On arborescence packing augmentation in hypergraphs

Pierre Hoppenot, Zoltán Szigeti
University Grenoble Alpes, Grenoble INP, CNRS,
Laboratory G-SCOP

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Abstract

We deepen the link between two classic areas of combinatorial optimization: augmentation and packing arborescences. We consider the following type of questions: What is the minimum number of arcs to be added to a digraph so that in the resulting digraph there exists some special kind of packing of arborescences? We answer this question for two problems: h-regular M-independent-rooted (f, g)-bounded (α, β)-limited packing of mixed hyperarborescences and h-regular (ℓ, ℓ')-bordered (α, β)-limited packing of k hyperbranchings. We also solve the undirected counterpart of the latter, that is the augmentation problem for h-regular (ℓ, ℓ')-bordered (α, β)-limited packing of k rooted hyperforests. Our results provide a common generalization of a great number of previous results.

1 Introduction

The design of robust networks consists in improving existing networks so that the resulting networks resist to different types of failures. A typical problem is the global edge-connectivity augmentation problem where the goal is to add a minimum number of edges to a given undirected graph to obtain a graph that remains connected after deleting any set of edges of a given size. Watanabe and Nakamura [22], Cai and Sun [3] independently solved the problem in the sense of a minimax theorem and an efficient algorithm. Frank [7] developed a method to solve edge-connectivity augmentation problems in general which for instance provided the solution of the local edge-connectivity augmentation problem. His paper has stimulated further research in a great number of directions that led to many interesting generalizations. For a survey on the subject see [20].

Let us now consider the directed case. In [7], Frank solved the global arc-connectivity augmentation problem and proved that the local arc-connectivity augmentation problem is NP-complete. Further, in [8], Frank solved the rooted k-arc-connectivity augmentation problem, that is when the local arc-connectivity requirement is k from a given vertex s to all the other vertices and 0 otherwise. By the fundamental theorems of Edmonds [4] and Menger [15], this special case is equivalent to the augmentation problem for packing k spanning s-arborescences. In this paper we study some more complex arborescence packing augmentation problems. We mention that for any arborescence packing problem, the augmentation version extends the original problem. On the one hand, we propose the solution of the augmentation version of h-regular M-independent-rooted (f, g)-bounded (α, β)-limited packing of mixed hyperarborescences. For the definitions see Section 2. Our solution depends on the discovery of a new submodular function and on the theory of generalized polymatroids. This way we are able to unify the results of two previous papers [18] and [19] and to provide a simpler proof of this common extension. On the other hand, we propose the solution of the augmentation version of h-regular (ℓ, ℓ')-bordered (α, β)-limited packing of k hyperbranchings, and also its undirected counterpart. Our solution depends on a new augmentation lemma and the recent results of [13].

Our theorems generalize the results of the following papers: [1], [2], [4], [6], [9], [10], [11], [12], [13], [14], [16], [17], [18], [19], [21]. In order to explain this, we present lots of remarks that point out how the results imply the previous ones.

2 Definitions

We use the usual notation for the set $\mathbb{Z}$ of integers, for the set $\mathbb{Z}_+$ of non-negative integers and $\mathbb{Z}_k = \{1, \dots, k\}$. An inequality is called *tight* if it holds with equality, otherwise it is *strict*. Let V be a finite set. For a function $m : V \to \mathbb{R}$ and a subset X of V, we define $m(X) = \sum_{x \in X} m(x)$. Functions may have the values $+\infty$ or $-\infty$. For a function $g : V \to \mathbb{Z}_+$ and a non-negative integer h, the function $\boldsymbol{g_h}$ is defined as follows: $g_h(v) = \min\{g(v), h\}$ for every $v \in V$. The function $\boldsymbol{\infty_0}$ on V has value ∞ everywhere except for the emptyset where we set its value to 0. The function ∞_0 is considered as an integer valued function. For a subset X of V, its *complement* is denoted by $\overline{\boldsymbol{X}}$. A *multiset* of V is a set of elements of V taken with multiplicities. For a multiset S of V and a subset X of V, we denote by $\boldsymbol{S_X}$ the multiset of V which is the restriction of S on X. For $k \in \mathbb{Z}_+$, we denote by $\boldsymbol{k \times V}$ the multiset of V where every element of V is taken with multiplicity k. A set $\mathcal{S}$ of subsets of V is called a *family* if the subsets of V are taken with multiplicities in $\mathcal{S}$. For a family $\mathcal{S}$ of subsets of V and a subset X of V, we denote by $\boldsymbol{\mathcal{S}_X}$ the subfamily of $\mathcal{S}$ containing the sets in $\mathcal{S}$ that intersect X. We say that two subsets X and Y of V are *properly intersecting* if none of $X \cap Y, X - Y$, and $Y - X$ is empty. We mean by *uncrossing* two properly intersecting sets the operation that replaces the two sets by their intersection and their union.

A set of mutually disjoint subsets of V is called a *subpartition.* If $\mathcal{P}$ is a subpartition of V, then $\boldsymbol{\cup\mathcal{P}}$ denotes the union of the sets in $\mathcal{P}$. If $\mathcal{P}$ is a subpartition of V and $\cup\mathcal{P} = V$, then $\mathcal{P}$ is called a *partition.* Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two subpartitions of V, $P_1 = \cup\mathcal{P}_1$, $P_2 = \cup\mathcal{P}_2$, and $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Note that $\mathcal{P}$ covers each element in $P_1 \cap P_2$ twice and each element in $(P_1 \cup P_2) - (P_1 \cap P_2)$ once. Using the usual uncrossing method on $\mathcal{P}$, we obtain a family $\mathcal{P}'$ that contains no properly intersecting sets and that covers each element in $P_1 \cap P_2$ twice and each element in $(P_1 \cup P_2) - (P_1 \cap P_2)$ once. Then, by taking respectively the minimal and the maximal sets in $\mathcal{P}'$, we obtain a partition $\mathcal{P}'_1$ of $P_1 \cap P_2$ and a partition $\mathcal{P}'_2$ of $P_1 \cup P_2$. We mention that while $\mathcal{P}'_1$ depends on the particular execution of the uncrossing method, $\cup\mathcal{P}'_1$, $|\mathcal{P}'_1|$, and $\mathcal{P}'_2$ are uniquely defined. We define $\mathcal{P}'_1$ as the intersection $\boldsymbol{\mathcal{P}_1 \sqcap \mathcal{P}_2}$ of $\mathcal{P}_1$ and $\mathcal{P}_2$, and $\mathcal{P}'_2$ as the union $\boldsymbol{\mathcal{P}_1 \sqcup \mathcal{P}_2}$ of $\mathcal{P}_1$ and $\mathcal{P}_2$. We will only use properties on $\mathcal{P}_1 \sqcap \mathcal{P}_2$ and $\mathcal{P}_1 \sqcup \mathcal{P}_2$ that are true for every execution of the uncrossing method such as:

$$\text{If } U_1 \in \mathcal{P}_1 \text{ and } U_2 \in \mathcal{P}_2 \text{ intersect, then an element of } \mathcal{P}_1 \sqcap \mathcal{P}_2 \text{ contains } U_1 \cap U_2. \tag{1}$$

$$\text{If } U \in \mathcal{P}_1 \cup \mathcal{P}_2 \text{, then an element of } \mathcal{P}_1 \sqcup \mathcal{P}_2 \text{ contains } U. \tag{2}$$

$$|\mathcal{P}_1| + |\mathcal{P}_2| = |\mathcal{P}_1 \sqcap \mathcal{P}_2| + |\mathcal{P}_1 \sqcup \mathcal{P}_2|. \tag{3}$$

Let S be a finite ground set. A set function b on S is called *non-decreasing* if $b(X) \leq b(Y)$ for all $X \subseteq Y \subseteq S$, *subcardinal* if $b(X) \leq |X|$ for every $X \subseteq S$, and *submodular* if $b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y)$ for all $X, Y \subseteq S$. A set function p on S is called *supermodular* if $-p$ is submodular. A set function m on S is called *modular* if it is submodular and supermodular. Let r be a non-negative integer-valued function on S such that r is subcardinal, non-decreasing and submodular. Then $\mathsf{M} = (S, r)$ is called a *matroid.* The function r is called the *rank function* of the matroid M. If a matroid

M is given, then we denote its rank function by $\boldsymbol{r_{\mathsf{M}}}$. An *independent set* of M is a subset X of S such that $r_{\mathsf{M}}(X) = |X|$. The set of independent sets of M is denoted by $\boldsymbol{\mathcal{I}_{\mathsf{M}}}$. A maximal independent set of M is called a *basis*. The *free matroid* is the matroid where every subset of S is independent. The *uniform matroid* $U_{S,k}$ of rank k is the matroid whose independent sets are the subsets of S of size at most k.

Let $\boldsymbol{D} = (V, A)$ be a directed graph or *digraph* with *vertex set* $\boldsymbol{V}$ and *arc set* $\boldsymbol{A}$. An *arc* $e = uv$ is an ordered pair of different vertices u (the *tail* of e) and v (the *head* of e). For a subset X of V, the set of arcs in A *entering* X, that is their heads are in X and their tails are in $\overline{X}$, is denoted by $\boldsymbol{\delta_A^-(X)}$. The *in-degree* of X is $\boldsymbol{d_A^-(X)} = |\delta_A^-(X)|$. A digraph $H = (U, B)$ is called an *s-arborescence* if $s \in U$ and every vertex of H is reachable from s via a unique path in H. The vertex s is called the *root* of the s-arborescence. We say that H is an *S-branching* if $S \subseteq U$ and there exists a unique path from S to every $v \in U$ in H. The vertex set S is called the *root set* of the S-branching. A branching H is called a *spanning branching* of D if $U = V$ and $B \subseteq A$. Note that if $S = \{s\}$, then an S-branching is an s-arborescence. For non-negative integer-valued functions f' and g' on V, an arc set F is called *(f', g')-indegree-bounded* if $f'(v) \leq d_F^-(v) \leq g'(v)$ for every $v \in V$. For non-negative integers q and q', an arc set F is called *(q, q')-size-limited* if $q \leq |F| \leq q'$.

Let $\boldsymbol{\mathcal{D}} = (V, \mathcal{A})$ be a directed hypergraph or *dypergraph* with dyperedge set $\boldsymbol{\mathcal{A}}$. A *dyperedge* $e = Zz$ is an ordered pair of a non-empty subset Z of $V - z$ (the set of *tails* of e) and a vertex z in V (the *head* of e). For a subset X of V, a dyperedge Zz *enters* X if $z \in X$ and $Z \cap \overline{X} \neq \emptyset$. The set of dyperedges in $\mathcal{A}$ *entering* X is denoted by $\boldsymbol{\delta_{\mathcal{A}}^-(X)}$ and the *in-degree* of X is $\boldsymbol{d_{\mathcal{A}}^-(X)} = |\delta_{\mathcal{A}}^-(X)|$. By *trimming* a dyperedge Zz, we mean the operation that replaces Zz by an arc yz for some $y \in Z$. We say that $\mathcal{H} = (U, \mathcal{B})$ is an *S-hyperbranching* if $\mathcal{H}$ can be trimmed to an S-branching. If $S = \{s\}$, then an S-hyperbranching is called an *s-hyperarborescence*. An S-hyperbranching $\mathcal{H}$ is called a *spanning S-hyperbranching* of $\mathcal{D}$ if $U = V$, $\mathcal{B} \subseteq \mathcal{A}$, and $|S| + |\mathcal{B}| = |V|$.

Let $\boldsymbol{\mathcal{F}} = (V, \mathcal{E} \cup \mathcal{A})$ be a *mixed hypergraph* with hyperedge set $\boldsymbol{\mathcal{E}}$ and dyperedge set $\boldsymbol{\mathcal{A}}$. A *hyperedge* is a subset Z of V containing at least two distinct elements. In this paper digraphs may have multiple arcs, hypergraph may have multiple hyperedges, dypergraph may have multiple dyperedges, and mixed hypergraphs may have multiple hyperedges and multiple dyperedge. For a subset X of V, a hyperedge Z *enters* X if $Z \cap X \neq \emptyset \neq Z \cap \overline{X}$. By *orienting* a hyperedge $Z \in \mathcal{E}$, we mean the operation that replaces the hyperedge Z by a dyperedge $Z'z$ where $z \in Z$ and $Z' = Z - z$. An *orientation* of $\mathcal{F}$ is obtained from $\mathcal{F}$ by orienting every hyperedge in $\mathcal{E}$. A *mixed (spanning) S-hyperbranching* is a mixed hypergraph that has an orientation that is a (spanning) S-hyperbranching. In particular, if $S = \{s\}$, then we are speaking of a mixed (spanning) s-hyperarborescence. A mixed S-hyperbranching is called a *rooted S-hyperforest* if it contains no dyperedge. For a subpartition $\mathcal{P}$ of V, we denote by $\boldsymbol{e_{\mathcal{E} \cup \mathcal{A}}(\mathcal{P})}$ the number of hyperedges in $\mathcal{E}$ and dyperedges in $\mathcal{A}$ that enter at least one member of $\mathcal{P}$. By a *packing* of mixed hyperbranchings in $\mathcal{F}$, we mean a set of mixed hyperbranchings that are hyperedge- and dyperedge-disjoint.

Let $\mathcal{B}$ be a packing of arborescences in a digraph D. For a positive integer h, the packing $\mathcal{B}$ is called *h-regular* if each vertex of D belongs to exactly h arborescences in $\mathcal{B}$. For non-negative integer-valued functions f and g on V, the packing $\mathcal{B}$ is called *(f, g)-bounded* if the number of arborescences in $\mathcal{B}$ rooted at v is at least $f(v)$ and at most $g(v)$ for every vertex v of D. For non-negative integers α and β, the packing $\mathcal{B}$ is called *(α, β)-limited* if the number of arborescences in $\mathcal{B}$ is at least α and at most β. For a multiset S of vertices in V and a matroid M on S, the packing $\mathcal{B}$ is called *M-independent-rooted* if the root set of the arborescences in $\mathcal{B}$ forms an independent set in M. If the root set of the arborescences in $\mathcal{B}$ forms a basis in M, then the packing $\mathcal{B}$ is called *M-basis-rooted*. For $\ell, \ell' : \mathbb{Z}_k \to \mathbb{Z}_+$, a packing of k branchings with root sets $S_1, \dots, S_k$ in D is said to be *(ℓ, ℓ')-bordered* if $\ell(i) \leq |S_i| \leq \ell'(i)$ for every $1 \leq i \leq k$.

Let $\mathcal{D}$ be a dypergraph, $\mathcal{F}$ a mixed hypergraph and P a subset of the properties of M-independent-rooted, M-bases-rooted, (f,g)-bounded, h-regular, (α,β)-limited, and (ℓ,ℓ')-bordered. We say that $\mathcal{D}$ has a P *packing* of hyperarborescences if $\mathcal{D}$ can be trimmed to a digraph that has a P packing of arborescences. We say that $\mathcal{F}$ has a P *packing* of mixed hyperarborescences if $\mathcal{F}$ can be oriented to a dypergraph that has a P packing of hyperarborescences.

3 Generalized polymatroids

We present the necessary definitions and results from the theory of generalized polymatroids. In this section let p and b be a supermodular and a submodular set function on V such that $p(\emptyset) = 0 = b(\emptyset)$. Let f and g be integer-valued functions on V and α and β integers. We will use the following polyhedra.

$$\begin{aligned}
\boldsymbol{Q(p,b)} &= \{x \in \mathbb{R}^V : p(Z) \leq x(Z) \leq b(Z) \text{ for every } Z \subseteq V\},\\
\boldsymbol{T(f,g)} &= \{x \in \mathbb{R}^V : f(v) \leq x(v) \leq g(v) \text{ for every } v \in V\},\\
\boldsymbol{K(\alpha,\beta)} &= \{x \in \mathbb{R}^V : \alpha \leq x(V) \leq \beta\},\\
\boldsymbol{Q_1 + Q_2} &= \{x_1 + x_2 : x_1 \in Q_1, x_2 \in Q_2\}.
\end{aligned}$$

If $b(X) - p(Y) \geq b(X-Y) - p(Y-X)$ for all $X, Y \subseteq V$, then $Q(p,b)$ is called a *generalized polymatroid*, shortly *g-polymatroid*. The sum $Q_1 + Q_2$ is defined only for non-empty polyhedra Q_1 and Q_2.

We need the following results on generalized polymatroids.

Theorem 1 (Frank [8]). *Let $Q(p,b)$ be a g-polymatroid, $f,g : V \to \mathbb{Z}$ functions, $\alpha,\beta \in \mathbb{Z}$, $Q(p_1,b_1)$ and $Q(p_2,b_2)$ two non-empty g-polymatroids.*

1. (Proposition 14.1.2 in [8])

   (a) $T(f,g) \neq \emptyset$ if and only if $f \leq g$.

   (b) If $T(f,g) \neq \emptyset$, then it is a g-polymatroid $Q(f,g)$.

2. (Theorems 14.3.9, 14.3.14 in [8])

   (a) $Q(p,b) \cap T(f,g) \neq \emptyset$ if and only if $\max\{p(Z), f(Z)\} \leq \min\{b(Z), g(Z)\}$ for every $Z \subseteq V$.

   (b) If $Q(p,b) \cap T(f,g) \neq \emptyset$, then it is a g-polymatroid $Q(p_f^g, b_f^g)$ with

$$\begin{aligned}
\boldsymbol{p_f^g(Z)} &= \max\{p(X) - g(X-Z) + f(Z-X) : X \subseteq V\},\\
\boldsymbol{b_f^g(Z)} &= \min\{b(X) - f(X-Z) + g(Z-X) : X \subseteq V\}.
\end{aligned}$$

3. (Corollary 14.3.18, Theorem 14.3.13 in [8])

   (a) $Q(p,b) \cap K(\alpha,\beta) \neq \emptyset$ if and only if $p \leq b$, $\alpha \leq \beta$, $\beta \geq p(V)$ and $\alpha \leq b(V)$.

   (b) If $Q(p,b) \cap K(\alpha,\beta) \neq \emptyset$, then it is a g-polymatroid $Q(p_\alpha^\beta, b_\alpha^\beta)$ with

$$\begin{aligned}
\boldsymbol{p_\alpha^\beta(Z)} &= \max\{p(Z), \alpha - b(\overline{Z})\},\\
\boldsymbol{b_\alpha^\beta(Z)} &= \min\{b(Z), \beta - p(\overline{Z})\}.
\end{aligned}$$

4. (Theorem 14.2.15 in [8])

(a) $Q(p_1, b_1) + Q(p_2, b_2) = Q(p_1 + p_2, b_1 + b_2)$.

(b) *If p_1, b_1, p_2, b_2 are integral, then every integral element of $Q(p_1 + p_2, b_1 + b_2)$ arises as the sum of an integral element of $Q(p_1, b_1)$ and an integral element of $Q(p_2, b_2)$.*

5. (Theorem 14.4.2 in [8])

(a) $Q(p_1, b_1) \cap Q(p_2, b_2) \neq \emptyset$ *if and only if* $p_1 \leq b_2$ *and* $p_2 \leq b_1$.

(b) *If p_1, b_1, p_2, b_2 are integral and the intersection is not empty, then it contains an integral element.*

4 Packing problems

This section lists some results on packing arborescences or more generally on packing mixed hyperbranchings relevant to this paper. It contains three subsections: the first one on packing mixed arborescences, the second one on packing mixed hyperarborescences, and the last one on packing mixed hyperbranchings.

4.1 Packing arborescences and mixed arborescences

We start by the classic result of Edmonds [4] on packing arborescences with fixed roots.

Theorem 2 (Edmonds [4]). *Let $D = (V, A)$ be a digraph and $S = \{s_1, \ldots, s_k\}$ a multiset of vertices in V. There exists a packing of k spanning arborescences with roots $s_1, \ldots, s_k$ in D if and only if*

$$|S_X| + d_A^-(X) \geq k \quad \text{for every non-empty } X \subseteq V. \tag{4}$$

The next result on packing arborescences with flexible roots is due to Frank [6]. Recall that, for a subpartition $\mathcal{P}$ of V, $e_A(\mathcal{P})$ denotes the number of arcs entering a member of $\mathcal{P}$. Note that $e_A(\mathcal{P}) = \sum_{X \in \mathcal{P}} d_A^-(X)$.

Theorem 3 (Frank [6]). *Let $D = (V, A)$ be a digraph and $k \in \mathbb{Z}_+$. There exists a packing of k spanning arborescences in D if and only if*

$$e_A(\mathcal{P}) + k \geq k|\mathcal{P}| \quad \text{for every subpartition } \mathcal{P} \text{ of } V. \tag{5}$$

Remark 1. *It is well-known that Theorems 2 and 3 are equivalent.*

Theorem 3 was generalized for (f, g)-bounded packings as follows.

Theorem 4 (Frank [6], Cai [2]). *Let $D = (V, A)$ be a digraph, $f, g : V \to \mathbb{Z}_+$ functions, and $k \in \mathbb{Z}_+$. There exists an (f, g)-bounded packing of k spanning arborescences in D if and only if*

$$g(v) \geq f(v) \quad \text{for every } v \in V, \tag{6}$$

$$e_A(\mathcal{P}) + \min\{k - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\} \geq k|\mathcal{P}| \quad \text{for every subpartition } \mathcal{P} \text{ of } V. \tag{7}$$

Remark 2. *For $f(v) = 0$ and $g(v) = k$ for every $v \in V$, Theorem 4 reduces to Theorem 3.*

Proof. In this case an (f, g)-bounded packing of k spanning arborescences is a packing of k spanning arborescences. Further, (6) trivially holds and, for every subpartition $\mathcal{P}$ of V, we have $\min\{k - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\} = k$, so (5) and (7) are equivalent. $\square$

Theorem 25 of Bérczi and Frank [1] provides the following extension of Theorem 4.

Theorem 5 (Bérczi, Frank [1]). *Let $D = (V, A)$ be a digraph, $f, g : V \to \mathbb{Z}_+$ functions, and $h, \alpha, \beta \in \mathbb{Z}_+$. There is an h-regular (f, g)-bounded (α, β)-limited packing of arborescences in D if and only if*

$$\begin{aligned} g_h(v) &\geq f(v) \quad \text{for every } v \in V, && (8)\\ \min\{\beta, g_h(V)\} &\geq \alpha, && (9)\\ e_A(\mathcal{P}) + \min\{\beta - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\} &\geq h|\mathcal{P}| \quad \text{for every subpartition } \mathcal{P} \text{ of } V. && (10) \end{aligned}$$

Remark 3. *For $h = \alpha = \beta = k$, Theorem 5 reduces to Theorem 4.*

Proof. In this case an h-regular (α, β)-limited packing of arborescences is a packing of k spanning arborescences. Further, (7) applied for $\mathcal{P} = \emptyset$ implies that $f(V) \leq k$, so by $f \geq 0$, we have $f(v) \leq k$ for every $v \in V$. (7) applied for $\mathcal{P} = \{V\}$ implies that $g(V) \geq k$. Then, (6)–(7) are equivalent to (8)–(10). □

Theorem 3 was extended to mixed graphs as follows.

Theorem 6 (Frank [6]). *Let $F = (V, E \cup A)$ be a mixed graph and $k \in \mathbb{Z}_+$. There exists a packing of k spanning mixed arborescences in F if and only if*

$$e_{E\cup A}(\mathcal{P}) + k \geq k|\mathcal{P}| \quad \text{for every subpartition } \mathcal{P} \text{ of } V. \tag{11}$$

Remark 4. *If $E = \emptyset$, then Theorem 6 reduces to Theorem 3.*

Theorem 4 was also extended to mixed graphs as follows.

Theorem 7 (Gao,Yang [12]). *Let $F = (V, E \cup A)$ be a mixed graph, $f, g : V \to \mathbb{Z}_+$ functions, and $k \in \mathbb{Z}_+$. There exists an (f, g)-bounded packing of k spanning mixed arborescences in F if and only if (6) is satisfied and*

$$e_{E\cup A}(\mathcal{P}) + \min\{k - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\} \geq k|\mathcal{P}| \quad \text{for every subpartition } \mathcal{P} \text{ of } V. \tag{12}$$

Remark 5. *If $E = \emptyset$, then Theorem 7 reduces to Theorem 4.*

Remark 6. *If $f(v) = 0$ and $g(v) = k$ for every $v \in V$, then Theorem 7 reduces to Theorem 6.*

Proof. In this case an (f, g)-bounded packing of k spanning mixed arborescences is a packing of k spanning mixed arborescences. Further, for every subpartition $\mathcal{P}$ of V, we have $\min\{k - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\} = k$, so (11) and (12) are equivalent. □

4.2 Packing hyperarborescences and mixed hyperarborescences

Theorem 2 was extended for dypergraphs by Frank, Király, and Király [9].

Theorem 8 (Frank, Király, Király [9]). *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph and $S = \{s_1, \dots, s_k\}$ a multiset of vertices in V. There exists a packing of k spanning hyperarborescences with roots $s_1, \dots, s_k$ in $\mathcal{D}$ if and only if*

$$|S_X| + d^-_{\mathcal{A}}(X) \geq k \quad \text{for every non-empty } X \subseteq V. \tag{13}$$

Remark 7. *If $\mathcal{D}$ is a digraph, then Theorem 8 reduces to Theorem 2.*

Hörsch and Szigeti [14] extended Theorem 7 to mixed hypergraphs.

Theorem 9 (Hörsch, Szigeti [14]). *Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be a mixed hypergraph, $f, g : V \to \mathbb{Z}_+$ functions, and $k \in \mathbb{Z}_+$. There exists an (f, g)-bounded packing of k spanning mixed hyperarborescences in $\mathcal{F}$ if and only if (6) is satisfied and*

$$e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}) + \min\{k - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\} \geq k|\mathcal{P}| \quad \text{for every subpartition } \mathcal{P} \text{ of } V. \tag{14}$$

Remark 8. *If $\mathcal{F}$ is a mixed graph, then Theorem 9 reduces to Theorem 7.*

Remark 9. *If $\mathcal{E} = \emptyset$, $k = |S|$ for a multiset S of vertices in V, and $f(v) = g(v) = |S_{\{v\}}|$ for every $v \in V$, then Theorem 9 reduces to Theorem 8.*

Proof. In this case an (f, g)-bounded packing of k spanning mixed hyperarborescences is a packing of k spanning hyperarborescences with roots $s_1, \dots, s_k$, where $S = \{s_1, \dots, s_k\}$. Further, for every subpartition $\mathcal{P}$ of V, we have

$$e_{\mathcal{A}}(\mathcal{P}) + \min\{k - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\} = e_{\mathcal{A}}(\mathcal{P}) + |S_{\cup\mathcal{P}}| = \sum_{X\in\mathcal{P}} (d^-_{\mathcal{A}}(X) + |S_X|),$$

so (13) and (14) are equivalent. □

An extension with a matroid constraint of a common generalization of Theorems 4 and 8 was given in Szigeti [19].

Theorem 10 (Szigeti [19]). *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph, $h \in \mathbb{Z}_+$, $f, g : V \to \mathbb{Z}_+$ functions, S a multiset of vertices in V, and $\mathsf{M} = (S, r_{\mathsf{M}})$ a matroid. There exists an h-regular M-basis-rooted (f, g)-bounded packing of hyperarborescences in $\mathcal{D}$ if and only if (8) holds and for all $X, Z \subseteq V$ and subpartition $\mathcal{P}$ of Z,*

$$r_{\mathsf{M}}(S_X) + g_h(\overline{X}) \geq r_{\mathsf{M}}(S), \tag{15}$$

$$e_{\mathcal{A}}(\mathcal{P}) + r_{\mathsf{M}}(S_X) - f(X - Z) + g_h(Z - X) \geq h|\mathcal{P}|. \tag{16}$$

Remark 10. *If $\mathcal{D}$ is digraph, $\mathsf{M} = U_{k\times V,k}$ is the uniform matroid of rank k on $k \times V$, and $h = k$, then Theorem 10 reduces to Theorem 4.*

Proof. In this case an h-regular M-basis-rooted packing of hyperarborescences is a packing of k spanning arborescences. Further, $r_{\mathsf{M}}(S_X) = k$ for every $\emptyset \neq X \subseteq V$ and $r_{\mathsf{M}}(S_\emptyset) = 0$. We have already seen that (7) implies that $f(v) \leq k$ for every $v \in V$ and that $g(V) \geq k$. Hence (6) is equivalent to (8), and (15) holds. (16) is equivalent to $e_{\mathcal{A}}(\mathcal{P}) + g(Z) \geq k|\mathcal{P}|$ (for $X = \emptyset$) and $e_{\mathcal{A}}(\mathcal{P}) + k - f(X - Z) + g_k(Z - X) \geq k|\mathcal{P}|$ (for $X \neq \emptyset$). The first one, by $g \geq 0$, is equivalent to $e_{\mathcal{A}}(\mathcal{P}) + g(\cup\mathcal{P}) \geq k|\mathcal{P}|$. The second one, by $f, g \geq 0$, is equivalent to $e_{\mathcal{A}}(\mathcal{P}) + k - f(\overline{\cup\mathcal{P}}) \geq k|\mathcal{P}|$. It follows that (16) is equivalent to (7). □

Remark 11. *For a multiset $S = \{s_1, \dots, s_k\}$ of V, if M is the free matroid on S, $f(v) = g(v) = |S_{\{v\}}|$ for every $v \in V$, and $h = |S|$, then Theorem 10 reduces to Theorem 8.*

Proof. In this case an h-regular M-basis-rooted (f, g)-bounded packing of hyperarborescences is a packing of k spanning hyperarborescences with roots $s_1, \dots, s_k$. (8) trivially holds and, by $r_{\mathsf{M}}(S_X) + g_h(\overline{X}) = |S_X| + |S_{\overline{X}}| = |S| = r_{\mathsf{M}}(S)$, (15) also holds. By $r_{\mathsf{M}}(S_X) - f(X - Z) + g_h(Z - X) = |S_X| - |S_{X-Z}| + |S_{Z-X}| = |S_Z| \geq |S_{\cup\mathcal{P}}|$, (16) is equivalent to $e_{\mathcal{A}}(\mathcal{P}) + |S_{\cup\mathcal{P}}| \geq k|\mathcal{P}|$ which is equivalent to (13). □

Theorem 10 and an orientation result of Gao [11] provide the following theorem.

Theorem 11. *Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be a mixed hypergraph, $h \in \mathbb{Z}_+$, $f, g : V \to \mathbb{Z}_+$ functions, S a multiset of vertices in V, and $\mathsf{M} = (S, r_\mathsf{M})$ a matroid. There exists an h-regular M-basis-rooted (f, g)-bounded packing of mixed hyperarborescences in $\mathcal{F}$ if and only if* (8) *and* (15) *hold and for all $X, Z \subseteq V$ and subpartition $\mathcal{P}$ of Z,*

$$e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}) + r_\mathsf{M}(S_X) - f(X - Z) + g_h(Z - X) \quad \geq \quad h|\mathcal{P}|. \tag{17}$$

Remark 12. *If $\mathcal{F}$ is a mixed graph, M is the uniform matroid of rank k on $k \times V$ and $h = k$, then Theorem 11 reduces to Theorem 7.*

Proof. In this case an h-regular M-basis-rooted packing of mixed hyperarborescences is a packing of k spanning mixed arborescences. Further, (8), (15), and (17) are equivalent to (6) and (12). This can be proved similarly as in Remark 10. □

Remark 13. *If $\mathcal{F}$ is a dypergraph, then Theorem 11 reduces to Theorem 10.*

An extension for mixed hypergraphs of a common generalization of Theorems 5 and 9 was given in Szigeti [18].

Theorem 12 (Szigeti [18])**.** *Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be a mixed hypergraph, $f, g : V \to \mathbb{Z}_+$ functions, and $h, \alpha, \beta \in \mathbb{Z}_+$. There exists an h-regular (f, g)-bounded (α, β)-limited packing of mixed hyperarborescences in $\mathcal{F}$ if and only if* (8) *and* (9) *hold and*

$$e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}) + \min\{\beta - f(\overline{\cup\mathcal{P}}), g_h(\cup\mathcal{P})\} \quad \geq \quad h|\mathcal{P}| \qquad \text{for every subpartition } \mathcal{P} \text{ of } V. \tag{18}$$

Remark 14. *If $\mathcal{F}$ is digraph, then Theorem 12 reduces to Theorem 5.*

Remark 15. *If $h = \alpha = \beta = k$, then Theorem 12 reduces to Theorem 9.*

As a new result we now provide a common generalization of the previous two theorems.

Theorem 13. *Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be a mixed hypergraph, $f, g : V \to \mathbb{Z}_+$ functions, $h, \alpha, \beta \in \mathbb{Z}_+$, S a multiset of vertices in V, and $\mathsf{M} = (S, r_\mathsf{M})$ a matroid. There exists an h-regular M-independent-rooted (f, g)-bounded (α, β)-limited packing of mixed hyperarborescences in $\mathcal{F}$ if and only if* (8) *and* (17) *hold and for all $X, Z \subseteq V$ and subpartition $\mathcal{P}$ of Z,*

$$\alpha \quad \leq \quad \beta, \tag{19}$$
$$\max\{h, \alpha\} - r_\mathsf{M}(S_{\overline{X}}) + f(Z - X) - g_h(X - Z) \quad \leq \quad h|Z|, \tag{20}$$
$$e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}) + \beta - f(\overline{Z}) \quad \geq \quad h|\mathcal{P}|. \tag{21}$$

Theorem 13 will easily follow from Corollary 2.

Remark 16. *If $\alpha = \beta = r_\mathsf{M}(S)$, then Theorem 13 reduces to Theorem 11.*

Proof. In this case M-independent-rooted (α, β)-limited is equivalent to M-basis-rooted. Further, by $\alpha = \beta$, (19) holds. By (17) applied for $X = Z = V$ and $\mathcal{P} = \{V\}$, we get $h \leq r_\mathsf{M}(S)$. Then, by $\max\{h, \alpha\} = r_\mathsf{M}(S)$, (8), (15) applied for $\overline{X}$, and $h \geq g_h$, we get that

$$\begin{aligned} & \max\{h, \alpha\} - r_\mathsf{M}(S_{\overline{X}}) + f(Z - X) - g_h(X - Z) \\ \leq \; & r_\mathsf{M}(S) - r_\mathsf{M}(S_{\overline{X}}) + h|Z - X| - g_h(X) + h|Z \cap X| \\ \leq \; & h|Z|, \end{aligned}$$

so (20) holds. On the other hand, by $\max\{h, \alpha\} = r_\mathsf{M}(S)$ and (20) applied for $Z = \emptyset$, we get that (15) holds. Hence (15) and (20) are equivalent. Finally, (17) applied for $X = V$ provides (21). □

Remark 17. *If* M *is the free matroid on* $h \times V$*, then Theorem 13 reduces to Theorem 12.*

Proof. In this case an h-regular M-independent-rooted packing is an h-regular packing. Further, $r_{\mathsf{M}}(S_X) = h|X|$ for every $X \subseteq V$. On the one hand, by (19) and (20) applied for $X = V$ and $Z = \emptyset$, we get that (9) holds. By (21) and (17) applied for $X = \emptyset$ and $Z = \cup\mathcal{P}$, we get that (18) holds. On the other hand, (9) implies (19). By $h \geq f \geq 0, h \geq g_h \geq 0$, and $\cup\mathcal{P} \subseteq Z$, we get that

$$\begin{aligned} & h|X| - f(X-Z) + g_h(Z-X) \\ \geq\ & h|X-Z| - f(X-Z) + h|X \cap (\cup\mathcal{P})| + g_h((\cup\mathcal{P}) - X) \\ \geq\ & g_h(\cup\mathcal{P}), \end{aligned}$$

so (18) implies (17). By (9) and (18) applied for $\mathcal{P} = \{V\}$, we get that $g_h(V) \geq \max\{h, \alpha\}$. By $h \geq f, h \geq g_h$ and $\overline{X} \cup Z \cup (X - Z) = V$, we have

$$\begin{aligned} & h|Z| + h|\overline{X}| - f(Z-X) + g_h(X-Z) \\ =\ & h|Z-X| - f(Z-X) + h|\overline{X} \cup Z| + g_h(X-Z) \\ \geq\ & 0 + g_h(V). \end{aligned}$$

Hence (20) holds. By $f \geq 0$ and $\cup\mathcal{P} \subseteq Z$, we get that (18) implies (21). ☐

4.3 Packing branchings and mixed hyperbranchings

Edmonds [4] also gave the characterization of the existence of a packing of spanning branchings with fixed root sets.

Theorem 14 (Edmonds [4])**.** *Let* $D = (V, A)$ *be a digraph and* $\mathcal{S} = \{S_1, \dots, S_k\}$ *a family of subsets of* V*. There exists a packing of* k *spanning branchings with root sets* $S_1, \dots, S_k$ *in* D *if and only if*

$$|\mathcal{S}_X| + d_A^-(X) \geq k \quad \textit{for every non-empty } X \subseteq V.$$

Remark 18. *If* $\mathcal{S} = \{s : s \in S\}$*, then Theorem 14 reduces to Theorem 2.*

Theorem 25 of Bérczi and Frank [1] provides the following extension of Theorem 3. Let us recall that for a function $\ell : \mathbb{Z}_k \to \mathbb{Z}_+$, $\ell(\mathbb{Z}_k) = \sum_{i=1}^k \ell(i)$.

Theorem 15 (Bérczi, Frank [1])**.** *Let* $D = (V, A)$ *be a digraph,* $k, \alpha, \beta \in \mathbb{Z}_+$*, and* $\ell, \ell' : \mathbb{Z}_k \to \mathbb{Z}_+$ *such that*

$$\begin{aligned} \ell'(\mathbb{Z}_k) \geq \beta \geq \alpha \geq \ell(\mathbb{Z}_k), & \qquad (22) \\ |V| \geq \ell'(i) \geq \ell(i) \quad \textit{for every } 1 \leq i \leq k. & \qquad (23) \end{aligned}$$

There exists an (ℓ, ℓ')*-bordered* (α, β)*-limited packing of* k *spanning branchings in* D *if and only if*

$$\beta - \ell(\mathbb{Z}_k) + \sum_{i=1}^{k} \min\{|\mathcal{P}|, \ell(i)\} + e_A(\mathcal{P}) \geq k|\mathcal{P}| \quad \textit{for every subpartition } \mathcal{P} \textit{ of } V, \qquad (24)$$

$$\sum_{i=1}^{k} \min\{|\mathcal{P}|, \ell'(i)\} + e_A(\mathcal{P}) \geq k|\mathcal{P}| \quad \textit{for every subpartition } \mathcal{P} \textit{ of } V. \qquad (25)$$

Remark 19. *If* $\alpha = \beta = k$*, and* $\ell(i) = \ell'(i) = 1$ *for every* $1 \leq i \leq k$*, then Theorem 15 reduces to Theorem 3.*

Proof. In this case an (ℓ,ℓ')-bordered (α,β)-limited packing of k spanning branchings is a packing of k spanning arborescences. Further, (22) and (23) trivially hold, (24) and (25) coincide and are equivalent to (5). $\square$

It was mentioned in [13] that Theorem 15 can be generalized as follows to h-regular packings in dypergraphs.

Theorem 16 (Hoppenot, Martin, Szigeti [13]). *Let $\mathcal{D}=(V,\mathcal{A})$ be a dypergraph, $h,k,\alpha,\beta\in\mathbb{Z}_+$, and $\ell,\ell':\mathbb{Z}_k\to\mathbb{Z}_+$ such that (22) and (23) hold. There exists an h-regular (ℓ,ℓ')-bordered (α,β)-limited packing of k hyperbranchings in $\mathcal{D}$ if and only if*

$$h|V| \;\geq\; \alpha, \tag{26}$$

$$\beta-\ell(\mathbb{Z}_k)+\sum_{i=1}^{k}\min\{|\mathcal{P}|,\ell(i)\}+e_{\mathcal{A}}(\mathcal{P}) \;\geq\; h|\mathcal{P}| \quad \textit{for every subpartition } \mathcal{P} \textit{ of } V, \tag{27}$$

$$\sum_{i=1}^{k}\min\{|\mathcal{P}|,\ell'(i)\}+e_{\mathcal{A}}(\mathcal{P}) \;\geq\; h|\mathcal{P}| \quad \textit{for every subpartition } \mathcal{P} \textit{ of } V. \tag{28}$$

Remark 20. *If $\mathcal{D}$ is a digraph and $h=k$, then Theorem 16 reduces to Theorem 15.*

The undirected counterpart of Theorem 16 follows.

Theorem 17 (Hoppenot, Martin, Szigeti [13]). *Let $\mathcal{G}=(V,\mathcal{E})$ be a hypergraph, $h,k,\alpha,\beta\in\mathbb{Z}_+$, and $\ell,\ell':\mathbb{Z}_k\to\mathbb{Z}_+$ such that (22) and (23) hold. There exists an h-regular (ℓ,ℓ')-bordered (α,β)-limited packing of k rooted hyperforests in $\mathcal{G}$ if and only if (26) holds and*

$$\beta-\ell(\mathbb{Z}_k)+\sum_{i=1}^{k}\min\{|\mathcal{P}|,\ell(i)\}+e_{\mathcal{E}}(\mathcal{P}) \;\geq\; h|\mathcal{P}| \quad \textit{for every partition } \mathcal{P} \textit{ of } V, \tag{29}$$

$$\sum_{i=1}^{k}\min\{|\mathcal{P}|,\ell'(i)\}+e_{\mathcal{E}}(\mathcal{P}) \;\geq\; h|\mathcal{P}| \quad \textit{for every partition } \mathcal{P} \textit{ of } V. \tag{30}$$

We mention that the natural extension of Theorems 16 and 17 to mixed hypergraphs does not hold, see [13].

We need to present the following result in order to deduce Corollary 1 of it.

Theorem 18 (Fortier et al. [5]). *Let $\mathcal{F}=(V,\mathcal{E}\cup\mathcal{A})$ be a mixed hypergraph and $\mathcal{S}=\{S_1,\dots,S_k\}$ a family of subsets in V. There exists a packing of spanning mixed hyperbranchings with root sets $S_1,\dots,S_k$ in $\mathcal{F}$ if and only if*

$$e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P})+\sum_{X\in\mathcal{P}}|\mathcal{S}_X| \;\geq\; k|\mathcal{P}| \quad \textit{for every subpartition } \mathcal{P} \textit{ of } V. \tag{31}$$

Remark 21. *If $\mathcal{F}$ is digraph, then Theorem 18 reduces to Theorem 14.*

The following result for digraphs was observed by Frank, we present it for dypergraphs to apply it later in the proof of Theorem 20.

Corollary 1. *Let $\mathcal{F}=(V\cup\{s\},\mathcal{E}\cup\mathcal{A})$ be a mixed hypergraph such that no hyperedge is incident to s, no dyperedge enters s and every dyperedge leaving s is an arc, and let F be a set of arcs in $\mathcal{A}$ leaving s. There exists a packing of $|F|$ spanning mixed s-hyperarborescences in $\mathcal{F}$ each containing an arc of F if and only if*

$$e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}) \;\geq\; |F||\mathcal{P}| \quad \textit{for every subpartition } \mathcal{P} \textit{ of } V. \tag{32}$$

Proof. Corollary 1 follows from Theorem 18 by applying it for $\mathcal{S} = \{\{s, v\} : sv \in F\}$ and $\mathcal{F}' = (V \cup \{s\}, \mathcal{E} \cup \mathcal{A}')$ where $\mathcal{A}' = \mathcal{A} - F$. Indeed, let $\mathcal{P}$ be a subpartition of $V \cup \{s\}$. Note that

$$e_{\mathcal{E}\cup\mathcal{A}'}(\mathcal{P}) + \sum_{X\in\mathcal{P}} |\mathcal{S}_X| \geq e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}) - e_F(\mathcal{P}) + \sum_{X\in\mathcal{P}} d_F^-(X) = e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}).$$

Thus, by (32), we get that (31) holds for $\mathcal{F}'$ and $\mathcal{S}$. Then, by Theorem 18, there exists a packing of spanning mixed hyperbranchings $\mathcal{B}'_v$ in $\mathcal{F}'$ with root sets $\{s, v\}$ for every $sv \in F$. Then $\mathcal{B}_v = \mathcal{B}'_v + sv$ for every $sv \in F$ is the desired packing of $|F|$ spanning mixed s-hyperarborescences in $\mathcal{F}$ each containing an arc of F. □

5 Augmentation problems

This section contains the augmentation versions of the results presented in Section 4.

The first results on arborescence packing augmentation appeared in Frank [8].

Theorem 19 (Frank [8])**.** *Let $D = (V, A)$ be a digraph, $k, \gamma \in \mathbb{Z}_+$, and $S = \{s_1, \dots, s_k\}$ a multiset of vertices in V. We can add γ arcs to D to have a packing of*

(a) k spanning arborescences with roots $s_1, \dots, s_k$ if and only if

$$\gamma + e_A(\mathcal{P}) + |S_{\cup\mathcal{P}}| \geq k|\mathcal{P}| \quad \text{for every subpartition } \mathcal{P} \text{ of } V. \tag{33}$$

(b) k spanning arborescences if and only if

$$\gamma + e_A(\mathcal{P}) + k \geq k|\mathcal{P}| \quad \text{for every subpartition } \mathcal{P} \text{ of } V. \tag{34}$$

Remark 22. *If $\gamma = 0$, then Theorem 19(a) reduces to Theorem 2.*

Proof. It follows from the fact that $\sum_{X\in\mathcal{P}} \big(|S_X| + d_A^-(X)\big) = e_A(\mathcal{P}) + \sum_{X\in\mathcal{P}} |S_X| \geq k|\mathcal{P}|$ for every subpartition $\mathcal{P}$ of V is equivalent to (4). □

Remark 23. *If $\gamma = 0$, then Theorem 19(b) reduces to Theorem 3.*

5.1 Augmentation to have a packing of mixed hyperarborescences

We now provide the first main result of the paper which is a slight extension of the augmentation version of Theorem 13.

Theorem 20. *Let $\mathcal{F} = (V, \mathcal{E}\cup\mathcal{A})$ be a mixed hypergraph, $f, g, f', g' : V \to \mathbb{Z}_+$ functions, $h, \alpha, \beta, q, q' \in \mathbb{Z}_+$, S a multiset of vertices in V, and $\mathsf{M} = (S, r_\mathsf{M})$ a matroid. We can add an (f', g')-indegree-bounded (q, q')-size-limited arc set F to $\mathcal{F}$ to have an h-regular M-independent-rooted (f, g)-bounded (α, β)-limited packing of mixed hyperarborescences that contains F if and only if* (8) *and* (19) *hold and for all $v \in V$, $X, Z \subseteq V$ and subpartition $\mathcal{P}$ of Z,*

$$f'(v) \leq g'(v), \tag{35}$$

$$q \leq q', \tag{36}$$

$$\max\{f'(Z), q - g'(\overline{Z})\} + \max\{f(Z), \max\{h, \alpha\} - r_\mathsf{M}(S_{\overline{X}}) + f(Z - X) - g_h(X - Z)\} \leq h|Z|, \tag{37}$$

$$e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}) + \min\{g'(Z), q' - f'(\overline{Z})\} + \min\{\beta - f(\overline{Z}), r_\mathsf{M}(S_X) - f(X - Z) + g_h(Z - X)\} \geq h|\mathcal{P}|. \tag{38}$$

Proof. The theory of generalized polymatroids provides the tools to be applied in the proof. The discovery of the submodularity of the function $e_{\mathcal{E}\cup\mathcal{A}}$ on subpartitions makes the application of generalized polymatroids possible.

Claim 1. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be subpartitions of V and $X \in \mathcal{E} \cup \mathcal{A}$.*

(a) If X enters an element of $\mathcal{P}_1 \sqcap \mathcal{P}_2$ or an element of $\mathcal{P}_1 \sqcup \mathcal{P}_2$, then it enters an element of $\mathcal{P}_1$ or an element of $\mathcal{P}_2$.

(b) If X enters an element of $\mathcal{P}_1 \sqcap \mathcal{P}_2$ and an element of $\mathcal{P}_1 \sqcup \mathcal{P}_2$, then it enters an element of $\mathcal{P}_1$ and an element of $\mathcal{P}_2$.

(c) Consequently,

$$e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}_1) + e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}_2) \geq e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}_1 \sqcap \mathcal{P}_2) + e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}_1 \sqcup \mathcal{P}_2). \tag{39}$$

Proof. We prove (a) and (b) together.

First suppose that $X \in \mathcal{E}$. If X enters a subset Y of V, then there exist $x \in X \cap Y$ and $z \in X - Y$.

Suppose that $Y \in \mathcal{P}_1 \sqcup \mathcal{P}_2$. Since $x \in Y \subseteq (\cup\mathcal{P}_1) \cup (\cup\mathcal{P}_2)$, there exist $U_1 \in \mathcal{P}_1$ or $U_2 \in \mathcal{P}_2$ containing x, say $U_1 \in \mathcal{P}_1$. By (2) and $x \in Y \cap U_1$, we have $U_1 \subseteq Y$. Since $x \in U_1 \cap X$ and $z \in X - Y \subseteq X - U_1$, we get that X enters U_1.

Suppose now that $Y \in \mathcal{P}_1 \sqcap \mathcal{P}_2$. Since $x \in Y \subseteq (\cup\mathcal{P}_1) \cap (\cup\mathcal{P}_2)$, there exist $U_1' \in \mathcal{P}_1$ and $U_2' \in \mathcal{P}_2$ containing x. By (1) and $x \in U_1' \cap U_2' \cap Y$, we have $U_1' \cap U_2' \subseteq Y$. Since $x \in U_1' \cap U_2' \cap X$ and $z \in X - Y \subseteq X - (U_1' \cap U_2')$, we get that X enters $U_1' \cap U_2'$. Hence, X enters either U_1' or U_2'. Thus, the proof of (a) is complete.

Further, if X enters say U_1' but not U_2', then, since $x \in X \cap U_2'$, we have $X \subseteq U_2'$. Then, by (2), X does not enter any element of $\mathcal{P}_1 \sqcup \mathcal{P}_2$. Hence, the proof of (b) is complete.

The case for $X \in \mathcal{A}$ is the same except that x is the head of X and z is a tail of X not in Y.

It is clear that (c) follows from (a) and (b). $\square$

We introduce a set function $\hat{p}$: for every $Z \subseteq V$,

$$\hat{\boldsymbol{p}}(Z) = \max\{h|\mathcal{P}| - e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}) : \mathcal{P} \text{ subpartition of } Z\}. \tag{40}$$

Claim 2. *The function $\hat{p}$ is supermodular.*

Proof. Let $X_1, X_2 \subseteq V$. For $i = 1, 2$, let $\mathcal{P}_i$ be the subpartition of X_i such that $\hat{p}(X_i) = h|\mathcal{P}_i| - e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}_i)$. Then, by (3), (39), $\mathcal{P}_1 \sqcap \mathcal{P}_2$ and $\mathcal{P}_1 \sqcup \mathcal{P}_2$ are subpartitions of $X_1 \cap X_2$ and $X_1 \cup X_2$, we have

$$\begin{aligned}
\hat{p}(X_1) + \hat{p}(X_2) &= h|\mathcal{P}_1| - e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}_1) + h|\mathcal{P}_2| - e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}_2) \\
&\leq h|\mathcal{P}_1 \sqcap \mathcal{P}_2| - e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}_1 \sqcap \mathcal{P}_2) + h|\mathcal{P}_1 \sqcup \mathcal{P}_2| - e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}_1 \sqcup \mathcal{P}_2) \\
&\leq \hat{p}(X_1 \cap X_2) + \hat{p}(X_1 \cup X_2),
\end{aligned}$$

and the claim follows. $\square$

We naturally consider the following g-polymatroids, where $\boldsymbol{b}_{\mathsf{M}}(X) = r_{\mathsf{M}}(S_X)$ for every $X \subseteq V$, which are well-defined because b_{M} is submodular and, by Claim 2, $\hat{p}$ is supermodular.

$\boldsymbol{Q_1} = T(f', g') \cap K(q, q')$,
$\boldsymbol{Q_2} = Q(-\infty_0, b_{\mathsf{M}}) \cap K(\max\{h, \alpha\}, \beta) \cap T(f, g_h)$,
$\boldsymbol{Q_3} = Q(\hat{p}, \infty_0) \cap T(0, h)$, where 0 and h are seen as constant valued functions,
$\boldsymbol{Q_4} = (Q_1 + Q_2) \cap Q_3$.

Lemma 1. *The following hold.*

(a) An integral vector m is in Q_4 if and only if there exist an (f', g')-indegree-bounded (q, q')-size-limited new arc set F and an h-regular M-independent-rooted (f, g)-bounded (α, β)-limited packing of mixed hyperarborescences in $\mathcal{F} + F$ that contains F with root set R such that $m(v) = d_F^-(v) + |R_v|$ for every $v \in V$.

(b) Q_4 is well-defined and non-empty if and only if (8), (19), (35), (36), (37), *and* (38) *hold.*

Proof. (a) To prove the **necessity**, suppose that there exist an (f', g')-indegree-bounded (q, q')-size-limited new arc set $\boldsymbol{F}$ and an h-regular M-independent-rooted (f, g)-bounded (α, β)-limited packing $\boldsymbol{\mathcal{B}}$ of mixed hyperarborescences in $\mathcal{F} + F$ that contains F. Let $\boldsymbol{R}$ be the root set of $\mathcal{B}$. Then we have the following.

$$
\begin{aligned}
|R_X| \leq r_{\mathsf{M}}(S_X) &= b_{\mathsf{M}}(X) && \text{for every } X \subseteq V, && (41)\\
\max\{h, \alpha\} \leq |R| &\leq \beta, && && (42)\\
f(v) \leq |R_v| &\leq g(v) && \text{for every } v \in V, && (43)\\
q \leq |F| &\leq q', && && (44)\\
f'(v) \leq d_F^-(v) &\leq g'(v) && \text{for every } v \in V, && (45)\\
0 \leq |R_v| + d_F^-(v) &\leq h, && \text{for every } v \in V. && (46)
\end{aligned}
$$

Let $\boldsymbol{m_1}(v) = d_F^-(v)$, $\boldsymbol{m_2}(v) = |R_v|$ for every $v \in V$ and $\boldsymbol{m} = m_1 + m_2$. Then, by (41), (42), (43), and (46), we have $m_2 \in \mathbb{Z}^V \cap Q_2$, by (44) and (45), we have $m_1 \in \mathbb{Z}^V \cap Q_1$, and, by (46), we have $m \in T(0, h)$. Since $m = m_1 + m_2$, we have $m \in \mathbb{Z}^V \cap (Q_1 + Q_2)$. It remains to show that $m \in Q(\hat{p}, \infty_0)$. By Theorem 12 applied for $f(v) = g(v) = |R_v|$ for every $v \in V$ and $\alpha = \beta = |R|$, we get that, for every $\emptyset \neq X \subseteq V$, we have

$$
\begin{aligned}
\hat{p}(X) &= \max\{h|\mathcal{P}| - e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}) : \mathcal{P} \text{ subpartition of } X\}\\
&\leq \max\{h|\mathcal{P}| - e_{\mathcal{E}\cup\mathcal{A}\cup F}(\mathcal{P}) + \sum_{v \in X} d_F^-(v) : \mathcal{P} \text{ subpartition of } X\}\\
&\leq |R_{\cup\mathcal{P}}| + \sum_{v \in X} d_F^-(v) \leq |R_X| + \sum_{v \in X} d_F^-(v) = m_2(X) + m_1(X) = m(X),
\end{aligned}
$$

so $m \in Q(\hat{p}, \infty_0)$. Thus, $m \in \mathbb{Z}^V \cap Q_4$.

To prove the **sufficiency**, now suppose that $\boldsymbol{m} \in \mathbb{Z}^V \cap Q_4$. By $m \in \mathbb{Z}^V \cap (Q_1 + Q_2)$ and Theorem 1.4, there exist $\boldsymbol{m_1} \in \mathbb{Z}^V \cap Q_1$ and $\boldsymbol{m_2} \in \mathbb{Z}^V \cap Q_2$ such that $m = m_1 + m_2$. Then, by $f, f' \geq 0$, we have $m_1, m_2 \geq 0$. Let the mixed hypergraph $\boldsymbol{\mathcal{F}_1}$ be obtained from $\mathcal{F}$ by adding a new vertex $\boldsymbol{s}$ and a new arc set $\boldsymbol{F_3} = F_1 \cup F_2$, where for $i = 1, 2$, $\boldsymbol{F_i}$ is a new arc set containing $m_i(v)$ arcs from s to every $v \in V$. Then $d_{F_1}^- = m_1, d_{F_2}^- = m_2$ and $d_{F_3}^- = m$.

We first show that $\mathcal{F}_1$ contains a packing of h spanning mixed s-hyperarborescences, each containing at least one arc of F_2, and their union containing F_3. By $d_{F_2}^- = m_2 \in K(\max\{h, \alpha\}, \beta)$, we have $|F_2| = m_2(V) \geq h$. Thus, there exists a subset $\boldsymbol{F_4}$ of F_2 of size h. By $m \in Q_3$, we have

$$
e_{F_3}(\mathcal{P}) = d_{F_3}^-(\cup\mathcal{P}) = m(\cup\mathcal{P}) \geq \hat{p}(\cup\mathcal{P}) \geq h|\mathcal{P}| - e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}) = |F_4||\mathcal{P}| - e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P})
$$

for every subpartition $\mathcal{P}$ of V, so (32) holds for $\mathcal{F}_1$ and F_4. Then, by Corollary 1, there exists a packing $\boldsymbol{\mathcal{B}^1}$ of h spanning mixed s-hyperarborescences in $\mathcal{F}_1$ each containing an arc of F_4. By $m \in T(0, h)$, we have $d_{F_3}^-(v) = m(v) \leq h$ for every $v \in V$. We may hence modify $\mathcal{B}^1$ to obtain a packing $\boldsymbol{\mathcal{B}^2}$ of h

spanning mixed s-hyperarborescences in $\mathcal{F}_1$ each containing an arc of F_4 and their union containing all the arcs in F_3.

We now modify each spanning mixed s-hyperarborescence B_i^2 in $\mathcal{B}^2$ by deleting s and adding an arc vu for every arc $su \in F_1$ contained in B_i^2 where sv is the unique arc of F_4 contained in B_i^2. Let $\boldsymbol{F}$ be the set of new arcs. This way we obtained an h-regular packing $\boldsymbol{\mathcal{B}^3}$ of mixed hyperarborescences in $\mathcal{F}+F$ that contains F. Note that the root set R^3 of the packing $\mathcal{B}^3$ satisfies $|R_v^3| = d_{F_2}^-(v) = m_2(v)$ for every $v \in V$. Then, since $m_2 \in T(f, g_h)$, $\mathcal{B}^3$ is (f, g)-bounded. By $m_2 \in K(\max\{h, \alpha\}, \beta)$, $\mathcal{B}^3$ is (α, β)-limited. Further, by $m_2 \in Q(-\infty_0, b_{\mathsf{M}})$ and Theorem 13.1.2 in [8], there exists an independent set $\boldsymbol{S^*}$ in M such that for every $v \in V$, $|S_v^*| = m_2(v) = |R_v^3|$ and hence the $m_2(v)$ mixed v-hyperarborescences in $\mathcal{B}^3$ can be rooted at S_v^*. It follows that the packing $\mathcal{B}^3$ is M-independent-rooted. Since $d_F^-(v) = d_{F_1}^-(v) = m_1(v)$ for every $v \in V$ and $m_1 \in T(f', g') \cap K(q, q')$, F is (f', g')-indegree-bounded and (q, q')-size-limited, so the proof of (a) is completed.

(b) Q_4 is well-defined and non-empty if and only if Q_1+Q_2 is well-defined and $(Q_1+Q_2)\cap Q_3 \neq \emptyset$. By Theorems 1.3 and 1.4, this is equivalent to $Q_1 \neq \emptyset$, $Q_2 \neq \emptyset$, $Q_3 \neq \emptyset$ and $(Q_1 + Q_2) \cap Q_3 \neq \emptyset$.

$\boldsymbol{Q_1 \neq \emptyset}$: By Theorem 1.1 and 1.3(a), $Q_1 = T(f', g') \cap K(q, q') \neq \emptyset$ if and only if $f' \leq g'$ (which is (35)), $q \leq q'$ (which is (36)), $g'(V) \geq q$ (which is implied by (37) applied for $Z = \emptyset$) and $f'(V) \leq q'$ (which is implied by (38) applied for $X = Z = \mathcal{P} = \emptyset$).

If $Q_1 \neq \emptyset$, then, by Theorem 1.3(b), $Q_1 = Q(p_1, b_1)$ where, for every $X \subseteq V$,

$$\boldsymbol{p_1}(X) = \max\{f'(X), q - g'(\overline{X})\}, \tag{47}$$

$$\boldsymbol{b_1}(X) = \min\{g'(X), q' - f'(\overline{X})\}. \tag{48}$$

$\boldsymbol{Q_2 \neq \emptyset}$: By Theorem 1.3(a), $Q' = Q(-\infty_0, b_{\mathsf{M}}) \cap K(\max\{h, \alpha\}, \beta) \neq \emptyset$ if and only if $\max\{h, \alpha\} \leq b_{\mathsf{M}}(V) = r_{\mathsf{M}}(S)$ (which is implied by (37) applied for $X = \emptyset = Z$), and $\max\{h, \alpha\} \leq \beta$ (which is implied by (38) applied for $\mathcal{P} = \{V\}$ and $Z = V$ and (19)).

If $Q' \neq \emptyset$, then by Theorem 1.3(b), $Q' = Q(p', b')$ where, for every $X \subseteq V$,

$$\begin{aligned} \boldsymbol{p'}(X) &= \max\{-\infty_0(X), \max\{h, \alpha\} - b_{\mathsf{M}}(\overline{X})\} \\ &= \max\{h, \alpha\} - b_{\mathsf{M}}(\overline{X}) \text{ if } X \neq \emptyset \text{ and } 0 \text{ if } X = \emptyset, \end{aligned} \tag{49}$$

$$\begin{aligned} \boldsymbol{b'}(X) &= \min\{b_{\mathsf{M}}(X), \beta + \infty_0(\overline{X})\} \\ &= b_{\mathsf{M}}(X) \text{ if } X \neq V \text{ and } \min\{b_{\mathsf{M}}(V), \beta\} \text{ if } X = V. \end{aligned} \tag{50}$$

By Theorem 1.2(a), $Q_2 = Q' \cap T(f, g_h) = Q(p', b') \cap T(f, g_h) \neq \emptyset$ if and only if $\max\{h, \alpha\} \leq b_{\mathsf{M}}(V) = r_{\mathsf{M}}(S)$, $f \leq g_h$ (which is (8)), $f(X) \leq b'(X)$ for every $X \subseteq V$ (which, by (50), is (38) applied for $Z = \emptyset = \mathcal{P}$) and $p'(X) \leq g_h(X)$ for every $X \subseteq V$ (which, by (49), is (37) applied for $Z = \emptyset$).

If $Q_2 \neq \emptyset$, then, by Theorem 1.2(b), $Q_2 = Q(p_2, b_2)$ where, for every $Z \subseteq V$,

$$\boldsymbol{p_2}(Z) = \max\{p'(X) - g_h(X - Z) + f(Z - X) : X \subseteq V\}, \tag{51}$$

$$\boldsymbol{b_2}(Z) = \min\{b'(X) - f(X - Z) + g_h(Z - X) : X \subseteq V\}. \tag{52}$$

By Theorem 1.4(a), $Q_1 + Q_2 = Q(p_+, b_+)$ where, for every $Z \subseteq V$,

$$\boldsymbol{p_+}(Z) = p_1(Z) + p_2(Z), \tag{53}$$

$$\boldsymbol{b_+}(Z) = b_1(Z) + b_2(Z). \tag{54}$$

$\boldsymbol{Q_3 \neq \emptyset}$: As the vector $(h, \ldots, h) \in Q(\hat{p}, \infty_0) \cap T(0, h) = Q_3$, $Q_3 \neq \emptyset$. Then, by Theorem 1.2(b) and the fact that $\hat{p}$ is non-decreasing, $Q_3 = Q(p_3, b_3)$ where, for every $Z \subseteq V$,

$$\boldsymbol{b_3}(Z) = h|Z|, \tag{55}$$

$$\boldsymbol{p_3}(Z) = \max\{\hat{p}(X) - h|X - Z| : X \subseteq V\}. \tag{56}$$

We now show that (56) can be simplified.

Claim 3. *We have*

$$p_3(Z) = \hat{p}(Z) \quad \textit{for every } Z \subseteq V. \tag{57}$$

Proof. For a given $Z \subseteq V$, by (56) and (40), let $X \subseteq V$ and $\mathcal{P}$ a subpartition of X such that

$$p_3(Z) = h|\mathcal{P}| - e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}) - h|X - Z|. \tag{58}$$

Let $\mathcal{P}'$ be the set of the elements of $\mathcal{P}$ contained in Z. Then, by (56), since $\mathcal{P}'$ is a subpartition of Z, by $e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}) \geq e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}')$, (58), and $|X - Z| \geq |\mathcal{P}| - |\mathcal{P}'|$, we have

$$\begin{aligned} p_3(Z) \;\geq\; \hat{p}(Z) \;&\geq\; h|\mathcal{P}'| - e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}') \\ &\geq\; h|\mathcal{P}| - e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}) - h|X - Z| + h(|X - Z| - (|\mathcal{P}| - |\mathcal{P}'|)) \\ &\geq\; p_3(Z). \end{aligned}$$

Thus, equality holds everywhere and therefore (57) holds. □

$\boldsymbol{(Q_1 + Q_2) \cap Q_3 \neq \emptyset}$: By Theorem 1.5(a), $Q_4 \neq \emptyset$ if and only if the above conditions hold and $p_3 \leq b_+$ (which, by (57), (54), (48), (52), is equivalent to (38)) and $p_+ \leq b_3$ (which, by (53), (47), (51), and (55), is equivalent to (37)).

By summarizing the above arguments we may conclude that (b) holds. □

Let $m \in \mathbb{Z}^V$. By Lemma 1(a), there exists an (f', g')-indegree-bounded (q, q')-size-limited new arc set F such that $\mathcal{F} + F$ has an (f, g)-bounded M-independent-rooted (α, β)-limited h-regular packing of mixed hyperarborescences containing F such that $m(v) = d_F^-(v) + |R_v|$ for every $v \in V$ if and only if $m \in Q_4$. By Theorem 1.5(b), this is equivalent to $Q_4 \neq \emptyset$. This holds, by Lemma 1(b), if and only if (8), (19), (35), (36), (37), and (38) hold. This completes the proof of Theorem 20. □

Remark 24. *If $\mathcal{F} = (V, A)$ is a digraph, S is a multiset of vertices in V, $f(v) = |S_{\{v\}}| = g(v), f'(v) = 0, g'(v) = \infty$ for every $v \in V$, $h = \alpha = \beta = k = |S|$, $q = 0, q' = \gamma$, M is the free matroid on S, then Theorem 20 reduces to Theorem 19(a).*

Proof. In this case an (f', g')-indegree-bounded (q, q')-size-limited arc set is an arc set of size at most γ and an h-regular M-independent-rooted (f, g)-bounded (α, β)-limited packing of mixed hyperarborescences is a packing of k spanning arborescences with roots $s_1, \ldots, s_k$, where $S = \{s_1, \ldots, s_k\}$, $\max\{f'(Z), q - g'(\overline{Z})\} = 0$ for every $Z \subseteq V$, $\max\{h, \alpha\} = k$, $r_{\mathsf{M}}(S_X) = |S_X|$ for every $X \subseteq V$, $f(Y) = |S_Y| = g(Y)$ for every $Y \subseteq V$, $\min\{g'(Z), q' - f'(\overline{Z})\} = \gamma$ for every non-empty $Z \subseteq V$ and 0 for $Z = \emptyset$, (35), (36), and (38) for $Z = \emptyset$ trivially hold. By $|S| = k = h$, we have

$$\begin{aligned} &\max\{f'(Z), q - g'(\overline{Z})\} + \max\{f(Z), \max\{h, \alpha\} - r_{\mathsf{M}}(S_{\overline{X}}) + f(Z - X) - g_h(X - Z)\} \\ =\; &0 + \max\{|S_Z|, k - |S_{\overline{X}}| + |S_{Z-X}| - |S_{X-Z}|\} = \max\{|S_Z|, |S_{X\cap Z}| + |S_{Z-X}|\} = |S_Z| \\ \leq\; &h|Z|, \end{aligned}$$

so (37) also holds. Finally, (38) for $\emptyset \neq Z \subseteq V$ becomes $\gamma + e_A(\mathcal{P}) + |S_Z| \geq k|\mathcal{P}|$ for every subpartition $\mathcal{P}$ of Z which is clearly equivalent to (33). □

We now present the augmentation version of Theorem 13 as an easy corollary of Theorem 20.

Corollary 2. *Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be a mixed hypergraph, $f, g : V \to \mathbb{Z}_+$ functions, $h, \alpha, \beta, \gamma \in \mathbb{Z}_+$, S a multiset of vertices in V, and $\mathsf{M} = (S, r_{\mathsf{M}})$ a matroid. We can add an arc set F of size at most γ to $\mathcal{F}$ to have an h-regular M-independent-rooted (f, g)-bounded (α, β)-limited packing of mixed* *hyperarborescences that contains F if and only if* (8), (19), *and* (20) *hold and for all $X, Z \subseteq V$ and subpartition $\mathcal{P}$ of Z,*

$$
\begin{aligned}
\beta &\geq f(V), & (59)\\
r_{\mathsf{M}}(S_X) &\geq f(X), & (60)\\
\gamma + e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}) + \min\{\beta - f(\overline{Z}), r_{\mathsf{M}}(S_X) - f(X-Z) + g_h(Z-X)\} &\geq h|\mathcal{P}|. & (61)
\end{aligned}
$$

Proof. We show that if $q = 0, q' = \gamma$, and $f'(v) = 0, g'(v) = \infty$ for every $v \in V$, then Theorem 20 reduces to Corollary 2. In this case an (f', g')-indegree-bounded (q, q')-size-limited arc set is an arc set of size at most γ, $\max\{f'(Z), q - g'(\overline{Z})\} = 0$ for every $Z \subseteq V$, $\min\{g'(Z), q' - f'(\overline{Z})\} = \gamma$ for every non-empty $Z \subseteq V$ and 0 for $Z = \emptyset$, (35) and (36) trivially hold. Further, by (8), we have $f(Z) \leq h|Z|$ for every $Z \subseteq V$. Hence (37) is equivalent to (20). (38) is equivalent to (61) for non-empty $Z \subseteq V$ and to (59) and (60) for $Z = \emptyset$. □

Remark 25. *If $\gamma = 0$, then Corollary 2 reduces to Theorem 13.*

Proof. In this case the arc set F to be added must be empty. Further, (61) applied for $Z = \emptyset = \mathcal{P}$ provides (59) and (60). Note that (17) and (21) are equivalent to (61). □

Remark 26. *If $\mathcal{F} = (V, A)$ is a digraph, $f(v) = 0, g(v) = k$ for every $v \in V$, $h = \alpha = \beta = k$, M is the free matroid on $k \times V$, then Corollary 2 reduces to Theorem 19(b).*

Proof. In this case an h-regular M-independent-rooted (f, g)-bounded (α, β)-limited packing of mixed hyperarborescences is a packing of k spanning arborescences, $r_{\mathsf{M}}(S_X) = k|X|$ for every $X \subseteq V$, (8), (19), (59), (60), and (61) for $Z = \emptyset$ trivially hold. Further, since

$$
\begin{aligned}
&\max\{h, \alpha\} - r_{\mathsf{M}}(S_{\overline{X}}) + f(Z-X) - g_h(X-Z)\\
= \;& k - k|\overline{X}| - k|X-Z| = k - k|V| + k|X \cap Z|\\
\leq \;& k|Z| \;=\; h|Z|,
\end{aligned}
$$

(20) also holds. (61) for $Z \neq \emptyset$ becomes $\gamma + e_A(\mathcal{P}) + \min\{k, k|X \cup Z|\} \geq k|\mathcal{P}|$ for every subpartition $\mathcal{P}$ of Z which is clearly equivalent to (34). □

5.2 Augmentation to have a packing of hyperbranchings

We now provide the second main result of the paper which is the augmentation version of Theorem 16.

Theorem 21. *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph, $h, k, \alpha, \beta, \gamma \in \mathbb{Z}_+$, and $\ell, \ell' : \mathbb{Z}_k \to \mathbb{Z}_+$ such that* (22) *and* (23) *hold. We can add at most γ arcs to $\mathcal{D}$ to have an h-regular (ℓ, ℓ')-bordered (α, β)-limited packing of k hyperbranchings if and only if* (26) *holds and*

$$
\begin{aligned}
k &\geq h, && (62)\\
\gamma + \beta - \ell(\mathbb{Z}_k) + \sum_{i=1}^{k} \min\{|\mathcal{P}|, \ell(i)\} + e_{\mathcal{A}}(\mathcal{P}) &\geq h|\mathcal{P}| && \textit{for every subpartition } \mathcal{P} \textit{ of } V, \quad (63)\\
\gamma + \sum_{i=1}^{k} \min\{|\mathcal{P}|, \ell'(i)\} + e_{\mathcal{A}}(\mathcal{P}) &\geq h|\mathcal{P}| && \textit{for every subpartition } \mathcal{P} \textit{ of } V. \quad (64)
\end{aligned}
$$

Proof. The **necessity** easily follows from Theorem 16 and the fact that $e_{\mathcal{A}\cup F}(\mathcal{P}) \le e_{\mathcal{A}}(\mathcal{P}) + \gamma$, where F is the new arc set added to $\mathcal{D}$.

To prove the **sufficiency**, suppose that (22), (23), (26), (62), (63), and (64) hold. We use Theorem 16 to show the existence of an intermediate packing that has more roots (and thus fewer hyperarcs) than the desired packing. We then tranform it by adding γ arcs to obtain the desired packing. To do so, we prove Lemma 2 which shows the existence of $\hat{\alpha}, \hat{\beta} \in \mathbb{Z}_+$, and $\hat{\ell}, \hat{\ell}' : \mathbb{Z}_k \to \mathbb{Z}_+$ for which the conditions of Theorem 16 are satisfied. In Lemma 2, the conditions (65) and (66) ensure that the new parameters are greater or equal to the old ones but not by more than γ, while the conditions (67)–(71) ensure that Theorem 16 can be applied.

Lemma 2. *Let $h, k, n, \alpha, \beta, \gamma \in \mathbb{Z}_+$, $\ell, \ell' : \mathbb{Z}_k \to \mathbb{Z}_+$, and $e : \{1, \ldots, n\} \to \mathbb{Z}_+$. There exist $\hat{\alpha}, \hat{\beta} \in \mathbb{Z}_+$, and $\hat{\ell}, \hat{\ell}' : \mathbb{Z}_k \to \mathbb{Z}_+$ such that*

$$\hat{\ell}(i) - \ell(i) \;\ge\; \hat{\ell}'(i) - \ell'(i) \;\ge\; 0 \qquad \text{for every } 1 \le i \le k, \tag{65}$$

$$\gamma \;\ge\; \hat{\ell}(\mathbb{Z}_k) - \ell(\mathbb{Z}_k) \;=\; \hat{\beta} - \beta \;=\; \hat{\alpha} - \alpha, \tag{66}$$

$$\hat{\ell}'(\mathbb{Z}_k) \;\ge\; \hat{\beta} \;\ge\; \hat{\alpha} \;\ge\; \hat{\ell}(\mathbb{Z}_k), \tag{67}$$

$$n \;\ge\; \hat{\ell}'(i) \;\ge\; \hat{\ell}(i) \qquad \text{for every } 1 \le i \le k, \tag{68}$$

$$hn \;\ge\; \hat{\alpha}, \tag{69}$$

$$\hat{\beta} + \sum_{i=1}^{k} \min\{p - \hat{\ell}(i), 0\} \;\ge\; hp - e(p) \qquad \text{for every } 1 \le p \le n, \tag{70}$$

$$\hat{\ell}'(\mathbb{Z}_k) + \sum_{i=1}^{k} \min\{p - \hat{\ell}'(i), 0\} \;\ge\; hp - e(p) \qquad \text{for every } 1 \le p \le n, \tag{71}$$

if and only if

$$\ell'(\mathbb{Z}_k) \;\ge\; \beta \;\ge\; \alpha \;\ge\; \ell(\mathbb{Z}_k), \tag{72}$$

$$n \;\ge\; \ell'(i) \;\ge\; \ell(i) \qquad \text{for every } 1 \le i \le k, \tag{73}$$

$$hn \;\ge\; \alpha, \tag{74}$$

$$\gamma + \beta + \sum_{i=1}^{k} \min\{p - \ell(i), 0\} \;\ge\; hp - e(p) \qquad \text{for every } 1 \le p \le n, \tag{75}$$

$$\gamma + \ell'(\mathbb{Z}_k) + \sum_{i=1}^{k} \min\{p - \ell'(i), 0\} \;\ge\; hp - e(p) \qquad \text{for every } 1 \le p \le n, \tag{76}$$

$$kp \;\ge\; hp - e(p) \qquad \text{for every } 1 \le p \le n. \tag{77}$$

Proof. We first show the **necessity**. Suppose that (65)–(71) hold for some $\hat{\alpha}, \hat{\beta}, \hat{\ell}$, and $\hat{\ell}'$. By summing up (65) and applying (66), we get that $\ell'(\mathbb{Z}_k) - \hat{\ell}'(\mathbb{Z}_k) \ge \beta - \hat{\beta} = \alpha - \hat{\alpha} = \ell(\mathbb{Z}_k) - \hat{\ell}(\mathbb{Z}_k)$. By adding this to (67), we obtain (72). By (68) and (65), we have, for every $1 \le i \le k$, $\ell(i) \le \hat{\ell}'(i) - \hat{\ell}(i) + \ell(i) \le \ell'(i) \le \hat{\ell}'(i) \le n$, so (73) holds. By applying (69), (66), and summing up (65), we get that $hn - \alpha \ge \hat{\alpha} - \alpha = \hat{\ell}(\mathbb{Z}_k) - \ell(\mathbb{Z}_k) \ge 0$, so (74) holds. By (66), (65), and (70), we have $\gamma + \beta + \sum_{i=1}^{k} \min\{p - \ell(i), 0\} \ge \hat{\beta} + \sum_{i=1}^{k} \min\{p - \hat{\ell}(i), 0\} \ge hp - e(p)$ for every $1 \le p \le n$, so (75) holds. By (66), (65), and (71), we have $\gamma + \ell'(\mathbb{Z}_k) + \sum_{i=1}^{k} \min\{p - \ell'(i), 0\} \ge \hat{\ell}'(\mathbb{Z}_k) + \sum_{i=1}^{k} \min\{p - \hat{\ell}'(i), 0\} \ge hp - e(p)$ for every $1 \le p \le n$, so (76) holds. By (71), we have $kp \ge \sum_{i=1}^{k} \min\{p, \hat{\ell}'(i)\} \ge hp - e(p)$, for every $1 \le p \le n$, so (77) holds.

To prove the **sufficiency**, suppose that (72)–(77) hold. Let $\hat{\boldsymbol{\gamma}}$ be the smallest non-negative integer such that there exist $\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}} \in \mathbb{Z}_+$, and $\hat{\boldsymbol{\ell}}, \hat{\boldsymbol{\ell}}' : \mathbb{Z}_k \to \mathbb{Z}_+$ satisfying (65), (67), (68), (69) and

$$\gamma - \hat{\gamma} \ \geq \ \hat{\ell}(\mathbb{Z}_k) - \ell(\mathbb{Z}_k) \ = \ \hat{\beta} - \beta \ = \ \hat{\alpha} - \alpha, \tag{78}$$

$$\hat{\gamma} + \hat{\beta} + \sum_{i=1}^{k} \min\{p - \hat{\ell}(i), 0\} \ \geq \ hp - e(p) \qquad \text{for every } 1 \leq p \leq n, \tag{79}$$

$$\hat{\gamma} + \hat{\ell}'(\mathbb{Z}_k) + \sum_{i=1}^{k} \min\{p - \hat{\ell}'(i), 0\} \ \geq \ hp - e(p) \qquad \text{for every } 1 \leq p \leq n. \tag{80}$$

Since (72)–(77) hold for $\alpha, \beta, \gamma, \ell$, and ℓ', it follows that $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\ell}$, and $\hat{\ell}'$ exist. We show that $\hat{\gamma} = 0$, and hence, by (78)–(80), it follows that $\hat{\alpha}, \hat{\beta}, \hat{\ell}$, and $\hat{\ell}'$ also satisfy (66), (70), (71) and we are done. Suppose that $\hat{\gamma} > 0$. By the minimality of $\hat{\gamma}$, there exists a minimum $\hat{\boldsymbol{p}} \in \{1, \ldots, n\}$ such that at least one of (79) and (80) is tight for $\hat{p}$. Let $\boldsymbol{X}$ be the set of indices where $\hat{p} - \hat{\ell}(i) \leq 0$ and $\boldsymbol{X'}$ the set of indices where $\hat{p} - \hat{\ell}'(i) \leq 0$. We first prove the following claim.

Claim 4. *The following hold.*

(a) $hn > \hat{\alpha}$.

(b) If (79) *is tight for* $\hat{p}$, *then* $X \neq \mathbb{Z}_k$.

(c) If $X' = \mathbb{Z}_k$, *then* (80) *is strict for every* $1 \leq p \leq n$ *and* $\hat{\ell}'(\mathbb{Z}_k) > \hat{\beta}$.

Proof. **(a)** If $\hat{\alpha} \geq hn$, then, by (67), we have

$$\hat{\ell}'(\mathbb{Z}_k) \ \geq \ \hat{\beta} \ \geq \ hn. \tag{81}$$

Case 1. If (79) is tight for $\hat{p}$, then, by (81), $e \geq 0$, and $\hat{\gamma} > 0$, we have

$$h\hat{p} - hn \geq h\hat{p} - \hat{\beta} > \sum_{i=1}^{k} \min\{\hat{p} - \hat{\ell}(i), 0\} = |X|\hat{p} - \hat{\ell}(X). \tag{82}$$

If $|X| \geq h$, then, by (82), $\hat{p}, \hat{\ell} \geq 0$, and (67), we have a contradiction: $-\hat{\beta} > (|X| - h)\hat{p} - \hat{\ell}(X) \geq -\hat{\ell}(\mathbb{Z}_k) \geq -\hat{\beta}$. If $|X| \leq h$, then, by (82), $n \geq \hat{p}$, and (73), we have a contradiction: $0 > (h - |X|)(n - \hat{p}) + |X|n - \hat{\ell}(X) \geq \sum_{i \in X}(n - \hat{\ell}(i)) \geq 0$.

Case 2. If (80) is tight for $\hat{p}$, then, by (81), $e \geq 0$, and $\hat{\gamma} > 0$, we have

$$h\hat{p} - hn \geq h\hat{p} - \hat{\ell}'(\mathbb{Z}_k) > \sum_{i=1}^{k} \min\{\hat{p} - \hat{\ell}'(i), 0\} = |X'|\hat{p} - \hat{\ell}'(X'). \tag{83}$$

If $|X'| \geq h$, then, by (83) and $\hat{p}, \hat{\ell}' \geq 0$, we have a contradiction: $h\hat{p} > \hat{\ell}'(\mathbb{Z}_k) + |X'|\hat{p} - \hat{\ell}'(X') \geq h\hat{p}$. If $|X'| \leq h$, then, by (83), $n \geq \hat{p}$, and (73), we have a contradiction: $0 > (h - |X'|)(n - \hat{p}) + |X'|n - \hat{\ell}'(X') \geq \sum_{i \in X'}(n - \hat{\ell}'(i)) \geq 0$.

(b) If $X = \mathbb{Z}_k$, then, by $\hat{\gamma} > 0$, (67), (79) is tight for $\hat{p}$, and (77), we have a contradiction: $k\hat{p} < \hat{\gamma} + \hat{\beta} - \hat{\ell}(\mathbb{Z}_k) + k\hat{p} = \hat{\gamma} + \hat{\beta} + \sum_{i=1}^{k} \min\{\hat{p} - \hat{\ell}(i), 0\} = h\hat{p} - e(\hat{p}) \leq k\hat{p}$.

(c) Suppose that $X' = \mathbb{Z}_k$. If $p < \hat{p}$, then, by the minimality of $\hat{p}$, (80) is strict for p.

If $p=\hat{p}$, then, by (77) and $X'=\mathbb{Z}_k$, we have $h\hat{p}-e(\hat{p})\leq k\hat{p}=\hat{\ell}'(\mathbb{Z}_k)+\sum_{i=1}^k \min\{\hat{p}-\hat{\ell}'(i),0\}$, so, by $\hat{\gamma}>0$, we get that (80) is strict for $\hat{p}$. Then, by the definition of $\hat{p}$, (79) is tight for $\hat{p}$. Hence, we have

$$\hat{\gamma}+\hat{\beta}+\sum_{i=1}^k \min\{\hat{p}-\hat{\ell}(i),0\}=h\hat{p}-e(\hat{p})<\hat{\gamma}+\hat{\ell}'(\mathbb{Z}_k)+\sum_{i=1}^k \min\{\hat{p}-\hat{\ell}'(i),0\}. \tag{84}$$

By (84) and (68), we get that $\hat{\beta}<\hat{\ell}'(\mathbb{Z}_k)$.

If $p\geq\hat{p}$, then, by (68) and Claim 8 of [13],

$$\sum_{i=1}^k \min\{p-\hat{\ell}(i),0\}-\sum_{i=1}^k \min\{\hat{p}-\hat{\ell}(i),0\}\leq\sum_{i=1}^k \min\{p-\hat{\ell}'(i),0\}-\sum_{i=1}^k \min\{\hat{p}-\hat{\ell}'(i),0\}. \tag{85}$$

By adding (84) and (85), we get $\hat{\gamma}+\hat{\beta}+\sum_{i=1}^k \min\{p-\hat{\ell}(i),0\}<\hat{\gamma}+\hat{\ell}'(\mathbb{Z}_k)+\sum_{i=1}^k \min\{p-\hat{\ell}'(i),0\}$. Thus, by (79), we get that (80) is strict for p.

This completes the proof of Claim 4. □

We now distinguish two cases and we give a contradiction in both cases.

Case I. Suppose that $X'\neq\mathbb{Z}_k$. Let $m\in\overline{X'}$. Then, by the definition of X' and (68), we have

$$n \quad\geq\quad \hat{p} \quad>\quad \hat{\ell}'(m) \quad\geq\quad \hat{\ell}(m). \tag{86}$$

We modify the values as follows. Let $\tilde{\gamma}=\hat{\gamma}-1$, $\tilde{\alpha}=\hat{\alpha}+1$, $\tilde{\beta}=\hat{\beta}+1$, $\tilde{\ell}(m)=\hat{\ell}(m)+1, \tilde{\ell}'(m)=\hat{\ell}'(m)+1$ and $\tilde{\ell}(i)=\hat{\ell}(i), \tilde{\ell}'(i)=\hat{\ell}'(i)$ for $i\in\mathbb{Z}_k-\{m\}$. We show that (65), (67), (68), (69), (78), (79), and (80) hold for $\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\ell}$, and $\tilde{\ell}'$ that contradicts the minimality of $\hat{\gamma}$.

- (i) By (65), we have $\tilde{\ell}(i)-\ell(i)\geq\tilde{\ell}'(i)-\ell'(i)\geq 0$ for every $1\leq i\leq k$, so (65) holds for $\tilde{\ell}$ and $\tilde{\ell}'$.
- (ii) By (67), we have $\tilde{\ell}'(\mathbb{Z}_k)\geq\tilde{\beta}\geq\tilde{\alpha}\geq\tilde{\ell}(\mathbb{Z}_k)$, so (67) holds for $\tilde{\alpha},\tilde{\beta},\tilde{\ell}$ and $\tilde{\ell}'$.
- (iii) By (68) and (86), we have $n\geq\tilde{\ell}'(i)\geq\tilde{\ell}(i)$ for every $1\leq i\leq k$, so (68) holds for $\tilde{\ell}$ and $\tilde{\ell}'$.
- (iv) By Claim 4(a), (69) holds for $\tilde{\alpha}$.
- (v) By (78), we have $\gamma-\tilde{\gamma}\geq\tilde{\ell}(\mathbb{Z}_k)-\ell(\mathbb{Z}_k)=\tilde{\beta}-\beta=\tilde{\alpha}-\alpha$, so (78) holds for $\tilde{\alpha},\tilde{\beta},\tilde{\ell}$, and $\tilde{\gamma}$.
- (vi) Note that, by (86), we have $\sum_{i=1}^k \min\{p-\tilde{\ell}(i),0\}=\sum_{i=1}^k \min\{p-\hat{\ell}(i),0\}$ for every $\hat{p}\leq p\leq n$ and $\sum_{i=1}^k \min\{p-\tilde{\ell}(i),0\}\geq\sum_{i=1}^k \min\{p-\hat{\ell}(i),0\}-1$ for every $1\leq p<\hat{p}$. Hence, by $\tilde{\gamma}+\tilde{\beta}=\hat{\gamma}+\hat{\beta}$, and (79) (which is strict for $p<\hat{p}$), we have $\tilde{\gamma}+\tilde{\beta}+\sum_{i=1}^k \min\{p-\tilde{\ell}(i),0\}\geq hp-e(p)$ for every $1\leq p\leq n$, so (79) holds for $\tilde{\beta},\tilde{\ell}$, and $\tilde{\gamma}$.
- (vii) Finally, by (86), we have $\sum_{i=1}^k \min\{p-\tilde{\ell}'(i),0\}=\sum_{i=1}^k \min\{p-\hat{\ell}'(i),0\}$ for every $\hat{p}\leq p\leq n$ and $\sum_{i=1}^k \min\{p-\tilde{\ell}'(i),0\}\geq\sum_{i=1}^k \min\{p-\hat{\ell}'(i),0\}-1$ for every $1\leq p<\hat{p}$. Hence, by $\tilde{\gamma}+\tilde{\ell}'(\mathbb{Z}_k)=\hat{\gamma}+\hat{\ell}'(\mathbb{Z}_k)$, and (80) (which is strict for $p<\hat{p}$), we have $\tilde{\gamma}+\tilde{\ell}'(\mathbb{Z}_k)+\sum_{i=1}^k \min\{p-\tilde{\ell}'(i),0\}\geq hp-e(p)$ for every $1\leq p\leq n$, so (79) holds for $\tilde{\ell}'$, and $\tilde{\gamma}$.

Case II. Suppose that $X'=\mathbb{Z}_k$. By Claim 4(c), (80) is strict for every $1\leq p\leq n$. Then, (79) is tight for $\hat{p}$, so, by Claim 4(b), we have $X\neq\mathbb{Z}_k$. Let $m\in\overline{X}$. Then, by (68), we have

$$n \quad\geq\quad \hat{\ell}'(m) \quad\geq\quad \hat{p} \quad>\quad \hat{\ell}(m). \tag{87}$$

We modify the values again. Let $\tilde{\gamma} = \hat{\gamma} - 1$, $\tilde{\alpha} = \hat{\alpha} + 1$, $\tilde{\beta} = \hat{\beta} + 1$, $\tilde{\ell}(m) = \hat{\ell}(m) + 1$, $\tilde{\ell}(i) = \hat{\ell}(i)$ for $i \in \mathbb{Z}_k - \{m\}$, and $\tilde{\ell}'(i) = \hat{\ell}'(i)$ for $i \in \mathbb{Z}_k$. We show similarly as in Case I that (65), (67), (68), (69), (78), (79), and (80) hold for $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\ell}$, and $\tilde{\ell}'$ that contradicts again the minimality of $\hat{\gamma}$. More precisely, $(i), (iv), (v)$, and (vi) are the same in Case II. (ii) is similar to (ii) in Case I using that, by Claim 4(c), we have $\hat{\ell}'(\mathbb{Z}_k) > \hat{\beta}$. (iii) is similar to (iii) in Case I using (87) instead of (86). Finally, (vii) for every $1 \leq p \leq n$, by $\tilde{\gamma} + \tilde{\ell}'(\mathbb{Z}_k) = \hat{\gamma} + \hat{\ell}'(\mathbb{Z}_k) - 1$, $\sum_{i=1}^k \min\{p - \tilde{\ell}'(i), 0\} = \sum_{i=1}^k \min\{p - \hat{\ell}'(i), 0\}$ and (80) is strict, we have $\tilde{\gamma} + \tilde{\ell}'(\mathbb{Z}_k) + \sum_{i=1}^k \min\{p - \tilde{\ell}'(i), 0\} \geq hp - e(p)$, so (79) holds for $\tilde{\ell}'$, and $\tilde{\gamma}$.

This completes the proof of Lemma 2. □

We now are in a position to prove Theorem 21. Let $n = |V|$. By (22), (23), and (26), we have that (72), (73) and (74) hold. Let $e(p) = \min\{e_{\mathcal{A}}(\mathcal{P}) : \mathcal{P}$ subpartition of V such that $|\mathcal{P}| = p\}$ for every $1 \leq p \leq n$. Note that $e(p) \geq 0$ for every $1 \leq p \leq n$. Then, by (62), (63), and (64), we obtain (77), (75) and (76). Hence, by Lemma 2, there exist $\hat{\alpha}, \hat{\beta} \in \mathbb{Z}_+$ and $\hat{\ell}, \hat{\ell}' : \mathbb{Z}_k \to \mathbb{Z}_+$ such that (65)–(71) hold. Then (22), (23) (26), (27), and (28) hold for $\hat{\alpha}, \hat{\beta}, \hat{\ell}$ and $\hat{\ell}'$. Hence, by Theorem 16, there exists an h-regular $(\hat{\ell}, \hat{\ell}')$-bordered $(\hat{\alpha}, \hat{\beta})$-limited packing of hyperbranchings $\hat{\mathcal{B}}_i$ with root sets $\hat{S}_1, \dots, \hat{S}_k$ in $\mathcal{D}$. Let $i \in \mathbb{Z}_k$. By (65), $\hat{\ell}(i) \leq |\hat{S}_i| \leq \hat{\ell}'(i)$, and $\ell \geq 0$, there exists $S_i \subseteq \hat{S}_i$ with $|S_i| = |\hat{S}_i| - \hat{\ell}(i) + \ell(i)$. Further, $\ell(i) \leq |S_i| \leq \ell'(i)$. Let $\mathcal{B}_i$ be obtained from $\hat{\mathcal{B}}_i$ by adding a set A_i of new arcs consisting of an arc from any vertex $s_i \in S_i$ to every vertex in $\hat{S}_i - S_i$. Then $\mathcal{B}_i$ is an S_i-hyperbranching. Let $\mathcal{D}'$ be obtained from $\mathcal{D}$ by adding $\bigcup_{i=1}^k A_i$. Since $\sum_{i=1}^k (|\hat{S}_i| - |S_i|) = \sum_{i=1}^k (\hat{\ell}(i) - \ell(i))$ and $\hat{\alpha} \leq \sum_{i=1}^k |\hat{S}_i| \leq \hat{\beta}$, (66) implies that $\alpha \leq \sum_{i=1}^k |S_i| \leq \beta$. By (66), the number of new arcs is $|\bigcup_{i=1}^k A_i| = \sum_{i=1}^k (\hat{\ell}(i) - \ell(i)) \leq \gamma$. Hence, $\mathcal{B}_1, \dots, \mathcal{B}_k$ is the desired h-regular (ℓ, ℓ')-bordered (α, β)-limited packing of hyperbranchings in $\mathcal{D}'$ that completes the proof of Theorem 21. □

Remark 27. *If $\gamma = 0$, then Theorem 21 reduces to Theorem 16.*

5.3 Augmentation to have a packing of rooted hyperforests

In Subsection 5.2, we have seen that Theorem 21 can be proved by Theorem 16 and Lemma 2. We finish the paper by another application of Lemma 2 to get the augmentation version of Theorem 17 which is the following undirected version of Theorem 21. It can be proved similarly as Theorem 17, by applying Theorem 17 and Lemma 2.

Theorem 22. *Let $\mathcal{G} = (V, \mathcal{E})$ be a hypergraph, $h, k, \alpha, \beta, \gamma \in \mathbb{Z}_+$ and $\ell, \ell' : \mathbb{Z}_k \to \mathbb{Z}_+$ such that* (22) *and* (23) *hold. We can add at most γ edges to $\mathcal{G}$ to have an h-regular (ℓ, ℓ')-bordered (α, β)-limited packing of k rooted hyperforests if and only if* (26) *and* (62) *hold and*

$$\gamma + \beta - \ell(\mathbb{Z}_k) + \sum_{i=1}^k \min\{|\mathcal{P}|, \ell(i)\} + e_{\mathcal{E}}(\mathcal{P}) \geq h|\mathcal{P}| \quad \textit{for every partition } \mathcal{P} \textit{ of } V,$$

$$\gamma + \sum_{i=1}^k \min\{|\mathcal{P}|, \ell'(i)\} + e_{\mathcal{E}}(\mathcal{P}) \geq h|\mathcal{P}| \quad \textit{for every partition } \mathcal{P} \textit{ of } V.$$

Remark 28. *If $\gamma = 0$, then Theorem 22 reduces to Theorem 17.*

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