

Edge-splittings preserving local edge-connectivity of graphs[☆]

Zoltán Szigeti

Laboratoire G-SCOP, 46 Avenue Félix Viallet, Grenoble, France

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Abstract

Let $G = (V + s, E)$ be a 2-edge-connected graph with a designated vertex s . A pair of edges rs, st is called admissible if splitting off these edges (replacing rs and st by rt) preserves the local edge-connectivity (the maximum number of pairwise edge disjoint paths) between each pair of vertices in V . The operation splitting off is very useful in graph theory, it is especially powerful in the solution of edge-connectivity augmentation problems as it was shown by Frank [Augmenting graphs to meet edge-connectivity requirements, SIAM J. Discrete Math. 5(1) (1992) 22–53]. Mader [A reduction method for edge-connectivity in graphs, Ann. Discrete Math. 3 (1978) 145–164] proved that if $d(s) \neq 3$ then there exists an admissible pair incident to s . We generalize this result by showing that if $d(s) \geq 4$ then there exists an edge incident to s that belongs to at least $\lfloor d(s)/3 \rfloor$ admissible pairs. An infinite family of graphs shows that this bound is best possible. We also refine a result of Frank [On a theorem of Mader, Discrete Math. 101 (1992) 49–57] by describing the structure of the graph if an edge incident to s belongs to no admissible pairs. This provides a new proof for Mader's theorem.

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1. Introduction

In this paper, $G = (V + s, E)$ denotes a 2-edge-connected graph, s being a vertex not in V . (It would be enough to suppose that no cut edge is incident to s but for the sake of simplicity we suppose that G contains no cut edge at all.)

For two vertices $u, v \in V$, the *local edge-connectivity*, $\lambda_G(u, v)$, between u and v is the maximum number of edge disjoint paths between u and v . If $\lambda_G(u, v) \geq k$ for all pairs $u, v \in V$, then G is called *k-edge-connected in V*.

The operation *splitting off* is defined as follows: two edges rs and st are replaced by a new edge rt . The graph obtained from G by splitting off a pair of edges rs, st is denoted by G_{rt} . A pair of edges rs, st is called *k-admissible* if G_{rt} is *k-edge-connected in V*. The pair of edges rs, st is called *admissible* if $\lambda_{G_{rt}}(u, v) \geq \lambda_G(u, v)$ for all pairs $u, v \in V$. An edge incident to s is called *admissible* if it belongs to an admissible pair, otherwise it is called *non-admissible*.

The first splitting off result is due to Lovász [6].

Theorem 1.1. *If $G = (V + s, E)$ is k-edge-connected in V for some $k \geq 2$ and $d(s)$ is even then each edge incident to s belongs to a k-admissible pair.*

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E-mail address: Zoltan.Szigeti@g-scop.inpg.fr.

Cai and Sun [3] showed how to apply this result to solve the following global edge-connectivity augmentation problem: given a graph H and an edge-connectivity requirement $k \in \mathbb{Z}_+$, find the minimum number of new edges whose addition makes the graph k -edge-connected.

Theorem 1.1 was extended in Bang-Jensen et al. [1].

Theorem 1.2. *If $G = (V + s, E)$ is k -edge-connected in V for some $k \geq 2$ and $d(s)$ is even then each edge incident to s belongs to at least $d(s)/2$ (resp. $d(s)/2 - 1$) k -admissible pairs if k is even (resp. odd).*

In [1], we applied Theorem 1.2 to solve the global edge-connectivity augmentation problem in bipartite graphs: given a connected bipartite graph H and an edge-connectivity requirement $k \in \mathbb{Z}_+$, what is the minimum number of new edges whose addition results in a bipartite k -edge-connected graph.

It is easy to construct examples to show that the bounds of Theorem 1.2 are best possible.

Mader [7] generalized Theorem 1.1 on local edge-connectivity.

Theorem 1.3. *If $G = (V + s, E)$ is 2-edge-connected and $d(s) \neq 3$ then there exists an admissible pair incident to s .*

Applying this result, Frank [5] solved the local edge-connectivity augmentation problem: given a graph $H = (V, E)$ and a requirement function $r: V \times V \rightarrow \mathbb{Z}_+$, find the minimum number of new edges F such that $\lambda_{H+F}(u, v) \geq r(u, v)$ for all pairs $u, v \in V$.

The main contribution of the present paper is the following strengthening of Theorem 1.3. It can be considered as the counterpart of Theorem 1.2 for local edge-connectivity.

Theorem 1.4. *If $G = (V + s, E)$ is a 2-edge-connected graph and $d(s) \geq 4$ then there is an edge sr that belongs to at least $\lfloor d(s)/3 \rfloor$ admissible pairs incident to s .*

We present, in Section 3, an infinite family of graphs showing that our bound is best possible.

Theorem 1.3 implies that at most three edges incident to s are non-admissible. Frank [5] provided a slight generalization of this result.

Theorem 1.5. *If $G = (V + s, E)$ is 2-edge-connected and $d(s) \neq 3$ then at most one edge incident to s belongs to no admissible pair.*

We refine this result by describing the structure of the graph if it contains a non-admissible edge incident to s . (For definitions, see Section 2.)

Theorem 1.6. *Let st be an edge of a 2-edge-connected graph $G = (V + s, E)$. The following are equivalent:*

- (a) *The edge st is non-admissible.*
- (b) *There exist two dangerous sets M_1 and M_2 such that $t \in M_1 \cap M_2$ and $M_1 \cup M_2$ contains all the neighbours of s .*
- (c) *The degree $d(s)$ of s is odd and there exist two disjoint tight sets C_1 and C_2 in $V - t$ such that $d(s, C_1) = d(s, C_2) = (d(s) - 1)/2$.*

As an application of Theorem 1.6 we present the following result.

Theorem 1.7. *Let $G = (V + s, E)$ be a 2-edge-connected graph with $d(s) \neq 3$. If an edge st is non-admissible then each edge $sr \neq st$ belongs to exactly $(d(s) - 1)/2$ admissible pairs.*

The proofs of Theorems 1.6 and 1.7, given in Sections 4 and 5, together provide a new proof of Theorem 1.5 and hence of Theorem 1.3.

We mention a related interesting result of Bang-Jensen and Jordán [2].

Theorem 1.8. *Let $G = (V + s, E)$ be a 2-edge-connected graph. Then, for every edge st , the number of edges rs for which the pair of edges rs, st is non-admissible is at most $2k^2 - 2k$, where $k = \max\{\lambda_G(u, v) : u, v \in V\}$.*

2. Notation and preliminary results

Let $G = (V + s, E)$ be a graph, with s a vertex not in V . Let $\Gamma(s)$ denote the set of neighbours of s . We use the notation $\subset$ for proper subset. For a set $T \subset V$, $T \neq \emptyset$, we denote the graph obtained from G by contracting T into one vertex v_T by G/T .

Let $X, Y \subseteq V + s$. Let $d(X, Y)$ denote the number of edges between $X - Y$ and $Y - X$. Let $\bar{d}(X, Y)$ denote the number of edges between $X \cap Y$ and $V + s - (X \cup Y)$. We define the degree of the set X by $d(X) = d(X, V + s - X)$. The degree function satisfies the following two well-known equalities:

- (1) $d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d(X, Y)$,
- (2) $d(X) + d(Y) = d(X - Y) + d(Y - X) + 2\bar{d}(X, Y)$.

Observe that, by Menger's theorem, $\lambda_G(x, y) = \lambda(x, y) = \min\{d(Z) : Z \subset V + s, x \in Z, y \notin Z\}$ for all $x, y \in V$. We define the function $R(X)$ as follows: $R(\emptyset) = R(V) = 0$ and, for a set $X \subset V$, $X \neq \emptyset$, let

$$R(X) = \max\{\lambda_G(x, y) : x \in X, y \in V - X\}.$$

Observe that the function $R(X)$ satisfies (3) and (4) for $X, Y \subset V$:

- (3) $R(X) = R(V - X)$,
- (4) $R((X - Y) \cup (Y - X)) \leq \max\{R(X - Y), R(Y - X)\}$.

The following property of $R(X)$ can be found in [4, Proposition 5.4]: for $X, Y \subset V$, at least one of (5) and (6) hold. If $X \cup Y = V$ then (2) holds:

- (5) $R(X) + R(Y) \leq R(X \cap Y) + R(X \cup Y)$,
- (6) $R(X) + R(Y) \leq R(X - Y) + R(Y - X)$.

Finally, we define the function

$$h(X) := d(X) - R(X).$$

Note that the function $h(X)$ satisfies (7) and (8) for $X, Y \subset V$.

- (7) $h(X) \geq 0$,
- (8) $h(X) = h(V - X) + 2d(s, X) - d(s)$.

The properties above imply:

Proposition 2.1. *For $X, Y \subset V$, at least one of (9) and (10) hold. If $X \cup Y = V$ then (10) holds:*

- (9) $h(X) + h(Y) \geq h(X \cap Y) + h(X \cup Y) + 2d(X, Y)$,
- (10) $h(X) + h(Y) \geq h(X - Y) + h(Y - X) + 2\bar{d}(X, Y)$.

A set $\emptyset \neq X \subset V$ is called *tight* if $h(X) = 0$ and it is called *dangerous* if $h(X) \leq 1$. Note that tight and dangerous sets are, by definition, subsets of V .

The following claim is due to Mader.

Claim 2.2. *Let T be a tight set in a graph $G = (V + s, E)$ and $G' := G/T$:*

- (a) *If a pair of edges e', f' incident to s is admissible in G' then the corresponding pair of edges e, f is admissible in G .*
- (b) *If $X' \subseteq V(G') - s$ then $h_{G'}(X') = h_G(X)$, where $X = X' - v_T \cup T$ if $v_T \in X'$ and $X = X'$ otherwise.*

The reduction method of Claim 2.2 will be applied in our proofs and hence we will be able to assume that:

(11) every tight set is a singleton.

We need the following claims.

Claim 2.3 (Frank [5, Claim 3.1]). *A pair of edges us, sv of a graph $G = (V + s, E)$ is admissible if and only if there is no dangerous set M with $u, v \in M$.*

Claim 2.4 (Frank [5, Claim 4.1]). *Let $G = (V + s, E)$ be a graph and $t \in \Gamma(s)$ be a vertex of minimum degree. Suppose that (11) holds. If a set $M \subseteq V$ contains t and $|\Gamma(s) \cap M| \geq 2$, then $R(M - t) \geq R(M)$.*

Claim 2.5 (Bang-Jensen and Jordán [2, Lemma 5.4]). *Let $G = (V + s, E)$ be a 2-edge-connected graph. If M is a dangerous set then*

- (a) $d(s, M) \leq (d(s) + 1)/2$, with equality only if $V - M$ is tight, and
- (b) $d(X, M - X) \geq 1$ for every $\emptyset \neq X \subset M$.

Proof. (a) By (8), since M is dangerous and by applying (8) for $V - M$, $d(s, M) = (d(s) + h(M) - h(V - M))/2 \leq (d(s) + 1)/2$ and (a) follows. $\square$

We close this section with a technical lemma.

Lemma 2.6. *Let $G = (V + s, E)$ be a 2-edge-connected graph, $st \in E$ and $S \subseteq V$. Let $\mathcal{M}$ be a minimum collection of dangerous sets such that $t \in \cap \mathcal{M}$ and $S \subseteq \cup \mathcal{M}$. If $|\mathcal{M}| \geq 3$, (11) holds and $M_i, M_j \in \mathcal{M}$, then:*

- (a) (Bang-Jensen and Jordán [2, Lemma 5.4]) *Property (10) does not apply for M_i and M_j , and*
- (b) $M_i \cap M_j = t$.

Proof. (a) Suppose that (10) applies for M_i and M_j . Then, by $1 \geq h(M_i)$ and $1 \geq h(M_j)$, we have $h(M_i - M_j) = 0$ and $h(M_j - M_i) = 0$ (so by (11), $M_i - M_j = r_i$ and $M_j - M_i = r_j$ for some vertices $r_i, r_j \in V$) and $\bar{d}(M_i, M_j) = 1$. Let $M_k \in \mathcal{M} - \{M_i, M_j\}$ and $X = M_i \cap M_j \cap M_k$. Note that $t \in X$ so $X \neq \emptyset$. By the minimality of $\mathcal{M}$, $M_k - X \neq \emptyset$. Then, by Claim 2.5(b) and since st enters $M_i \cap M_j$, we have $1 \leq d(X, M_k - X) \leq d(M_i \cap M_j, M_k - (M_i \cap M_j)) \leq \bar{d}(M_i, M_j) - d(M_i \cap M_j, s) \leq 1 - 1 = 0$, a contradiction.

(b) By Proposition 2.1 and (a), (9) applies for M_i and M_j . Then, since $1 \geq h(M_i)$, $1 \geq h(M_j)$ and by the minimality of $\mathcal{M}$, $h(M_i \cup M_j) \geq 2$ (otherwise we could replace M_i and M_j by $M_i \cup M_j$), we have $h(M_i \cap M_j) = 0$ and, hence, by (11) and $t \in M_i \cap M_j$, (b) is satisfied. $\square$

3. Proof of Theorem 1.4

The proof is similar to the proof of Theorem 1.3 given by Frank in [5].

Proof. We prove the theorem by induction on $|V|$. We may assume, by Claim 2.2(a), that (11) is satisfied. Let t be a neighbour of s of minimum degree. Let S be the set of neighbours r of s such that $r = t$ or the pair of edges rs, st is not admissible. By Claim 2.3, there is a minimum collection $\mathcal{M}$ of dangerous sets such that $t \in \cap \mathcal{M}$ and $S \subseteq \cup \mathcal{M}$. Suppose that st belongs to less than $\lfloor d(s)/3 \rfloor$ admissible pairs (otherwise, we are done). Then:

(12) $d(s, \cup \mathcal{M}) \geq d(s, S) > d(s) - \lfloor d(s)/3 \rfloor = \lceil 2d(s)/3 \rceil$.

By Claim 2.5(a) and (12), for $M_i \in \mathcal{M}$, $d(s, M_i) \leq (d(s) + 1)/2 < \lceil 2d(s)/3 \rceil < d(s, \cup \mathcal{M})$ and hence $|\mathcal{M}| \geq 2$. Let $M_1, M_2 \in \mathcal{M}$. By the minimality of $\mathcal{M}$, each $M_i \in \mathcal{M}$ contains a neighbour $r_i \neq t$ of s that belongs to no other $M_j \in \mathcal{M}$. Let us choose such a vertex r_i for each $M_i \in \mathcal{M}$.

Claim 3.1. $\mathcal{M} = \{M_1, M_2\}$.

Proof. For $i = 1, 2$, M_i contains t and r_i , so $|\Gamma(s) \cap M_i| \geq 2$. Then, by Claim 2.4, $R(M_1 - t) \geq R(M_1)$ and $R(M_2 - t) \geq R(M_2)$. Suppose that $|\mathcal{M}| \geq 3$. Then, by Lemma 2.6(b), $M_1 \cap M_2 = t$, thus M_1 and M_2 satisfy (6) and hence (10), a contradiction by Lemma 2.6(a). $\square$

Claim 3.2. Property (10) applies for M_1 and M_2 .

Proof. Suppose that (10) does not hold for M_1 and M_2 . Then, by Proposition 2.1, $M_1 \cup M_2 \neq V$ and (9) applies for M_1 and M_2 . By (8), (7), Claim 3.1, (12) and $d(s) \geq 4$, $h(M_1 \cup M_2) \geq 2d(s, M_1 \cup M_2) - d(s) = 2d(s, \cup \mathcal{M}) - d(s) > 2\lceil 2d(s)/3 \rceil - d(s) \geq 2$.

It follows, by $1 \geq h(M_1)$, $1 \geq h(M_2)$, (9) and (7), that $1 + 1 \geq h(M_1) + h(M_2) \geq h(M_1 \cap M_2) + h(M_1 \cup M_2) > 0 + 2$, a contradiction. $\square$

Claim 3.3. $d(s, r_1) + d(s, r_2) \geq \lceil 2d(s)/3 \rceil$.

Proof. By $1 \geq h(M_1)$, $1 \geq h(M_2)$, Claim 3.2, (7), $st \in E$ and $t \in M_1 \cap M_2$, we have $1 + 1 \geq h(M_1) + h(M_2) \geq h(M_1 - M_2) + h(M_2 - M_1) + 2\bar{d}(M_1, M_2) \geq 0 + 0 + 2d(s, M_1 \cap M_2) \geq 2$, so $h(M_1 - M_2) = 0 = h(M_2 - M_1)$ and $d(s, M_1 \cap M_2) = 1$. It follows, by $r_1 \in M_1 - M_2$, $r_2 \in M_2 - M_1$ and (11), that $M_1 - M_2 = r_1$ and $M_2 - M_1 = r_2$. Then, by Claim 3.1 and (12), $d(s, r_1) + d(s, r_2) = d(s, M_1 \cup M_2) - d(s, M_1 \cap M_2) = d(s, \cup \mathcal{M}) - 1 \geq \lceil 2d(s)/3 \rceil$. $\square$

Let e_i be any edge connecting s and r_i for $1 \leq i \leq 2$.

Claim 3.4. The pair of edges e_1, e_2 is admissible.

Proof. Otherwise, by Claim 2.3, there is a dangerous set X with $r_1, r_2 \in X$, and then, by (8), (7), Claim 3.3 and $d(s) \geq 4$, we have $1 \geq h(X) \geq 2d(s, X) - d(s) \geq 2\lceil 2d(s)/3 \rceil - d(s) \geq 2$, a contradiction.

By Claim 3.3, we may assume without loss of generality that $d(s, r_1) \geq \lceil d(s)/3 \rceil \geq \lfloor d(s)/3 \rfloor$. Then, by Claim 3.4, e_2 belongs to at least $\lfloor d(s)/3 \rfloor$ admissible pairs and the proof of Theorem 1.4 is complete. $\square$

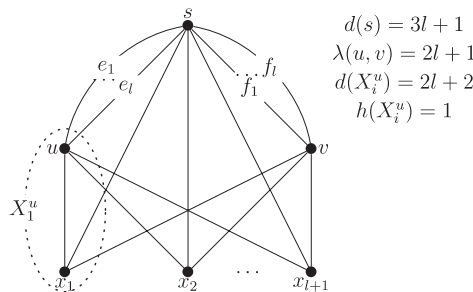


Fig. 1. Each edge incident to s belongs to exactly $\lfloor d(s)/3 \rfloor$ admissible pairs.

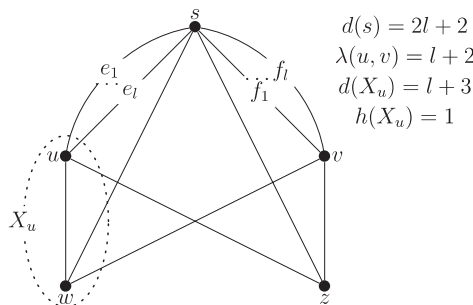


Fig. 2. The degree $d(s)$ of s is even and the edge ws belongs to a unique admissible pair ws, sz .

Examples. There exists an infinite class of graphs in which each edge incident to s belongs to exactly $\lfloor d(s)/3 \rfloor$ admissible pairs. See Fig. 1. We mention that it is not true in general, even if we suppose that the degree of s is even, that each edge incident to s belongs to many admissible pairs. In Fig. 2, the edge ws belongs to the unique admissible pair of edges ws, sz .

4. Proof of Theorem 1.6

Proof. We consider first the most complicated part, we prove that (a) implies (b) by induction on $|V|$.

Claim 4.1. *We may assume that (11) is satisfied.*

Proof. Suppose that there exists a tight set T with $|T| > 1$. Let $G' = G/T$. By Claim 2.2(a), st belongs to no admissible pair in G' , G' is 2-edge-connected and $|V(G')| < |V|$; hence, by induction, (b) is true for G' and, then, by Claim 2.2 (b), it is also true for G . $\square$

The edge st belongs to no admissible pair; thus, by Claim 2.3, there is a minimum collection $\mathcal{M}$ of dangerous sets such that $t \in \cap \mathcal{M}$ and $\Gamma(s) \subseteq \cup \mathcal{M}$. By the minimality of $\mathcal{M}$, each $M_i \in \mathcal{M}$ contains a neighbour $r_i \neq t$ of s that belongs to no other $M_j \in \mathcal{M}$. Let us choose such a vertex r_i for each $M_i \in \mathcal{M}$. By Claim 2.5(a), $d(s) \geq 2$ and $\Gamma(s) \subseteq \cup \mathcal{M}$, for $M_i \in \mathcal{M}$, $d(s, M_i) \leq (d(s) + 1)/2 < d(s) = d(s, \cup \mathcal{M})$ and hence $|\mathcal{M}| \geq 2$.

Suppose that $|\mathcal{M}| \geq 3$. We shall find a contradiction showing that this case can not happen and hence $|\mathcal{M}| = 2$. By Lemma 2.6(b), for all $M_i, M_j \in \mathcal{M}$, $M_i - M_j = M_i - t$. Let $T = V - \cup \mathcal{M}$. Note that $d(s, T) = 0$.

Claim 4.2. *If $R(M_1) = \lambda(a, b)$ with $a \in M_1$ and $b \in T$, then for some $M_k \in \mathcal{M} - M_1$, $R(M_k - t) > R(t)$.*

Proof. Note that $d(s) \geq |\mathcal{M}| + 1$ and $d(T) \geq \lambda(a, b) = R(M_1) \geq d(M_1) - 1$ because M_1 is dangerous. By repeated applications of (1) we get

$$\begin{aligned} \sum_{M_j \in \mathcal{M}} (d(M_j) - d(t)) &\geq d(s \cup T) - d(t) \\ &= d(s) + d(T) - d(t) \\ &\geq (|\mathcal{M}| + 1) + (d(M_1) - 1) - d(t) \\ &> (|\mathcal{M}| - 1) + (d(M_1) - d(t)), \end{aligned}$$

so there exists $M_k \in \mathcal{M} - M_1$ with $d(M_k) - d(t) > 1$. Then, since M_k is dangerous, $R(M_k) \geq d(M_k) - 1 > d(t) \geq R(t)$, so, by (4), $R(M_k - t) > R(t)$. $\square$

Claim 4.3. *There exists $M_i \in \mathcal{M}$ for which $R(M_i - t) \geq R(t)$.*

Proof. Let $Y = \{y \in V - t : R(t) = \lambda(t, y)\}$. By definition, $Y \neq \emptyset$. If there exists a vertex $y \in M_i \cap Y$ for some $M_i \in \mathcal{M}$, then $R(M_i - t) \geq \lambda(t, y) = R(t)$. Thus we may suppose that $Y \subseteq T$. Let $y \in Y$. Then $R(M_1) \geq \lambda(t, y) = R(t)$. If $R(M_1) = \lambda(t, y)$ then, by Claim 4.2, $R(M_1 - t) > R(t)$. Otherwise $R(M_1) > R(t)$, so, by (4), $R(M_1 - t) > R(t)$. $\square$

Claim 4.4. *If $M_j \in \mathcal{M} - M_i$, then $R(M_j - t) < R(M_j) \leq R(t)$.*

Proof. Suppose that $R(M_j - t) \geq R(M_j)$. By Claim 4.3 and (4), $R(M_i - t) \geq R(M_i)$. So (6) and hence (10) apply for M_i and M_j , contradicting Lemma 2.6(a). By $R(M_j - t) < R(M_j)$ and (4), $R(M_j) \leq R(t)$. $\square$

Claim 4.5. *If $R(M_i) = \lambda(a, b)$ with $a \in M_i$ and $b \in V - M_i$, then $b \in T$.*

Proof. Suppose that $b \in M_j \in \mathcal{M} - M_i$. Then, $R(M_j - t) \geq \lambda(a, b) = R(M_i)$. By Claims 4.4 and 4.3, $R(M_j) \leq R(t) \leq R(M_i - t)$. Thus (6) and hence (10) apply for M_i and M_j , a contradiction by Lemma 2.6(a). $\square$

By Claims 4.3 and 4.4, there exists $M_i \in \mathcal{M}$ such that $R(M_j - t) < R(t)$ for all $M_j \in \mathcal{M} - M_i$. However, by Claims 4.5 and 4.2, applied for $M_1 = M_i$, $R(M_j - t) > R(t)$ for some $M_j \in \mathcal{M} - M_i$. This contradiction completes the proof of (a) implies (b).

Obviously, (b) implies (a) by Claim 2.3.

We show now that (b) implies (c). Let $C_1 = M_1 - M_2$ and $C_2 = M_2 - M_1$. Clearly, $C_1 \cap C_2 = \emptyset$ and, by $t \in M_1 \cap M_2$, the sets C_1 and C_2 are in $V - t$.

Claim 4.6. $d(s)$ is odd and $d(s, C_1) = (d(s) - 1)/2 = d(s, C_2)$.

Proof. By (8), $\Gamma(s) \subseteq M_1 \cup M_2$ and $st \in E$, we have $2(d(s) + 1)/2 \geq d(s, M_1) + d(s, M_2) = d(s, M_1 \cup M_2) + d(s, M_1 \cap M_2) \geq d(s) + 1$. It follows that $d(s)$ is odd, $d(s, M_i) = (d(s) + 1)/2$ and $d(s, M_1 \cap M_2) = 1$. Then $d(s, C_i) = d(s, M_i) - d(s, M_1 \cap M_2) = (d(s) + 1)/2 - 1 = (d(s) - 1)/2$ for $i = 1, 2$. $\square$

Claim 4.7. Property (10) applies for M_1 and M_2 .

Proof. Suppose that (10) does not hold for M_1 and M_2 . Then, by Proposition 2.1, $M_1 \cup M_2 \neq V$ and (9) applies for M_1 and M_2 , so, by $1 \geq h(M_1)$, $1 \geq h(M_2)$ and (7), we have $2 \geq h(M_1 \cup M_2)$. It follows, by (8), (7) and $\Gamma(s) \subseteq M_1 \cup M_2$, that $2 \geq h(M_1 \cup M_2) = h(V - (M_1 \cup M_2)) + 2d(s, M_1 \cup M_2) - d(s) \geq d(s)$. However, since G is 2-edge-connected and $d(s)$ is odd, $d(s) \geq 3$, a contradiction. $\square$

Then, by $1 \geq h(M_1)$, $1 \geq h(M_2)$, (10), $t \in M_1 \cap M_2$ and $st \in E$, we get that $h(C_1) = 0 = h(C_2)$, that is, C_1 and C_2 are tight sets. This completes the proof of (b) implies (c).

Finally, we show that (c) implies (b). Suppose that $d(s)$ is odd and there exist two disjoint tight sets $C_1, C_2 \subseteq V - t$ such that $d(s, C_1) = (d(s) - 1)/2 = d(s, C_2)$. Then, by (8), $M_1 = V - C_1$ and $M_2 = V - C_2$ are dangerous sets. Note that $t \in M_1 \cap M_2$ and $\Gamma(s) \subseteq M_1 \cup M_2$. $\square$

5. Proof of Theorem 1.7

By Theorem 1.6, there exist two dangerous sets M_1 and M_2 with $t \in M_1 \cap M_2$ and $\Gamma(s) \subseteq M_1 \cup M_2$. It also follows from the proof above that $d(s, M_1 \cap M_2) = 1$ and $d(s, M_1) = d(s, M_2) = (d(s) + 1)/2$. Let $sr \neq st$ be an edge incident to s . Then, by Claim 2.3, the edge sr belongs to at most $d(s) - (d(s) + 1)/2 = (d(s) - 1)/2$ admissible pairs. To finish the proof we show the following lemma.

Lemma 5.1. *The edge sr belongs to at least $(d(s) - 1)/2$ admissible pairs.*

Proof. We prove the lemma by induction on $|V|$. We may assume, by Claim 2.2(a), that (11) is satisfied. By Theorem 1.6, $d(s)$ is odd and there exist two disjoint tight sets $C_1, C_2 \subseteq V - t$ such that $d(s, C_1) = d(s, C_2) = (d(s) - 1)/2$. Then, by (11), $C_1 = c_1$ and $C_2 = c_2$ for some vertices $c_1, c_2 \in V$. Since $sr \neq st$, either $r = c_1$ or c_2 . The lemma follows from the following claim.

Claim 5.2. *Let e_i be any edge connecting s and c_i for $1 \leq i \leq 2$. Then the pair of edges e_1, e_2 is admissible.*

Proof. Otherwise, by Claim 2.3, there is a dangerous set X containing c_1 and c_2 . Then, by $d(s, c_1) = d(s, c_2) = (d(s) - 1)/2$ and Claim 2.5(a), $2(d(s) - 1)/2 \leq d(s, X) \leq (d(s) + 1)/2$, that is, $d(s) \leq 3$. However, since G is 2-edge-connected and $d(s)$ is odd and $\neq 3$, $d(s) \geq 5$, a contradiction. $\square$

6. Open problems

For a summary on edge-connectivity augmentation problems in graphs we refer to [8]. We repeat one of the open problems proposed in [8], the problem of *local edge-connectivity augmentation in bipartite graphs*: given a connected bipartite graph $H = (V, E)$ and a requirement function $r: V \times V \rightarrow \mathbb{Z}_+$, find the minimum number of new edges F

such that $\lambda_{H+F}(u, v) \geq r(u, v)$ for all pairs $u, v \in V$ and $H + F$ is a bipartite graph. Theorem 1.4 could help to solve this problem.

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