/H)$. Then, by Theorem 9, $(G/H)/B$ is factor-critical. H/e_1 and H/e_2 are factor-critical by Claim 1. Let $B' := B + e_1$. Note that $|B'| = \varphi(G)$. Then, by Theorem 3(a), G/B' and $G/(B'-e_1+e_2)$ are factor-critical, thus, by Theorem 9, B' and $B'-e_1+e_2$ are in $\mathcal{B}(G)$, hence e_1 and e_2 belong to the same block of $\mathcal{M}(G)$.

(ii) Let e_1 and e_2 be two arbitrary edges of G_1. By induction, e_1 and e_2 belong to the same block of $\mathcal{M}(G_1)$, thus there exists a base $B \in \mathcal{B}(G_1)$ such that $e_1 \in B$ and $B - e_1 + e_2 \in \mathcal{B}(G_1)$. For each block G_i of G/H different from G_1 let $B_i \in \mathcal{B}(G_i)$. Furthermore, let $f \in E(H)$. Finally, let $D := B \cup (\cup B_i) + f$. Note that $|D| = \varphi(G)$. Then, by Theorem 3(a), G/D and $G/(D-e_1+e_2)$ are factor-critical, thus, by Theorem 9, D and $D-e_1+e_2 \in \mathcal{B}(G)$. Hence e_1 and e_2 belong to the same block of $\mathcal{M}(G)$.

(iii) Let e_1 and f_1 be two edges of G_1 such that the corresponding two edges in G are incident on two different vertices u and v of X. By the 2-vertex-connectivity of G, such edges exist. Let e_2 and f_2 be two edges of H incident on u and v, respectively. By (i), (ii) and Lemma 6, there exists an optimal ear-decomposition $P_1 + P_2 + \cdots + P_k$ ($P'_1 + P'_2 + \cdots + P'_l$) of H (of G_1) such that e_2 and f_2 (e_1 and f_1) belong to the starting even ear. Furthermore, let $P''_1 + P''_2 + \cdots + P''_m$ be an optimal ear-decomposition of $(G/H)/G_1$ such that the first ear contains the vertex corresponding to the contracted vertex set. Using these ear-decompositions we provide an optimal ear-decomposition of G such that the starting even ear will contain e_1 and e_2. By Lemma 1(b), u and v divide P_1 into two paths D_1 and D_2 of even length. Suppose D_1 contains e_2. Consider the following ear-decomposition of G: $(D_1 + P_1') + D_2 + P_2 + \cdots + P_k + P_2' + \cdots + P_l' + P_1'' + P_2'' + \cdots + P_m''$. It is clear that this is an optimal ear-decomposition of G, the first ear contains e_1 and e_2 and it is even. Hence, by Lemma 6, e_1 and e_2 belong to the same block of $\mathcal{M}(G)$.

(i), (ii) and (iii) imply the theorem. □

THEOREM 17. *The edge sets of the blocks of $D(G)$ and the blocks of $\mathcal{M}(G)$ coincide.*

PROOF. (a) Let e and f be two edges of G from the same block of $\mathcal{M}(G)$. By Lemma 6, there exists a good even circuit C that contains e and f. Since, by Theorem 11(b), every edge of C is φ-extreme, the edges of this circuit C belong to the same block of $D(G)$.

(b) Let e and f be two edges of G from the same block G_1 of $D(G)$. By Theorem 15(d), G_1 is φ-covered, thus, by Theorem 16 and Lemma 6, there exists an optimal ear-decomposition of G_1 such that the starting even ear contains e and f. By Theorem 15(c), $\varphi(G/G_1) = \varphi(G) - \varphi(G_1)$, so this ear-decomposition can be extended to an optimal ear-decomposition of G such that the starting even ear contains e and f. Then, by Lemma 6, e and f belong to the same block of $\mathcal{M}(G)$. □

8. φ-COVERED GRAPHS

The aim of this section is to extend earlier results on matching-covered graphs of Lovász and Plummer [5] to φ-covered graphs. First we prove a technical lemma.

LEMMA 7. *Let e be a φ-extreme edge of a 2-edge-connected graph G with $\varphi(G) \geq 2$. Then there exists a strong subgraph H of G such that $e \in E(G/H)$.*

PROOF. First suppose that G has a perfect matching. Then, by Theorem 12(c), G has two vertex disjoint strong subgraphs. Clearly, for one of them $e \in E(G/H)$. Secondly, suppose that G has no perfect matching. Then, by Theorem 1, there exists a set X with $c_o(G - X) > |X|$. Let X be such a maximal vertex set. Then each component of $G - X$ is factor-critical. Since G is not factor-critical by Theorem 2(a), $|X| \neq \emptyset$.

(i) If a component F of $G - X$ contains an end vertex of e, then by Theorem 10, G has a strong subgraph H such that $V(H) \subseteq V(G) - V(F)$ so we are done.

(ii) Otherwise, by Theorem 10, G has a strong subgraph H with strong barrier $Y \subseteq X$ such that each component of $H - Y$ is a component of $G - X$. We claim that $e \in E(G/H)$. If not then the two end vertices u and v of e belong to Y because we are in (ii). Then, by Lemma 1(a), e is not φ-extreme in H. This contradicts the fact that e is φ-extreme in H by Theorem 14. □

The following theorem generalizes Theorem 6(d) for φ-covered graphs. By Lemma 6, it is equivalent to Theorem 16.

THEOREM 18. *For a 2-vertex-connected φ-covered graph G, any two edges belong to a good even circuit of G.*

By Theorem 7(b), each matching-covered graph has a 2-graded ear-decomposition. This result can also be generalized for φ-covered graphs. A sequence $(G_0, G_1, \ldots, G_m)$ of subgraphs of G is a *generalized 2-graded ear-decomposition* of G if G_0 is a vertex, $G_m = G$, for every $i = 1, \ldots, m$: G_i is φ-covered; G_i is obtained from G_{i-1} by adding at most two disjoint paths (ears) which are openly disjoint from G_{i-1} but their end-vertices belong to G_{i-1}, if we add two ears then both are of odd length; and $\varphi(G_{i-1}) \leq \varphi(G_i)$. This is the natural extension of the original definition of Lovász and Plummer. Indeed, if G is matching-covered then $\varphi(G) = 1$, thus the first ear will be even and all the other ears will be odd; and for all i, $1 = \varphi(G_1) \leq \varphi(G_i) \leq \varphi(G) = 1$ and G_i is φ-covered so, by Claim 1, G_i is matching-covered.

THEOREM 19. *Let e be an arbitrary edge of a φ-covered graph G. Then G possesses a generalized 2-graded ear-decomposition such that the starting ear contains e.*

PROOF. If $\varphi(G) = 1$ then, by Claim 1, G is matching-covered so, by Theorem 7(c), we are done. From now on we assume that $\varphi(G) \geq 2$. We shall frequently use in the proof that a graph L is φ-covered if and only if each block of L is φ-covered. We prove the theorem by induction on $|V(G)|$. We may suppose that G is 2-vertex-connected because, by induction, for each block the theorem is true. By Lemma 7, there exists a strong subgraph H of G with strong barrier X such that $e \in E(G/H)$. By Theorem 14, H and G/H are φ-covered. By Theorem 12(b) and Claim 1, H is matching-covered. Let us denote by v the vertex of G/H corresponding to H. Let us denote by Q the block of G/H which contains e. Note that $v \in V(Q)$. Since G/H is φ-covered, Q is also φ-covered. By induction, Q has a generalized 2-graded ear-decomposition $(G_0, G_1, \ldots, G_k)$ such that the starting ear contains e. Let G_j be the first subgraph of Q which contains v and let a and b be the two edges of G_j incident on v. $(G/H)/Q$ is also φ-covered so, by induction, $(G/H)/Q$ has a generalized 2-graded ear-decomposition $(G_0^*, G_1^*, \ldots, G_p^*)$ such that the starting ear contains an edge incident on v.

(i) First suppose that a and b are incident on the same vertex u of X in G. Let c be an edge of H incident on u. By Theorem 7(c), H has a 2-graded ear-decomposition $(G_0', G_1', \ldots, G_l')$ such that the starting ear contains c. Let $G_i'' = G_i$ if $0 \leq i \leq j$, let G_i'' be the graph obtained from G_{i-j}' by replacing the vertex u by G_j if $j+1 \leq i \leq j+l$, let G_i'' be the graph obtained from G_{i-l} by replacing the vertex v by G_{j+l}'' if $j+l+1 \leq i \leq k+l$ and finally let G_i'' be the graph obtained from G_{i-k-l}^* by replacing the vertex v by G_{k+l}'' if $k+l+1 \leq i \leq k+l+p$. We show that $(G_0'', G_1'', \ldots, G_{k+l+p}'')$ is the desired generalized 2-graded ear-decomposition of G. The starting ear contains e, in each step we added at most two ears, when two ears were added then they were of odd length, $\varphi(G_i'') \leq \varphi(G_{i+1}'')$ and finally by Theorem 14, each subgraph G_i'' is φ-covered.

(ii) Secondly, suppose that a and b are incident on different vertices u and w of X in G. Let c and d be two edges of H incident on u and w, respectively. By Theorem 7(c), H has a 2-graded ear-decomposition $(G_0', G_1', \ldots, G_l')$ such that the starting ear P_1 contains c and d. u and w divide P_1 (which is an even ear) into two paths D_1 and D_2. By Lemma 1(b), D_1 and D_2 are of even length. Let $G_i'' = G_i$ if $0 \leq i \leq j-1$, G_j'' be the graph obtained from G_j by replacing the vertex v by D_1, G_{j+1}'' be the graph obtained from G_j by replacing the vertex v by P_1, let G_i'' be the graph obtained from G_{i-j}' by replacing P_1 by G_{j+1}'' if $j+2 \leq i \leq j+l$, let G_i'' be the graph obtained from G_{i-l} by replacing the vertex v by G_{j+l}'' if $j+l+1 \leq i \leq k+l$ and finally, as above, let G_i'' be the graph obtained from G_{i-k-l}^* by replacing the vertex v by G_{k+l}'' if $k+l+1 \leq i \leq k+l+p$. It is easy to see that $(G_0'', G_1'', \ldots, G_{k+l+p}'')$ is the desired generalized 2-graded ear-decomposition of G. □

The next theorem is the natural generalization of Theorem 7(a). However, we cannot prove Theorem 19 using this result.

THEOREM 20. *Let $F := \{e_1, \ldots, e_k\}$ be a set of non-edges of a φ-covered graph G. If $G + F$ is φ-covered and $\varphi(G) = \varphi(G + F)$, then there exist $i \leq j$ such that $G + e_i + e_j$ is φ-covered.*

PROOF. We prove the theorem by induction on $\varphi(G)$. If $\varphi(G) = 1$, then G is matching-covered by Claim 1 so, by Theorem 7(a), we are done. In the following we suppose that $\varphi(G) \geq 2$. Let $F' \subseteq F$ be a minimal non-empty set in F such that $G' := G + F'$ is φ-covered. Then $\varphi(G) = \varphi(G')$ because $\varphi(G) \geq \varphi(G + F') \geq \varphi(G + F) = \varphi(G)$. We claim that $|F'| \leq 2$. Suppose that $|F'| \geq 3$ and let $e_i \in F'$. By Lemma 7, there exists a strong subgraph H of G' such that $e_i \in E(G'/H)$. By Theorem 14, H and G'/H are φ-covered. Let $E_1 := E(H) \cap F'$ and $E_2 := E(G'/H) \cap F'$. Then $E_1 \cup E_2 = F'$ and $E_2 \neq \emptyset$.

First suppose $E_1 = \emptyset$. Then H is a strong subgraph of G, so by Theorem 14, G/H is φ-covered. By Theorem 12, $\varphi(G/H) = \varphi(G) - 1 = \varphi(G') - 1 = \varphi(G'/H)$, thus by induction for G/H and F', there exists $\emptyset \neq F'' \subseteq F'$ such that $|F''| \leq 2$ and $(G/H) + F''$ is φ-covered. By Theorem 14, $G + F''$ is φ-covered, and we are done.

Secondly suppose $E_1 \neq \emptyset$. Clearly, each edge of G is φ-extreme in $G + E_1$. Furthermore, each edge of E_1 is φ-extreme in H, so by Theorem 14, they are φ-extreme in $G + E_1$. Thus $G + E_1$ is φ-covered. Since $E_1 \subset F'$, this contradicts the minimality of F'. $\square$

EXAMPLE. The following example shows the necessity of the condition $\varphi(G) = \varphi(G+F)$ in Theorem 20. Let $G := (V, E)$, where $V = \{a, b, c, d\}$, $E = \{ab, ab, ac, ac, ad, ad\}$ and let $F := \{bc, bd, cd\}$. Then G and $G + F$ are φ-covered but for every $\emptyset \neq F' \subset F$, $G + F'$ is not φ-covered. Note that $\varphi(G) = 3$ and $\varphi(G + F) = 1$.

ACKNOWLEDGEMENT

This research was undertaken while the author visited the Research Institute for Discrete Mathematics in Bonn, supported by an Alexander von Humboldt fellowship. I thank the referee for his remarks that improved the presentation of the paper.

REFERENCES

1. J. Edmonds, L. Lovász and W. R. Pulleyblank, Brick decompositions and the matching rank of graphs, *Combinatorica*, **2** (1982), 247–274.
2. A. Frank, Conservative weightings and ear-decompositions of graphs, *Combinatorica*, **13** (1993), 65–81.
3. C. H. C. Little, A theorem on connected graphs in which every edge belongs to a 1-factor, *J. Aust. Math. Soc.*, **18** (1974), 450–452.
4. L. Lovász, A note on factor-critical graphs, *Stud. Sci. Math. Hung.*, **7** (1972), 279–280.
5. L. Lovász and M. D. Plummer, *Matching Theory*, Amsterdam, North Holland, 1986.
6. Z. Szigeti, On Lovász's Cathedral Theorem, *Proceedings of the 3rd IPCO Conference, Erice*, G. Rinaldi and L. A. Wolsey (eds), 1993, pp. 413–423.
7. Z. Szigeti, On a matroid defined by ear-decompositions of graphs, *Combinatorica*, **16** (1996), 233–241.
8. Z. Szigeti, On the two ear theorem of matching-covered graphs, *J. Comb. Theory, Ser. B*, **74** (1998a), 104–109.
9. Z. Szigeti, Perfect matchings versus odd cuts, Technical Report of Research Institute for Discrete Mathematics, 1998b, submitted for publication.
10. Z. Szigeti, On optimal ear-decompositions of graphs, *Proceedings of the 7th IPCO Conference, Graz, Lecture Notes in Computer Science*,**1610**, Springer, Heidelberg, 1999, pp. 415–428.

11. W. T. Tutte, The factorization of linear graphs, *J. London Math. Soc.*, **22** (1947), 107–111.

Received 5 December 1998 and accepted 12 March 2001, published electronically 21 May 2001

ZOLTÁN SZIGETI
Equipe Combinatoire,
Université Paris 6,
75252 Paris, Cedex 05,
France