ON PACKING T-CUTS

by

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ABSTRACT A short proof of a difficult theorem of P.D. Seymour on grafts with the max-flow min-cut property is given.

I. INTRODUCTION

The Chinese Postman problem, in other words the minimum T-join problem, consists of finding a minimum cardinality subset of edges of a graph satisfying prescribed parity constraints on the degrees of nodes. This minimum is bounded from below by the maximum value of a (fractional) packing of T-cuts. In the literature there are several min-max theorems for cases when equality actually holds. In this paper we list some of these results and exhibit new relationships among them.

To be more specific, P. Seymour's theorem [1977] on binary matroids with the max-flow min-cut property, when specialized to T-joins, provides a characterization of pairs (G, T) for which the minimum weight of a T-join is equal to the maximum packing of T-cuts for every integer weighting. Motivated by Seymour's theorem, A. Sebő [1988] proved a min-max theorem concerning minimum T-joins and maximum packing of T-borders. He also observed that his result, combined with a simple-sounding lemma on bi-critical graphs (Theorem 7 below), immediately implies Seymour's theorem.

The purpose of this note is two-fold. We show first that Sebő's theorem is an easy consequence of an earlier min-max theorem [Frank, Sebő, Tardos, 1984] and, second, we provide a simple proof of the above-mentioned statement on bi-critical graphs. This way we will have obtained a simple proof of Seymour's theorem.

A **graft** (G, T) is a pair consisiting of a connected undirected graph $G = (V, E)$ and a subset T of V of even cardinality. A subset J of edges is called a **T-join** if $d_J(v)$ is odd precisely when $v \in T$. Here $d_J(v)$ denotes the number of elements of J incident to v. J is called a **perfect matching** if $d_J(v) = 1$ for every $v \in V$. Note that a perfect matching is a T-join for which for $T = V$. A graph $G = (V, E)$ is called **bi-critical** if E is non-empty and every pair of nodes u, v, the graph $G - \{u, v\}$ contains a perfect matching. It follows immediately from Tutte's theorem (see Theorem 0 below) that G is bi-critical if and only if

$$q(X) \leq |X| - 2 \text{ for every subset } X \subseteq V \text{ with } |X| \geq 2 \tag{1}$$

where $q(X)$ denotes the number of odd-cardinality components of $G - X$.

Let us call a set $X \subseteq V$ **T-odd** if $|X \cap T|$ is odd. Given a partition $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ of V, by a **multicut** $B = B(\mathcal{P})$ we mean the set of edges connecting different parts of $\mathcal{P}$. If each V_i is T-odd and induces a connected subgraph, B is called a **T-border**. Then clearly k is even and $val(B) := k/2$ is called the **value** of the T-border. When $k = 2$ a T-border B is called a **T-cut**. Note that the value of a T-cut is one.

The **border graph** G_B of a T-border $B = B(\mathcal{P})$ is one obtained by contracting each V_i into one node. Let us call a T-border **bi-critical** if its border graph is bi-critical.

Note that the cardinality of the intersection of a T-cut and a T-join is always odd, in particular, at least one. Hence the cardinality of the intersection of a T-border B and a T-join J is always at least $val(B)$ and equality holds precisely when the edges in J connecting distinct $V_i's$ form a perfect matching in the border graph of B.

A list $\mathcal{B} = \{B_1, \ldots, B_l\}$ of T-borders is called a **packing** (**2-packing**) if each edge of G belongs to at most one (two) member(s) of $\mathcal{B}$. The **value** of a packing is $\sum(val(B) : B \in \mathcal{B})$ and the **value** of a 2-packing is $\sum(val(B) : B \in \mathcal{B})/2$. Note that a T-border of value t determines a 2-packing of T-cuts of value t.

For an edge $e = uv$ we define the **elementary T-contraction** as a graft (G', T') where G' arises from G by contracting e and $T' := T - \{u, v\}$ if $|\{u, v\} \cap T|$ is even and $T' := T - \{u, v\} + x_{uv}$ if $|\{u, v\} \cap T|$ is odd where x_{uv} denotes the contracted node. The **T-contraction** of a graph means a sequence of elementary T-contractions. If $X \subseteq V$ induces a connected subgraph of G, then by T-contracting X we mean the operation of T-contracting a spanning tree of X.

Let $\mathbf{K_4}$ denote a graft $(K_4, V(K_4))$ where K_4 is a complete graph on 4 nodes. Note that a graft (G, T) can be T-contracted to $\mathbf{K_4}$ precisely when there is a partition $\{V_1, V_2, V_3, V_4\}$ of V into T-odd sets so that each V_i induces a connected subgraph and there is an edge connecting V_i and V_j whenever $1 \leq i < j \leq 4$.

For a general account on matchings and T-joins, see [Lovász and Plummer, 1986].

II. RESULTS ON T-CUTS AND T-JOINS

Our starting point is Tutte's theorem [1947] on perfect matchings.

THEOREM 0 *A graph $G = (V, E)$ contains no perfect matching if and only if there is a set X of nodes so that $G - X$ includes more than $|X|$ components of odd cardinality.*

The perfect matching problem can be reformulated in terms of T-joins. Namely, by chosing $T := V$, one observes that G has a perfect matching precisely if the minimum cardinality of a T-join is $|V|/2$. Therefore it was natural to ask for theorems concerning the minimum cardinality of a T-join. Let us list some known results concerning this minimum. The first one was proved by L. Lovász [1975] (and was stated earlier in a more general form by J. Edmonds and E. Johnson [1970]).

THEOREM 1 *The minimum cardinality of a T-join is equal to the maximum value of a 2-packing of T-cuts.*

For example, in $\mathbf{K_4}$ a perfect matching is a T-join of 2 elements and a 2-packing of T-cuts with value 2 is provided by taking each T-cut once. Note that the value of the best integral T-cut packing is 1.

Although this theorem, when applied to $T := V$, provides a good characterization for the existence of a perfect matching (namely, *a graph $G = (V, E)$ with $|V|$ even has no perfect matching if and only if there is a list of more than $|V|$ V-cuts so that every edge belongs to at most two of them)*, Tutte's theorem does not seem to follow directly.

For bipartite graphs P. Seymour [1981] proved a stronger statement:

THEOREM 2 *In a bipartite graph the minimum cardinality of a T-join is equal to the maximum number of disjoint T-cuts.*

This theorem immediately implies Theorem 1 by subdividing each edge by a new node. In [Frank, Sebő, Tardos, 1984] the following sharpening of Theorem 2 was proved:

THEOREM 3 *In a bipartite graph $D = (U, V; F)$ the minimum cardinality of a T-join is equal to* $\max \sum q_T(V_i)$ *where the maximum is taken over all partitions $\{V_1, \ldots, V_l\}$ of V and $q_T(X)$ denotes the number of T-odd components of $D - X$.*

Let $G = (V, E)$ be an arbitrary graph. Subdivide each edge by a new node and let $D = (V, U; F)$ denote the resulting bipartite graph (where U denotes the set of new nodes). By applying Theorem 3 to this graph one can easily obtain the following.

THEOREM 4 *In a graph $G = (V, E)$ the minimum cardinality of a T-join is equal to* $\max \sum q_T(V_i)/2$ *where the maximum is taken over all partitions $\{V_1, \ldots, V_l\}$ of V.*

Observe that Theorem 3 implies Seymour's Theorem 2. In [Frank, Sebő, Tardos] we pointed out via an elementary construction that Theorem 3 also implies the Berge-Tutte formula, a slight generalization of Tutte's theorem.

Let us show now an even simpler derivation of the (non-trivial part of) Tutte's theorem.

THEOREM 4 → THEOREM 0

Apply Theorem 4 with the choice $T := V$. Notice that in this case a set is T-odd if its cardinality is odd. If there is no perfect matching, then the minimum cardinality of a T-join is larger than $|V|/2$. By Theorem 4 there is a partition $\{V_1, \ldots, V_l\}$ of V so that $\sum q_T(V_i)/2 > |V|/2$, that is, $\sum q_T(V_i) > \sum |V_i|$. Therefore there must be a subscript i so that $q_T(V_i) > |V_i|$, that is, the number of components in $G - V_i$ with odd cardinality is larger than $|V_i|$, as required. ♠

A. Sebő [1988] determined the minimal totally dual integral linear system defining the conical hull of T-joins. As a by-product, he derived the following integer min-max theorem concerning T-joins:

THEOREM 5 *In a graph $G = (V, E)$ the minimum cardinality of a T-join is equal to the maximum value of a T-border packing $\{B_1, \ldots, B_r\}$. Furthermore, if an optimal packing is chosen in such a way that r is as large as possible, then each B_i is bi-critical.*

Note that both Theorems 4 and 5 imply Theorem 1. The last theorem of our list is also due to P. Seymour [1977].

THEOREM 6 *If a graft (G, T) cannot be T-contracted to $\mathbf{K_4}$, then the minimum cardinality of a T-join is equal to the maximum number of disjoint T-cuts.*

This theorem is a special case of a very difficult result of Seymour concerning binary matroids with the max-flow min-cut property. It can be formulated in an apparently stronger form:

A graft (G, T) cannot be T-contracted to $\mathbf{K_4}$ if and only if for every integer weight-function w the minimum weight of a T-join is equal to the maximum number of T-cuts so that every edge belongs to at most $w(e)$ T-cuts.

Note, however, that the "if" part is trivial and the "only if" part easily follows from Theorem 6 if we delete each edge e with $w(e) = 0$ and subdivide each edge e by $w(e) - 1$ new nodes when $w(e) > 0$.

III. PROOFS

We are going to show first that Sebő's Theorem 5 is also an easy consequence of Theorem 3 and, second, using Sebő's theorem we provide a simple proof of Seymour's Theorem 6.

Let $G = (V, E)$ be an arbitrary graph and let $D = (V, U; F)$ be a bipartite graph arising from G by subdividing each edge by a new node. Here sets E and U are in a one-to-one correspondence and we will not distinguish between their corresponding elements. In particular, a subset of U will be considered as a subset of E and vice versa.

Observe that in Theorem 3 the two parts U and V of the bipartite graph play an asymmetric role. When one applies Theorem 3 to D and the maximum is taken over the partitions of V, Theorem 4 can be obtained. Sebő's theorem will follow from Theorem 3 by taking the maximum over the partitions of U.

Proof of Theorem 5

We have already seen that the value of a T-border packing is a lower bound for the minimum cardinality of a T-join. We are going to prove that there is a T-join J of G and a packing $\mathcal{F}$ of T-borders of G so that

$$|J| = val(\mathcal{F}). \tag{2}$$

By Theorem 3 there is a partition $\mathcal{U}$ of U and a T-join J' of D for which

$$|J'| = \sum (q_T(X) : X \in \mathcal{U}). \tag{3}$$

Assume that $l := |\mathcal{U}|$ is as large as possible and let Z be an arbitrary member of $\mathcal{U}$ with $q_T(Z) > 0$. Let $K_1, K_2, \ldots, K_h$ be the components of $D - Z$, $V_i := V \cap K_i$ and $\mathcal{P} := \{V_1, \ldots, V_h\}$.

Clearly, $Z \supseteq B(\mathcal{P})$ and, in fact, we have equality here since if an edge e induced by V_i belonged to Z, then $|Z| \geq 2$ and in $\mathcal{U}$ we could replace Z by two sets $Z - e$ and $\{e\}$ without destroying (3), contradicting the maximality of l. We also claim that each V_i is T-odd for otherwise $|Z| \geq 2$ and for an edge $e \in Z$ leaving V_i we could replace Z by $Z - e$ and $\{e\}$ without destroying (3), contradicting again the maximality of l.

Let $\mathcal{F} := \{Z \in \mathcal{U} : q_T(Z) > 0\}$. We have seen that each member Z of $\mathcal{F}$ is a T-border of G with $val(Z) = q_T(Z)/2$. Hence (2) and the first half of Theorem 5 follows by noticing that J' corresponds to a T-join J of G with $|J| = |J'|/2$.

To prove the second half of the theorem let $\mathcal{B}$ be a T-border packing of maximum value such that $r := |\mathcal{B}|$ is as large as possible. Suppose indirectly, that a member $B \in \mathcal{B}$ is not bi-critical. That is, the border graph G_B of B includes a subset X of nodes with $|X| \geq 2$ for which $q(X) \geq |X|$. (Here $q(X)$ denotes the number of odd-cardinality components of $G_B - X$.)

For any odd component K of $G_B - X$ let us define a partition of $V(G_B)$ consisting of the elements of K as singletons and a set $V(G_B) - K$. This partition defines a T-border of G with value $(|K| + 1)/2$. For any even component L of $G_B - X$ let us define a partition of $V(G_B)$ consisting of the elements of $L - v$ as singletons and the set $V(G_B) - (L - v)$ where v is an arbitrary element of L. This partition defines a T-border of G with value $|L|/2$. The T-borders defined this way are pairwise disjoint subsets of B and their total value is $|V(G_B)|/2$, the value of B. This contradicts the maximal choice of r. ♠♠♠

The following Theorem 7, interesting for its own right, was stated by A. Sebő [1988]. He noted that it follows from Seymour's Theorem 6 and observed that, conversely, Theorem 6 is an easy consequence of Theorems 5 and 7. We provide here a simple proof.

THEOREM 7 *The node set of an arbitrary bi-critical graph G_B on $k \geq 4$ nodes can be partitioned into four subsets V_1, V_2, V_3, V_4 of odd cardinality so that each V_i induces a connected subgraph and there is an edge connecting V_i and V_j whenever $1 \leq i < j \leq 4$.*

Proof. Let M be a perfect matching of G_B, $uv \in M$ and $M_{uv} := M - uv$. Let $z (\neq v)$ be a neighbour of u. Since G_B is bi-critical $G_B - \{v, z\}$ contains a perfect matching M_{vz}. The symmetric difference $M_{uv} \oplus M_{vz}$ consists of node-disjoint circuits and a path P connecting z and u. Now $C := P + uz$ is an odd circuit of G_B so that, starting at u and going along C, every second edge of C belongs to M.

Let $u, u_1, \ldots, u_h$ be the nodes of C (in this order). Because of the existence of M, the component K of $G_B - V(C)$ containing v is of odd cardinality while all the other components are of even cardinality.

Let $V_1 := K$. It follows from (1) that G_B is 2-connected and, moreover, contains no separating set X of two elements for which $q(X) > 0$. Hence K must have at least three distinct neighbours u, u_i, u_j in C.

If there is a matching edge $xy \in M$ on C so that u, u_i, x, y, u_j reflects the order of these nodes around C (where both $u_i = x$ and $u_j = y$ are possible), then define $V_2' := \{u_1, u_2, \ldots, x\}$, $V_3' := \{y, \ldots, u_{h-1}, u_h\}$, $V_4' := \{u\}$.

If there is no such matching edge, that is, $j = i+1$ and i is even, then define $V_2' := \{u_i\}$, $V_3' := \{u_{i+1}\}$, $V_4' := V(C) - \{u_i, u_{i+1}\}$.

In both cases $\{V_2', V_3', V_4'\}$ is a partition of $V(C)$. Let $\mathcal{L}$ denote the set of even components of $G_B - V(C)$. For each $L \in \mathcal{L}$ choose a subscript $s = s(L) (= 2, 3, 4)$ so that L is connected to a node in V_s'. For $t = 2, 3, 4$ define $V_t := V_t' \cup \cup(L \in \mathcal{L}\colon s(L) = t)$

The partition $\{V_1, V_2, V_3, V_4\}$ constructed this way satisfies the requirements. ♠♠♠

Proof of Theorem 6

Let $\mathcal{B}$ be an optimal packing of bi-critical T-borders provided by Theorem 5. We claim that each member of $\mathcal{B}$ is a T-cut. Indeed, if $B \in \mathcal{B}$ is a T-border determined by a partition $\mathcal{P}$ of V ($|\mathcal{P}| \geq 4$) into T-odd sets, then the graft $(G_B, V(G_B))$ arises from (G, T) by T-contracting each member of $\mathcal{P}$ and then, by Theorem 7, (G, T) can be T-contracted to $\mathbf{K_4}$, a contradiction. ♠♠♠

In order for the paper to be self-contained, we include here a proof of Theorem 3, due to A. Sebő [1987].

Proof of Theorem 3

We prove only the non-trivial direction max $\leq$ min. Let J be a T-join of minimum cardinality. Let w denote a weighting on F for which $w(e) = -1$ if $e \in J$ and $w(e) = 1$ if $e \in F - J$. Then w is clearly **conservative,** that is, there is no circuit of negative total weight. Actually, we prove the following:

THEOREM 3' *Let $D = (U, V; F)$ be a bipartite graph and $w : F \to \{+1, -1\}$ a conservative weighting. There is a partition $\mathcal{L}$ of V so that for each part $P \in \mathcal{L}$ and for each component C of $D - P$ there is at most one negative edge connecting P and C.*

Proof. We use induction on $|J|$ where J denotes the set of negative edges. If J is empty, $\mathcal{L} := \{V\}$ will do. Assume that J is non-empty and let s be an arbitrary node incident to an element of J. Let P be a path of D starting at s so that its weight $m := w(P)$ is minimum and, in addition, P has as few edges as possible. Let t denote the other end-node of P, xt the last edge of P and B the set of edges of D incident to t. Since B is a cut of

D, the graph $D' := D/B := (U', V'; F')$ arising from D by contracting the elements of B is bipartite. Let t' denote the contracted node of D' corresponding to t and let w' denote the weighting of D' determined by w. We call a subpath $P[y,t]$ of P an **end-segment.** Clearly $m < 0$ by the choice of s and

$$\text{each end-segment of } P \text{ has negative weight,} \qquad (*)$$

in particular, $w(xt) < 0$.

CLAIM *(i) xt is the only negative edge incident to t. (ii) In $D - t$ there is no negative path R connecting two neighbours u, v of t.*

Proof. (i) Let tz be another negative edge. If $z \in P$, then $P[z,t] + tz$ would form a negative circuit contradicting that w is conservative. If $z \notin P$, then $P' := P + tz$ would be a path with $w(P') < w(P)$ contradicting the minimal choice of P. Thus (i) follows.

(ii) Let R be a path for which $w(R)$ is minimum and suppose for a contradiction that $w(R) < 0$. Clearly u and v are distinct from x since otherwise $R + ut + tv$ would form a negative circuit in G.

An arbitrary node y of R subdivides R into two segments $R[y,u]$ and $R[y,v]$. Since $w(R) < 0$, at least one of the two segments has negative weight.

Suppose first that P and R have a node y in common. Choose y so that $P[y,t]$ has as few edges as possible. Assume that $w(R[u,y]) < 0$. Property (*) implies that $P[t,y] + R[y,u] + ut$ is a negative circuit in D, a contradiction.

Now let P and R be disjoint. Since D is bipartite, R has even length from which $w(R) \leq -2$. Hence $P' := P + tu + R$ is a simple path starting at s such that $w(P') < m$ contradicting the minimal choice of P. ♠

The claim is equivalent to saying that w' is a conservative weighting of D'. By the inductional hypothesis, there is a partition $\mathcal{L}'$ of V' satisfying the requirement of the theorem with respect to w'. If $t \in U$ (that is, $t' \in V'$), then $\mathcal{L}'$ determines a partition $\mathcal{L}$ of V. If $t \in V$, then define $\mathcal{L} := \mathcal{L}' \cup \{t\}$. In both cases it is easily seen that $\mathcal{L}$ satisfies the requirements of the theorem. ♠♠♠

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