ON PACKING $T$-CUTS

by

András Frank and Zoltán Szigeti

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ABSTRACT A short proof of a difficult theorem of P.D. Seymour on grafts with the max-flow min-cut property is given.

I. INTRODUCTION

The Chinese Postman problem, in other words the minimum $T$-join problem, consists of finding a minimum cardinality subset of edges of a graph satisfying prescribed parity constraints on the degrees of nodes. This minimum is bounded from below by the maximum value of a (fractional) packing of $T$-cuts. In the literature there are several min-max theorems for cases when equality actually holds. In this paper we list some of these results and exhibit new relationships among them.

To be more specific, P. Seymour’s theorem [1977] on binary matroids with the max-flow min-cut property, when specialized to $T$-joins, provides a characterization of pairs $(G, T)$ for which the minimum weight of a $T$-join is equal to the maximum packing of $T$-cuts for every integer weighting. Motivated by Seymour’s theorem, A. Sebő [1988] proved a min-max theorem concerning minimum $T$-joins and maximum packing of $T$-borders. He also observed that his result, combined with a simple-sounding lemma on bi-critical graphs (Theorem 7 below), immediately implies Seymour’s theorem.

The purpose of this note is two-fold. We show first that Sebő’s theorem is an easy consequence of an earlier min-max theorem [Frank, Sebő, Tardos, 1984] and, second, we provide a simple proof of the above-mentioned statement on bi-critical graphs. This way we will have obtained a simple proof of Seymour’s theorem.

A graft $(G, T)$ is a pair consisting of a connected undirected graph $G = (V, E)$ and a subset $T$ of $V$ of even cardinality. A subset $J$ of edges is called a $T$-join if $d_J(v)$ is odd precisely when $v \in T$. Here $d_J(v)$ denotes the number of elements of $J$ incident to $v$. $J$ is called a perfect matching if $d_J(v) = 1$ for every $v \in V$. Note that a perfect matching is a $T$-join for which for $T = V$. A graph $G = (V, E)$ is called bi-critical if $E$ is non-empty and every pair of nodes $u, v$, the graph $G - \{u, v\}$ contains a perfect matching. It follows immediately from Tutte’s theorem (see Theorem 0 below) that $G$ is bi-critical if and only if
\[ q(X) \leq |X| - 2 \quad \text{for every subset } X \subseteq V \text{ with } |X| \geq 2 \] (1)

where \( q(X) \) denotes the number of odd-cardinality components of \( G - X \).

Let us call a set \( X \subseteq V \) T-odd if \( |X \cap T| \) is odd. Given a partition \( \mathcal{P} = \{V_1, V_2, \ldots, V_k\} \) of \( V \), by a multicut \( B = B(\mathcal{P}) \) we mean the set of edges connecting different parts of \( \mathcal{P} \). If each \( V_i \) is T-odd and induces a connected subgraph, \( B \) is called a T-border. Then clearly \( k \) is even and \( \text{val}(B) := k/2 \) is called the value of the T-border. When \( k = 2 \) a T-border \( B \) is called a T-cut. Note that the value of a T-cut is one.

The border graph \( G_B \) of a T-border \( B = B(\mathcal{P}) \) is one obtained by contracting each \( V_i \) into one node. Let us call a T-border bi-critical if its border graph is bi-critical.

Note that the cardinality of the intersection of a T-cut and a T-join is always odd, in particular, at least one. Hence the cardinality of the intersection of a T-border \( B \) and a T-join \( J \) is always at least \( \text{val}(B) \) and equality holds precisely when the edges in \( J \) connecting distinct \( V_i \)'s form a perfect matching in the border graph of \( B \).

A list \( B = \{B_1, \ldots, B_l\} \) of T-borders is called a packing (2-packing) if each edge of \( G \) belongs to at most one (two) member(s) of \( B \). The value of a packing is \( \sum \{\text{val}(B) : B \in B\} \) and the value of a 2-packing is \( \sum \{\text{val}(B) : B \in B\}/2 \). Note that a T-border of value \( t \) determines a 2-packing of T-cuts of value \( t \).

For an edge \( e = uv \) we define the elementary T-contraction as a graft \( (G', T') \) where \( G' \) arises from \( G \) by contracting \( e \) and \( T' := T - \{u, v\} \) if \( \{|u, v\} \cap T| \) is even and \( T' := T - \{u, v\} + x_{uv} \) if \( \{|u, v\} \cap T| \) is odd where \( x_{uv} \) denotes the contracted node. The T-contraction of a graph means a sequence of elementary T-contractions. If \( X \subseteq V \) induces a connected subgraph of \( G \), then by T-contracting \( X \) we mean the operation of T-contracting a spanning tree of \( X \).

Let \( K_4 \) denote a graft \( (K_4, V(K_4)) \) where \( K_4 \) is a complete graph on 4 nodes. Note that a graft \( (G, T) \) can be T-contracted to \( K_4 \) precisely when there is a partition \( \{V_1, V_2, V_3, V_4\} \) of \( V \) into T-odd sets so that each \( V_i \) induces a connected subgraph and there is an edge connecting \( V_i \) and \( V_j \) whenever \( 1 \leq i < j \leq 4 \).

For a general account on matchings and T-joins, see [Lovász and Plummer, 1986].

II. RESULTS ON T-CUTS AND T-JOINS

Our starting point is Tutte’s theorem [1947] on perfect matchings.

**THEOREM 0** A graph \( G = (V, E) \) contains no perfect matching if and only if there is a set \( X \) of nodes so that \( G - X \) includes more than \( |X| \) components of odd cardinality.

The perfect matching problem can be reformulated in terms of T-joins. Namely, by choosing \( T := V \), one observes that \( G \) has a perfect matching precisely if the minimum cardinality of a T-join is \(|V|/2\). Therefore it was natural to ask for theorems concerning the minimum cardinality of a T-join. Let us list some known results concerning this minimum. The first one was proved by L. Lovász [1975] (and was stated earlier in a more general form by J. Edmonds and E. Johnson [1970]).
THEOREM 1 The minimum cardinality of a $T$-join is equal to the maximum value of a 2-packing of $T$-cuts.

For example, in $K_4$ a perfect matching is a $T$-join of 2 elements and a 2-packing of $T$-cuts with value 2 is provided by taking each $T$-cut once. Note that the value of the best integral $T$-cut packing is 1.

Although this theorem, when applied to $T := V$, provides a good characterization for the existence of a perfect matching (namely, a graph $G = (V, E)$ with $|V|$ even has no perfect matching if and only if there is a list of more than $|V|$ $V$-cuts so that every edge belongs to at most two of them), Tutte’s theorem does not seem to follow directly.

For bipartite graphs P. Seymour [1981] proved a stronger statement:

THEOREM 2 In a bipartite graph the minimum cardinality of a $T$-join is equal to the maximum number of disjoint $T$-cuts.

This theorem immediately implies Theorem 1 by subdividing each edge by a new node. In [Frank, Sebő, Tardos, 1984] the following sharpening of Theorem 2 was proved:

THEOREM 3 In a bipartite graph $D = (U, V; F)$ the minimum cardinality of a $T$-join is equal to $\max \sum q_T(V_i)$ where the maximum is taken over all partitions $\{V_1, \ldots, V_l\}$ of $V$ and $q_T(X)$ denotes the number of $T$-odd components of $D - X$.

Let $G = (V, E)$ be an arbitrary graph. Subdivide each edge by a new node and let $D = (V, U; F)$ denote the resulting bipartite graph (where $U$ denotes the set of new nodes). By applying Theorem 3 to this graph one can easily obtain the following.

THEOREM 4 In a graph $G = (V, E)$ the minimum cardinality of a $T$-join is equal to $\max \sum q_T(V_i)/2$ where the maximum is taken over all partitions $\{V_1, \ldots, V_l\}$ of $V$.

Observe that Theorem 3 implies Seymour’s Theorem 2. In [Frank, Sebő, Tardos] we pointed out via an elementary construction that Theorem 3 also implies the Berge-Tutte formula, a slight generalization of Tutte’s theorem.

Let us show now an even simpler derivation of the (non-trivial part of) Tutte’s theorem.

THEOREM 4 $\rightarrow$ THEOREM 0

Apply Theorem 4 with the choice $T := V$. Notice that in this case a set is $T$-odd if its cardinality is odd. If there is no perfect matching, then the minimum cardinality of a $T$-join is larger than $|V|/2$. By Theorem 4 there is a partition $\{V_1, \ldots, V_l\}$ of $V$ so that $\sum q_T(V_i)/2 > |V|/2$, that is, $\sum q_T(V_i) > \sum |V_i|$. Therefore there must be a subscript $i$ so that $q_T(V_i) > |V_i|$, that is, the number of components in $G - V_i$ with odd cardinality is larger than $|V_i|$, as required. ♠
A. Sebő [1988] determined the minimal totally dual integral linear system defining the conical hull of \(T\)-joins. As a by-product, he derived the following integer min-max theorem concerning \(T\)-joins:

**THEOREM 5** In a graph \(G = (V, E)\) the minimum cardinality of a \(T\)-join is equal to the maximum value of a \(T\)-border packing \(\{B_1, \ldots, B_r\}\). Furthermore, if an optimal packing is chosen in such a way that \(r\) is as large as possible, then each \(B_i\) is bi-critical.

Note that both Theorems 4 and 5 imply Theorem 1. The last theorem of our list is also due to P. Seymour [1977].

**THEOREM 6** If a graft \((G, T)\) cannot be \(T\)-contracted to \(K_4\), then the minimum cardinality of a \(T\)-join is equal to the maximum number of disjoint \(T\)-cuts.

This theorem is a special case of a very difficult result of Seymour concerning binary matroids with the max-flow min-cut property. It can be formulated in an apparently stronger form:

A graft \((G, T)\) cannot be \(T\)-contracted to \(K_4\) if and only if for every integer weight-function \(w\) the minimum weight of a \(T\)-join is equal to the maximum number of \(T\)-cuts so that every edge belongs to at most \(w(e)\) \(T\)-cuts.

Note, however, that the "if" part is trivial and the "only if" part easily follows from Theorem 6 if we delete each edge \(e\) with \(w(e) = 0\) and subdivide each edge \(e\) by \(w(e) - 1\) new nodes when \(w(e) > 0\).

**III. PROOFS**

We are going to show first that Sebő’s Theorem 5 is also an easy consequence of Theorem 3 and, second, using Sebő’s theorem we provide a simple proof of Seymour’s Theorem 6.

Let \(G = (V, E)\) be an arbitrary graph and let \(D = (V, U; F)\) be a bipartite graph arising from \(G\) by subdividing each edge by a new node. Here sets \(E\) and \(U\) are in a one-to-one correspondence and we will not distinguish between their corresponding elements. In particular, a subset of \(U\) will be considered as a subset of \(E\) and vice versa.

Observe that in Theorem 3 the two parts \(U\) and \(V\) of the bipartite graph play an asymmetric role. When one applies Theorem 3 to \(D\) and the maximum is taken over the partitions of \(V\), Theorem 4 can be obtained. Sebő’s theorem will follow from Theorem 3 by taking the maximum over the partitions of \(U\).

**Proof of Theorem 5**

We have already seen that the value of a \(T\)-border packing is a lower bound for the minimum cardinality of a \(T\)-join. We are going to prove that there is a \(T\)-join \(J\) of \(G\) and a packing \(\mathcal{F}\) of \(T\)-borders of \(G\) so that
\[ |J| = \text{val}(\mathcal{F}). \] (2)

By Theorem 3 there is a partition \( \mathcal{U} \) of \( U \) and a \( T \)-join \( J' \) of \( D \) for which

\[ |J'| = \sum (q_T(X) : X \in \mathcal{U}). \] (3)

Assume that \( l := |\mathcal{U}| \) is as large as possible and let \( Z \) be an arbitrary member of \( \mathcal{U} \) with \( q_T(Z) > 0 \). Let \( K_1, K_2, \ldots, K_h \) be the components of \( D - Z \), \( V_i := V \cap K_i \) and \( \mathcal{P} := \{V_1, \ldots, V_h\} \).

Clearly, \( Z \supseteq B(\mathcal{P}) \) and, in fact, we have equality here since if an edge \( e \) induced by \( V_i \) belonged to \( Z \), then \( |Z| \geq 2 \) and in \( \mathcal{U} \) we could replace \( Z \) by two sets \( Z - e \) and \( \{e\} \) without destroying (3), contradicting the maximality of \( l \). We also claim that each \( V_i \) is \( T \)-odd for otherwise \( |Z| \geq 2 \) and for an edge \( e \in Z \) leaving \( V_i \) we could replace \( Z \) by \( Z - e \) and \( \{e\} \) without destroying (3), contradicting again the maximality of \( l \).

Let \( \mathcal{F} := \{Z \in \mathcal{U} : q_T(Z) > 0\} \). We have seen that each member \( Z \) of \( \mathcal{F} \) is a \( T \)-border of \( G \) with \( \text{val}(Z) = q_T(Z)/2 \). Hence (2) and the first half of Theorem 5 follows by noticing that \( J' \) corresponds to a \( T \)-join of \( G \) with \( |J| = |J'|/2 \).

To prove the second half of the theorem let \( B \) be a \( T \)-border packing of maximum value such that \( r := |B| \) is as large as possible. Suppose indirectly, that a member \( B \in \mathcal{B} \) is not bi-critical. That is, the border graph \( G_B \) of \( B \) includes a subset \( X \) of nodes with \( |X| \geq 2 \) for which \( q(X) \geq |X| \). (Here \( q(X) \) denotes the number of odd-cardinality components of \( G_B - X \).)

For any odd component \( K \) of \( G_B - X \) let us define a partition of \( V(G_B) \) consisting of the elements of \( K \) as singletons and a set \( V(G_B) - K \). This partition defines a \( T \)-border of \( G \) with value \( (|K| + 1)/2 \). For any even component \( L \) of \( G_B - X \) let us define a partition of \( V(G_B) \) consisting of the elements of \( L - v \) as singletons and the set \( V(G_B) - (L - v) \) where \( v \) is an arbitrary element of \( L \). This partition defines a \( T \)-border of \( G \) with value \( |L|/2 \). The \( T \)-borders defined this way are pairwise disjoint subsets of \( B \) and their total value is \( |V(G_B)|/2 \), the value of \( B \). This contradicts the maximal choice of \( r \). ★★★★

The following Theorem 7, interesting for its own right, was stated by A. Sebő [1988]. He noted that it follows from Seymour’s Theorem 6 and observed that, conversely, Theorem 6 is an easy consequence of Theorems 5 and 7. We provide here a simple proof.

**THEOREM 7** The node set of an arbitrary bi-critical graph \( G_B \) on \( k \geq 4 \) nodes can be partitioned into four subsets \( V_1, V_2, V_3, V_4 \) of odd cardinality so that each \( V_i \) induces a connected subgraph and there is an edge connecting \( V_i \) and \( V_j \) whenever \( 1 \leq i < j \leq 4 \).

**Proof.** Let \( M \) be a perfect matching of \( G_B \), \( uv \in M \) and \( M_{uv} := M - uv \). Let \( z(\neq v) \) be a neighbour of \( u \). Since \( G_B \) is bi-critical \( G_B - \{v, z\} \) contains a perfect matching \( M_{vz} \). The symmetric difference \( M_{uv} \oplus M_{vz} \) consists of node-disjoint circuits and a path \( P \) connecting \( z \) and \( u \). Now \( C := P + uz \) is an odd circuit of \( G_B \) so that, starting at \( u \) and going along \( C \), every second edge of \( C \) belongs to \( M \).
Let $u, u_1, \ldots, u_h$ be the nodes of $C$ (in this order). Because of the existence of $M$, the component $K$ of $G_B - V(C)$ containing $v$ is of odd cardinality while all the other components are of even cardinality.

Let $V_1 := K$. It follows from (1) that $G_B$ is 2-connected and, moreover, contains no separating set $X$ of two elements for which $q(X) > 0$. Hence $K$ must have at least three distinct neighbours $u, u_i, u_j$ in $C$.

If there is a matching edge $xy \in M$ on $C$ so that $u, u_i, x, y, u_j$ reflects the order of these nodes around $C$ (where both $u_i = x$ and $u_j = y$ are possible), then define $V'_2 := \{u_1, u_2, \ldots, x\}$, $V'_3 := \{y, \ldots, u_{h-1}, u_h\}$, $V'_4 := \{u\}$.

If there is no such matching edge, that is, $j = i+1$ and $i$ is even, then define $V'_2 := \{u_i\}$, $V'_3 := \{u_{i+1}\}$, $V'_4 := V(C) - \{u_i, u_{i+1}\}$.

In both cases $\{V'_2, V'_3, V'_4\}$ is a partition of $V(C)$. Let $\mathcal{L}$ denote the set of even components of $G_B - V(C)$. For each $L \in \mathcal{L}$ choose a subscript $s = s(L)(= 2, 3, 4)$ so that $L$ is connected to a node in $V'_s$.

For $t = 2, 3, 4$ define $V'_t := V'_t \cup \cup (L \in \mathcal{L}: s(L) = t)$

The partition $\{V_1, V_2, V_3, V_4\}$ constructed this way satisfies the requirements. ♠♠♠

Proof of Theorem 6

Let $\mathcal{B}$ be an optimal packing of bi-critical $T$-borders provided by Theorem 5. We claim that each member of $\mathcal{B}$ is a $T$-cut. Indeed, if $B \in \mathcal{B}$ is a $T$-border determined by a partition $\mathcal{P}$ of $V$ ($|\mathcal{P}| \geq 4$) into $T$-odd sets, then the graft $(G_B, V(G_B))$ arises from $(G, T)$ by $T$-contracting each member of $\mathcal{P}$ and then, by Theorem 7, $(G, T)$ can be $T$-contracted to $K_4$, a contradiction. ♠♠♠

In order for the paper to be self-contained, we include here a proof of Theorem 3, due to A. Sebő [1987].

Proof of Theorem 3

We prove only the non-trivial direction $\max \leq \min$. Let $J$ be a $T$-join of minimum cardinality. Let $w$ denote a weighting on $F$ for which $w(e) = -1$ if $e \in J$ and $w(e) = 1$ if $e \in F - J$. Then $w$ is clearly conservative, that is, there is no circuit of negative total weight. Actually, we prove the following:

**THEOREM 3’** Let $D = (U, V; F)$ be a bipartite graph and $w : F \to \{+1, -1\}$ a conservative weighting. There is a partition $\mathcal{L}$ of $V$ so that for each part $P \in \mathcal{L}$ and for each component $C$ of $D - P$ there is at most one negative edge connecting $P$ and $C$.

**Proof.** We use induction on $|J|$ where $J$ denotes the set of negative edges. If $J$ is empty, $\mathcal{L} := \{V\}$ will do. Assume that $J$ is non-empty and let $s$ be an arbitrary node incident to an element of $J$. Let $P$ be a path of $D$ starting at $s$ so that its weight $m := w(P)$ is minimum and, in addition, $P$ has as few edges as possible. Let $t$ denote the other end-node of $P$, $xt$ the last edge of $P$ and $B$ the set of edges of $D$ incident to $t$. Since $B$ is a cut of
$D$, the graph $D' := D/B := (U', V'; F')$ arising from $D$ by contracting the elements of $B$ is bipartite. Let $t'$ denote the contracted node of $D'$ corresponding to $t$ and let $w'$ denote the weighting of $D'$ determined by $w$. We call a subpath $P[y, t]$ of $P$ an end-segment. Clearly $m < 0$ by the choice of $s$ and

$$w'(xt) < 0. \quad (*)$$

in particular, $w(xt) < 0$.

**CLAIM** (i) $xt$ is the only negative edge incident to $t$. (ii) In $D - t$ there is no negative path $R$ connecting two neighbours $u, v$ of $t$.

**Proof.** (i) Let $tz$ be another negative edge. If $z \in P$, then $P[z, t] + tz$ would form a negative circuit contradicting that $w$ is conservative. If $z \not\in P$, then $P' := P + tz$ would be a path with $w(P') < w(P)$ contradicting the minimal choice of $P$. Thus (i) follows.

(ii) Let $R$ be a path for which $w(R)$ is minimum and suppose for a contradiction that $w(R) < 0$. Clearly $u$ and $v$ are distinct from $x$ since otherwise $R + ut + tv$ would form a negative circuit in $G$.

An arbitrary node $y$ of $R$ subdivides $R$ into two segments $R[y, u]$ and $R[y, v]$. Since $w(R) < 0$, at least one of the two segments has negative weight.

Suppose first that $P$ and $R$ have a node $y$ in common. Choose $y$ so that $P[y, t]$ has as few edges as possible. Assume that $w(R[u, y]) < 0$. Property (*) implies that $P[t, y] + R[y, u] + ut$ is a negative circuit in $D$, a contradiction.

Now let $P$ and $R$ be disjoint. Since $D$ is bipartite, $R$ has even length from which $w(R) \leq -2$. Hence $P' := P + tu + R$ is a simple path starting at $s$ such that $w(P') < m$ contradicting the minimal choice of $P$. ♠

The claim is equivalent to saying that $w'$ is a conservative weighting of $D'$. By the inductional hypothesis, there is a partition $\mathcal{L}'$ of $V'$ satisfying the requirement of the theorem with respect to $w'$. If $t \in U$ (that is, $t' \in V'$), then $\mathcal{L}'$ determines a partition $\mathcal{L}$ of $V$. If $t \in V$, then define $\mathcal{L} := \mathcal{L}' \cup \{t\}$. In both cases it is easily seen that $\mathcal{L}$ satisfies the requirements of the theorem. ♠♠♠

**REFERENCES**


