Polymatroid-Based Capacitated Packing of Branchings

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July 12, 2017

Abstract

Edmonds (1973) characterized the condition for the existence of a packing of spanning arborescences and also that of spanning branchings in a directed graph and gave polynomial-time algorithms to find such packings if they exist. Durand de Gevigney, Nguyen and Szigeti (2013) generalized the first problem and solved the matroid-based arborescence packing problem.

In this paper, a generalization of this latter problem - the polymatroid-based arborescence packing problem - is considered. Two problem settings are formulated: the unsplittable version and the splittable version. The unsplittable version is shown to be strongly NP-complete. Whereas, the splittable version, which generalizes the capacitated version of the spanning arborescence packing problem, can be solved in strongly polynomial time. Actually, a strongly polynomial time algorithm for the problem of the polymatroid-based capacitated packing of branchings is provided for this version.

1 Introduction

Let $D = (V, A)$ be a directed graph. For $X \subseteq V$, let

$$\Delta_D^-(X) := \{uv \in A \mid u \in V \setminus X, v \in X\},$$

$\Delta_D^+(X) := \Delta_D^-(V \setminus X)$ and $\rho_D(X) := |\Delta_D^-(X)|$ the in-degree of X in D. We omit D and write $\rho(X)$ when the directed graph considered is clear from the context. We write a singleton $\{s\}$ simply as s. For $s \in V' \subseteq V$ and $A' \subseteq A$, the subgraph $D' = (V', A')$ of D is called an *s-arborescence* if $\rho_{D'}(s) = 0, \rho_{D'}(v) = 1$ for all $v \in V' \setminus s$ and D' is acyclic. The distinguished vertex s is called the *root* of the arborescence. The subgraph D' of D is said to be *spanning* if $V' = V$. A set of arc-disjoint arborescences is called a *packing* of arborescences.

Let us start with the following well-known problem.

Problem 1. *Given a directed graph $D = (V, A)$, $s \in V$ and $k \in \mathbb{Z}_{>0}$, find a packing of k spanning s-arborescences in D.*

Problem 1 has been solved by Edmonds [2] as follows.

Theorem 1 (Edmonds [2])**.** *Problem 1 has a solution if and only if $\rho(X) \geq k$ for all nonempty $X \subseteq V \setminus s$. Problem 1 can be solved in polynomial time.*

Actually, the following more general theorem on packing branchings with possibly distinct root-sets was also shown by Edmonds [2]. For $R \subseteq V' \subseteq V$ and $A' \subseteq A$, the subgraph $B = (V', A')$ of D is called an *R-branching* if it satisfies $\rho_B(v) = 0$ for all $v \in R$, $\rho_B(v) = 1$ for all $v \in V' \setminus R$ and B is acyclic.

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Theorem 2 (Edmonds [2]). *Let $D = (V, A)$ be a directed graph and $R_1, \dots, R_k \subseteq V$. There exist arc-disjoint spanning R_i-branchings ($i = 1, \dots, k$) if and only if $\rho(X) \geq |\{i \mid R_i \cap X = \emptyset\}|$ for all nonempty $X \subseteq V$.*

Note that Theorem 2 is a generalization of Theorem 1: this can be easily shown by setting $R_i = \{s\}$ for $i = 1, \dots, k$.

Problem 1 was also generalized to the capacitated problem.

Problem 2. *Given a directed graph $D = (V, A)$, a capacity function $c : A \to \mathbb{Z}_{\geq 0}$, a set $S \subseteq V$ of roots and a demand function $d : S \to \mathbb{Z}_{>0}$, find spanning s-arborescences $\{T^{s,j}\}$ and positive integers $\{\lambda^{s,j}\}$ for all $s \in S$ such that $\sum \lambda^{s,j} = d(s)$ for all $s \in S$ and $\sum\{\lambda^{s,j} \mid a \in A(T^{s,j})\} \leq c(a)$ for all $a \in A$.*

The following result is known for Problem 2. For a function $x : Q \to \mathbb{Z}_{\geq 0}$ and $Q' \subseteq Q$, we use the usual notation: $x(Q') = \sum_{q \in Q'} x(q)$.

Theorem 3 (cf. [11]). *Problem 2 has a solution if and only if $c(\Delta^-(X)) \geq d(S \setminus X)$ for all nonempty $X \subseteq V$. Problem 2 can be solved in strongly polynomial time.*

Edmonds' arborescence packing theorem was generalized in many other directions as well. In this paper, we focus on the direction when a matroid constraint is added to Problem 1, introduced by Durand de Gevigney, Nguyen and Szigeti [1].

Definition 1. *A* matroid $\mathcal{M} = (\mathsf{S}, r)$ *is a pair of a finite set* S *(called* ground set*) and a function* $r : 2^{\mathsf{S}} \to \mathbb{Z}_{\geq 0}$ *(called* rank function*) which satisfies the following four conditions:*

(r1)	$r(\emptyset) = 0$,		
(r2)	$r(\mathsf{S}_1) \leq \lvert\mathsf{S}_1\rvert$	*for all* $\mathsf{S}_1 \subseteq \mathsf{S}$	(subcardinality),
(r3)	$r(\mathsf{S}_2) \leq r(\mathsf{S}_1)$	*for all* $\mathsf{S}_2 \subseteq \mathsf{S}_1 \subseteq \mathsf{S}$	(monotonicity),
(r4)	$r(\mathsf{S}_1) + r(\mathsf{S}_2) \geq r(\mathsf{S}_1 \cup \mathsf{S}_2) + r(\mathsf{S}_1 \cap \mathsf{S}_2)$	*for all* $\mathsf{S}_1, \mathsf{S}_2 \subseteq \mathsf{S}$	(submodularity).

The family of independent sets $\mathcal{I}$ *of* $\mathcal{M}$ *is defined by*

$$\mathcal{I}(\mathcal{M}) := \{\mathsf{S}' \subseteq \mathsf{S} \mid r(\mathsf{S}') = |\mathsf{S}'|\}.$$

Each element of $\mathcal{I}$ *is called an* independent set *and each maximal independent set is called a* base.

Problem 3 (Matroid-Based Packing of Arborescences [1]). *Given a directed graph $D = (V, A)$, a matroid $\mathcal{M} = (\mathsf{S}, r)$ and a map $\pi : \mathsf{S} \to V$, find a packing of $\{\pi(\mathsf{s})$-arborescences $T^{\mathsf{s}} \mid \mathsf{s} \in \mathsf{S}\}$ such that $\{\mathsf{s} \in \mathsf{S} \mid v \in V(T^{\mathsf{s}})\}$ is a base of $\mathcal{M}$ for all $v \in V$.*

Note that Problem 1 can be reduced to Problem 3 by setting $|\mathsf{S}| = k, r(\mathsf{S}') = |\mathsf{S}'|$ for all $\mathsf{S}' \subseteq \mathsf{S}$ (this matroid is called a *free matroid*) and $\pi(\mathsf{s}) = s$ for all $\mathsf{s} \in \mathsf{S}$. For Problem 3, the following theorem was shown. For $\pi : \mathsf{S} \to V$ and $X \subseteq V$,

$$\pi^{-1}(X) := \{\mathsf{s} \in \mathsf{S} \mid \pi(\mathsf{s}) \in X\}.$$

Theorem 4 (Durand de Gevigney–Nguyen–Szigeti [1]). *Problem 3 has a solution if and only if*

$$\pi^{-1}(v) \text{ is independent in } \mathcal{M} \text{ for all } v \in V, \tag{1}$$

$$\rho(X) \geq r(\mathsf{S}) - r(\pi^{-1}(X)) \text{ for all nonempty } X \subseteq V. \tag{2}$$

Problem 3 can be solved in strongly polynomial time.

Definition 2. *If a function $r : 2^{\mathsf{S}} \to \mathbb{R}_{\geq 0}$ satisfies conditions (r1), (r3) and (r4), then $\mathcal{P} := (\mathsf{S}, r)$ is called a* polymatroid. *If the rank function is integral, then $\mathcal{P}$ is called an integral polymatroid. For $\mathsf{S}' \subseteq \mathsf{S}$, let*

$$\mathsf{Span}_r(\mathsf{S}') := \{\mathsf{s} \in \mathsf{S} \mid r(\mathsf{S}' \cup \mathsf{s}) = r(\mathsf{S}')\}.$$

It is well-known that Span_r is monotone, that is,

$$\mathsf{Span}_r(\mathsf{S}_2) \subseteq \mathsf{Span}_r(\mathsf{S}_1) \quad \text{for all } \mathsf{S}_2 \subseteq \mathsf{S}_1 \subseteq \mathsf{S}. \tag{3}$$

A function $r : 2^V \to \mathbb{R} \cup \{+\infty\}$ is *submodular* if it satisfies the submodularity condition (r4).

Theorem 5 ([7, 10]). *A submodular function can be minimized in strongly polynomial time in the size of the underlying set.*

The rest of the paper is organized as follows. We explain the problem settings in Section 2, where we introduce the unsplittable version and the splittable version. In Section 3, we show that the unsplittable version is strongly NP-complete. For the splittable version, a pseudo-polynomial time algorithm is given first and then a strongly polynomial time algorithm is given in Section 4. Section 5 concludes this paper.

2 Problem setting

In this section, we introduce two problem settings by replacing in Problem 3 the matroid constraint by a more general polymatroid constraint.

Definition 3. *For a polymatroid $\mathcal{P} = (\mathsf{S}, r)$,*

$$\mathrm{P}(r) := \left\{ x \in \mathbb{R}^{\mathsf{S}}_{\geq 0} \mid x(\mathsf{S}') \leq r(\mathsf{S}') \ \forall \mathsf{S}' \subseteq \mathsf{S} \right\}$$

is called an independent polyhedron *of $\mathcal{P}$ and*

$$\mathrm{B}(r) := \left\{ x \in \mathbb{R}^{\mathsf{S}}_{\geq 0} \mid x \in \mathrm{P}(r), x(\mathsf{S}) = r(\mathsf{S}) \right\}$$

is called a base polyhedron *of $\mathcal{P}$.*

Problem 4 (Unsplittable Version). *Given a directed graph $D = (V, A)$, a capacity function $c : A \to \mathbb{Z}_{\geq 0}$, an integral polymatroid $\mathcal{P} = (\mathsf{S}, r)$, a demand function $d : \mathsf{S} \to \mathbb{Z}_{\geq 0}$ and a map $\pi : \mathsf{S} \to V$, find a $\pi(\mathsf{s})$-arborescence T^{s} for all $\mathsf{s} \in \mathsf{S}$ such that*

(u1) $\sum\{d(\mathsf{s}) \mid a \in A(T^{\mathsf{s}})\} \leq c(a)$ *for all $a \in A$ and*

(u2) $\boldsymbol{t}^v \in \mathrm{B}(r)$ *for all $v \in V$, where $\boldsymbol{t}^v \in \mathbb{Z}^{|\mathsf{S}|}_{\geq 0}$ is defined by* $\boldsymbol{t}^v(\mathsf{s}) := \begin{cases} d(\mathsf{s}) & v \in V(T^{\mathsf{s}}) \\ 0 & v \notin V(T^{\mathsf{s}}) \end{cases}$ *for all $\mathsf{s} \in \mathsf{S}$.*

Problem 5 (Splittable Version). *Given a directed graph $D = (V, A)$, a capacity function $c : A \to \mathbb{Z}_{\geq 0}$, an integral polymatroid $\mathcal{P} = (\mathsf{S}, r)$, a demand function $d : \mathsf{S} \to \mathbb{Z}_{\geq 0}$ and a map $\pi : \mathsf{S} \to V$, find $\pi(\mathsf{s})$-arborescences $\{T^{\mathsf{s},j}\}$ and positive integers $\{\lambda^{\mathsf{s},j}\}$ for all $\mathsf{s} \in \mathsf{S}$ such that*

(s1) $\sum \lambda^{\mathsf{s},j} = d(\mathsf{s})$ *for all $\mathsf{s} \in \mathsf{S}$,*

(s2) $\sum\{\lambda^{\mathsf{s},j} \mid a \in A(T^{\mathsf{s},j})\} \leq c(a)$ *for all $a \in A$,*

(s3) $\boldsymbol{\lambda}^v \in \mathrm{B}(r)$ *for all $v \in V$, where $\boldsymbol{\lambda}^v \in \mathbb{Z}^{|\mathsf{S}|}_{\geq 0}$ is defined by $\boldsymbol{\lambda}^v(\mathsf{s}) := \sum\{\lambda^{\mathsf{s},j} \mid v \in V(T^{\mathsf{s},j})\}$ for all $\mathsf{s} \in \mathsf{S}$.*

The relation of Problem 5 to the previous problems is shown in Figure 1.

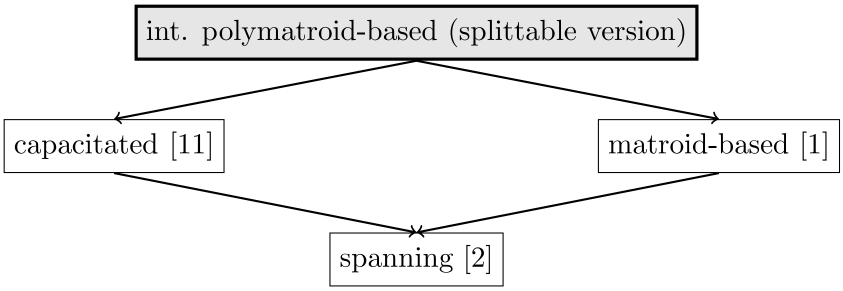

Figure 1: Relation between problems. The splittable version (Problem 5) is a common generalization of the capacitated problem (Problem 2) and the matroid-based problem (Problem 3).

Problem 3 can be reduced to Problem 4 by setting $c(a) = 1$ for all $a \in A$, $\mathcal{P} = \mathcal{M}$ and $d(\mathsf{s}) = 1$ for all $\mathsf{s} \in \mathsf{S}$. Problem 3 can also be reduced to Problem 5 by a similar setting. The difference between Problems 4 and 5 is that one must pack one $\pi(\mathsf{s})$-arborescence with width $d(\mathsf{s})$ for each $\mathsf{s} \in \mathsf{S}$ in Problem 4 and $d(\mathsf{s})$ (not necessarily distinct) $\pi(\mathsf{s})$-arborescences with width 1 for each $\mathsf{s} \in \mathsf{S}$ in Problem 5.

3 Unsplittable version

In this section, we show that the problem of unsplittable version is strongly NP-complete. This can be shown by the complexity of the following special case.

Problem 6 (Packing of Spanning Arborescences with Width)**.** *Given a directed graph $D = (V, A)$, a capacity function $c : A \to \mathbb{Z}_{\geq 0}$, a root $s \in V$ and demands $d_1, \ldots, d_k \in \mathbb{Z}_{>0}$, find k spanning s-arborescences $\{T_i\}$ $(i = 1, \ldots, k)$ such that $\sum\{d_i \mid a \in A(T_i)\} \leq c(a)$ for all $a \in A$.*

Problem 6 can be reduced to Problem 4 by setting $\mathsf{S} = \{\mathsf{s}^1, \ldots, \mathsf{s}^k\}$, $d(\mathsf{s}^i) = d_i$ $(i = 1, \ldots, k)$, $r(\mathsf{S}') = d(\mathsf{S}')$ for all $\mathsf{S}' \subseteq \mathsf{S}$, and $\pi(\mathsf{s}^i) = s$ for all $\mathsf{s}^i \in \mathsf{S}$. To prove the strong NP-completeness of Problem 6 we use 3-Partition.

Problem 7 (3-Partition)**.** *Given $3n$ positive integers $q_1, q_2, \ldots, q_{3n}$ satisfying $\frac{1}{4n}\sum_{i=1}^{3n} q_i < q_j < \frac{1}{2n}\sum_{i=1}^{3n} q_i$ $(j = 1, \ldots, 3n)$, find a partition of these integers into n components of 3 elements such that the sum of each component is the same value.*

It is known [5] that Problem 7 is strongly NP-complete. Problem 7 can be reduced to Problem 6 as follows. Let $q_1, \ldots, q_{3n}$ with $\frac{1}{4n}\sum_{i=1}^{3n} q_i < q_j < \frac{1}{2n}\sum_{i=1}^{3n} q_i$ $(j = 1, \ldots, 3n)$ be an instance of 3-Partition. Let $D = (V, A)$ be a directed graph with $V = \{v_1, \ldots, v_n, t\}$ and $A = \{v_{i+1}v_i \mid 1 \leq i \leq n-1\} \bigcup \{v_i t \mid 1 \leq i \leq n\}$, $c(a) = \sum_{i=1}^{3n} q_i$ for all $a \in \{v_{i+1}v_i \mid 1 \leq i \leq n-1\}$ and $c(a) = \frac{1}{n}\sum_{i=1}^{3n} q_i$ for all $a \in \{v_i t \mid 1 \leq i \leq n\}$, $s = v_n$, $k = 3n$ and $d_i = q_i$ for $i = 1, 2, \ldots, k$ (see Figure 2).

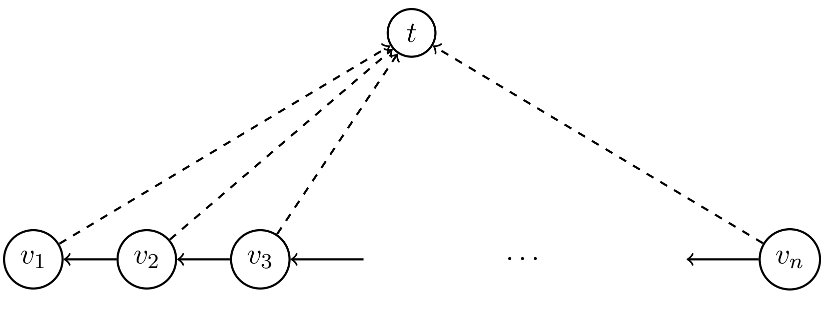

Figure 2: The directed graph D in the reduction of Problem 7 to Problem 6. The arcs of capacity $\sum_{i=1}^{3n} d_i$ are represented by thick lines and those of capacity $\frac{1}{n}\sum_{i=1}^{3n} d_i$ by dashed lines.

Note that T is a spanning v_n-arborescence of D if and only if $A(T)$ is the union of $\{v_{i+1}v_i \mid 1 \leq i \leq n-1\}$ and just one element of $\{v_i t \mid 1 \leq i \leq n\}$. This element of $\{v_i t \mid 1 \leq i \leq n\}$ corresponds to a component of the partition. Therefore Problem 7 has a solution $\{Q_1,\dots,Q_n\}$ (a partition of $\{q_1,\dots,q_{3n}\}$) if and only if Problem 6 arising from the reduction has a solution $\{T_1,\dots,T_k\}$ satisfying $\{j \mid v_i t \in A(T_j)\} = \{l \mid q_l \in Q_i\}$ for $i = 1,\dots,n$. Thus, if one has a solution $\{T_1,\dots,T_{3n}\}$ for the corresponding Problem 6, one can obtain a solution $\{Q_1,\dots,Q_n\}$ for Problem 7 in polynomial time. Since Problem 7 is strongly NP-complete, the following theorem holds.

Theorem 6. *Problem 6 is strongly NP-complete.*

The following corollary on Problem 4 can be obtained immediately by Theorem 6.

Corollary 1. *Problem 4 is strongly NP-complete.*

4 Splittable version

In this section, we first show that there is a pseudo-polynomial time algorithm for the problem of the splittable version. After that, we show that actually this problem can be solved in strongly polynomial time. Our algorithm uses ideas from the proof of Theorem 53.9 in [11] and those from the proof of Claim 2.1 and Theorem 1.6 in [1].

4.1 Pseudo-polynomial time algorithm

To make a pseudo-polynomial time algorithm for Problem 5, we deal first with the special case when $d(\mathsf{s}) = r(\mathsf{s})$ for $\mathsf{s} \in \mathsf{S}$. For this problem, the following theorem holds.

Theorem 7. *Problem 5 with the input $(D = (V, A), c, \mathcal{P} = (\mathsf{S}, r), d, \pi)$ satisfying $d(\mathsf{s}) = r(\mathsf{s})$ for all $\mathsf{s} \in \mathsf{S}$ has a solution if and only if*

$$\boldsymbol{S}^v \in \mathrm{P}(r) \qquad \textit{for all } v \in V, \tag{4}$$

$$c(\Delta^-(X)) \geq r(\mathsf{S}) - r(\pi^{-1}(X)) \qquad \textit{for all nonempty } X \subseteq V, \tag{5}$$

where $\boldsymbol{S}^v \in \mathbb{Z}_{\geq 0}^{|\mathsf{S}|}$ is defined by $\boldsymbol{S}^v(\mathsf{s}) := \begin{cases} r(\mathsf{s}) & (\pi(\mathsf{s}) = v), \\ 0 & (\pi(\mathsf{s}) \neq v) \end{cases}$ *for all $\mathsf{s} \in \mathsf{S}$.*

Proof. Here we prove only the sufficiency, the proof for the necessity will be given for Theorem 9 that handles the general case. We define an instance $(\tilde{D}, \tilde{\mathcal{M}} = (\tilde{\mathsf{S}}, \tilde{r}), \tilde{\pi})$ of Problem 3 as follows:

$$\tilde{D} := (V, \{a^1, \dots, a^{c(a)} (c(a) \text{ copies of } a) \mid a \in A\}), \tag{6}$$

$$\tilde{\mathsf{S}} := \bigcup_{\mathsf{s}^i \in \mathsf{S}} \tilde{\mathsf{S}}_i, \text{ where } \tilde{\mathsf{S}}_i = \{\tilde{\mathsf{s}}^i_1, \dots, \tilde{\mathsf{s}}^i_{r(\mathsf{s}^i)}\} \text{ are disjoint sets} \quad (\mathsf{s}^i \in \mathsf{S}), \tag{7}$$

$$\tilde{r}(\tilde{\mathsf{S}}') := \min_{\mathsf{S}' \subseteq \mathsf{S}} \left\{ \left| \tilde{\mathsf{S}}' \setminus \bigcup_{\mathsf{s}^i \in \mathsf{S}'} \tilde{\mathsf{S}}_i \right| + r(\mathsf{S}') \right\} \quad (\tilde{\mathsf{S}}' \subseteq \tilde{\mathsf{S}}), \tag{8}$$

$$\tilde{\pi}(\tilde{\mathsf{s}}^i_j) := \pi(\mathsf{s}^i) \quad (\tilde{\mathsf{s}}^i_j \in \tilde{\mathsf{S}}_i, \mathsf{s}^i \in \mathsf{S}). \tag{9}$$

It is known that this pair $(\tilde{\mathsf{S}}, \tilde{r})$ actually becomes a matroid (cf. Theorem 5.5.14 in [3], or [6], or Section 44.6b in [11]). By Theorem 4, Problem 3 with the input $(\tilde{D}, \tilde{\mathcal{M}}, \tilde{\pi})$ has a solution if and only if

$$\tilde{\pi}^{-1}(v) \in \mathcal{I}(\tilde{\mathcal{M}}) \qquad \text{for all } v \in V, \tag{10}$$

$$\rho_{\tilde{D}}(X) \geq \tilde{r}(\tilde{\mathsf{S}}) - \tilde{r}(\tilde{\pi}^{-1}(X)) \qquad \text{for all nonempty } X \subseteq V. \tag{11}$$

We show that conditions (10) and (11) are satisfied.

Claim 1. $\tilde{r}(\tilde{\pi}^{-1}(X)) = r(\pi^{-1}(X))$ *for all* $X \subseteq V$.

Proof. Let $X \subseteq V$. By the monotonicity of r, (8), (9) and (7), we have

$$\tilde{r}(\tilde{\pi}^{-1}(X)) = \min_{\mathsf{S}' \subseteq \mathsf{S}} \left\{ r(\mathsf{S}') + \left| \tilde{\pi}^{-1}(X) \setminus \bigcup_{\mathsf{s}^i \in \mathsf{S}'} \tilde{\mathsf{S}}_i \right| \right\} = \min_{\mathsf{S}' \subseteq \pi^{-1}(X)} \left\{ r(\mathsf{S}') + \sum_{\mathsf{s}^i \in \pi^{-1}(X) \setminus \mathsf{S}'} r(\mathsf{s}^i) \right\}. \quad (12)$$

By the submodularity of r, we have

$$r(\pi^{-1}(X)) \leq \min_{\mathsf{S}' \subseteq \pi^{-1}(X)} \left\{ r(\mathsf{S}') + \sum_{\mathsf{s}^i \in \pi^{-1}(X) \setminus \mathsf{S}'} r(\mathsf{s}^i) \right\} \leq r(\pi^{-1}(X)). \quad (13)$$

Hence equality holds everywhere in (13). Then, (12) and (13) implies the claim. □

Let $v \in V$. By the submodularity of r and (4), we have $r(\pi^{-1}(v)) \leq \sum_{\mathsf{s}^i \in \pi^{-1}(v)} r(\mathsf{s}^i) \leq r(\pi^{-1}(v))$, that is, by Claim 1, (7) and (9)

$$\tilde{r}(\tilde{\pi}^{-1}(v)) = r(\pi^{-1}(v)) = \sum_{\mathsf{s}^i \in \pi^{-1}(v)} r(\mathsf{s}^i) = \left| \bigcup_{\mathsf{s}^i \in \pi^{-1}(v)} \tilde{\mathsf{S}}_i \right| = |\tilde{\pi}^{-1}(v)|, \quad (14)$$

and hence (10) is satisfied.

By Claim 1 applied for V and for $X \subseteq V$, (5) and (6), we have

$$\tilde{r}(\tilde{\mathsf{S}}) - \tilde{r}(\tilde{\pi}^{-1}(X)) = r(\mathsf{S}) - r(\pi^{-1}(X)) \leq c(\Delta^-_D(X)) = \rho_{\tilde{D}}(X),$$

and hence (11) is satisfied.

Then, by Theorem 4, Problem 3 with the input $(\tilde{D}, \tilde{\mathcal{M}}, \tilde{\pi})$ has a solution, that is there exists in $\tilde{D}$ a packing of $\tilde{\pi}(\tilde{\mathsf{s}}^i_j)$-arborescences $\{\tilde{T}^i_j\}$ $(i = 1, \dots, k; j = 1, \dots, r(\mathsf{s}^i))$ such that $\{\tilde{\mathsf{s}}^i_j \in \tilde{\mathsf{S}} \mid v \in V(\tilde{T}^i_j)\}$ is a base of $\tilde{\mathcal{M}}$ for all $v \in V$. For $i = 1, \dots, k; j = 1, \dots, r(\mathsf{s}^i)$, let T'_{ij} be obtained from $\tilde{T}^i_j$ by replacing each arc a^ℓ of $\tilde{T}^i_j$ by the original arc a of D. Let $\{T_{ij}\}$ be obtained from $\{T'_{ij}\}$ by keeping one copy of each arborescence with λ_{ij} being equal to the multiplicity of that arborescence in $\{T'_{ij}\}$. Then, by (9), T_{ij} is a $\pi(\mathsf{s}^i)$-arborescence in D and, for $i = 1, \dots, k$, since $\{\tilde{T}^i_j\}$ exists for $j = 1, \dots, r(\mathsf{s}^i)$, $\sum \lambda_{ij} = r(\mathsf{s}^i) = d(\mathsf{s}^i)$, so (s1) is satisfied. Since the arborescences in $\{\tilde{T}^i_j\}$ are arc-disjoint in $\tilde{D}$ and each arc a of D has $c(a)$ copies in $\tilde{D}$, (s2) is satisfied. Finally, for $v \in V$ and $\mathsf{S}' \subseteq \mathsf{S}$, by the definitions of $\boldsymbol{\lambda}^v, \lambda_{ij}, \tilde{T}^i_j$ and $\tilde{r}$, we have

$$\begin{aligned} \sum \{\boldsymbol{\lambda}^v(i) \mid \mathsf{s}^i \in \mathsf{S}'\} &= \sum \{\lambda_{ij} \mid \mathsf{s}^i \in \mathsf{S}', v \in V(T_{ij})\} \\ &= |\{\tilde{T}^i_j \mid \mathsf{s}^i \in \mathsf{S}', v \in V(\tilde{T}^i_j)\}| \\ &= \left| \left\{ \tilde{\mathsf{s}}^i_j \in \bigcup_{\mathsf{s}^i \in \mathsf{S}'} \tilde{\mathsf{S}}_i \mid v \in V(\tilde{T}^i_j) \right\} \right| \\ &\leq \tilde{r}\left(\bigcup_{\mathsf{s}^i \in \mathsf{S}'} \tilde{\mathsf{S}}_i \right) \\ &\leq r(\mathsf{S}'), \end{aligned}$$

that is $\boldsymbol{\lambda}^v \in \mathrm{P}(r)$. Moreover, if $\mathsf{S}' = \mathsf{S}$, then, by $\{\tilde{\mathsf{s}}^i_j \in \tilde{\mathsf{S}} \mid v \in V(\tilde{T}^i_j)\}$ is a base of $\tilde{\mathcal{M}}$ and Claim 1, the last two inequalities are satisfied with equality, and hence $\boldsymbol{\lambda}^v \in \mathrm{B}(r)$, so (s3) is satisfied. This implies that $\{T_{ij}\}$ is a solution of Problem 5. □

To prove the general case we need the following result on the intersection of a polymatroid and a box. For the first part see Section 44.1 in [11] and the second part can be obtained by Theorem 5 since $r(\mathsf{S}') + d(\mathsf{S}^* \setminus \mathsf{S}') \quad (\mathsf{S}' \subseteq \mathsf{S}^*)$ is submodular.

Theorem 8. *The intersection of the integral polymatroid $\mathcal{P} = (\mathsf{S}, r)$ and the integral box $\times_1^{|\mathsf{S}|}[0, d(\mathsf{s})]$ (where $d : \mathsf{S} \to \mathbb{Z}_{\geq 0}$) is an integral polymatroid $\hat{\mathcal{P}} = (\mathsf{S}, \hat{r})$ whose rank function $\hat{r}$ satisfies:*

$$\hat{r}(\mathsf{S}^*) = \min_{\mathsf{S}' \subseteq \mathsf{S}^*} \{r(\mathsf{S}') + d(\mathsf{S}^* \setminus \mathsf{S}')\} \quad \mathsf{S}^* \subseteq \mathsf{S}. \tag{15}$$

Moreover, $\hat{r}(\mathsf{S}^*)$ can be computed in strongly polynomial time for all $\mathsf{S}^* \subseteq \mathsf{S}$.

By using these results, the following theorem can be shown.

Theorem 9. *Problem 5 has a solution if and only if the following conditions hold.*

$$\boldsymbol{L}^v \in \mathrm{P}(r) \qquad \textit{for all } v \in V, \tag{16}$$

$$c(\Delta^-(X)) \geq r(\mathsf{S}) - r(\mathsf{S}') - d(\pi^{-1}(X) \setminus \mathsf{S}') \qquad \textit{for all nonempty } X \subseteq V, \mathsf{S}' \subseteq \pi^{-1}(X), \tag{17}$$

where $\boldsymbol{L}^v \in \mathbb{Z}_{\geq 0}^{|\mathsf{S}|}$ is defined by $\boldsymbol{L}^v(\mathsf{s}) := \begin{cases} d(\mathsf{s}) & (\pi(\mathsf{s}) = v), \\ 0 & (\pi(\mathsf{s}) \neq v) \end{cases}$ *for all* $\mathsf{s} \in \mathsf{S}$.

Proof. We first prove the necessity. Let us fix a solution $\{T^{\mathsf{s},j}\}$ with values $\{\lambda^{\mathsf{s},j}\}$ of Problem 5.

For all $v \in V$, if $\pi(\mathsf{s}) \neq v$, then $\boldsymbol{L}^v(\mathsf{s}) = 0 \leq \boldsymbol{\lambda}^v(\mathsf{s})$, and if $\pi(\mathsf{s}) = v$, then, by (s1) and (s3), we have

$$\boldsymbol{L}^v(\mathsf{s}) = d(\mathsf{s}) = \sum \lambda^{\mathsf{s},j} \leq \sum \{\lambda^{\mathsf{s},j} \mid v \in V(T^{\mathsf{s},j})\} = \boldsymbol{\lambda}^v(\mathsf{s}).$$

Thus $\boldsymbol{\lambda}^v \in \mathrm{B}(r)$ implies (16).

Let us fix a nonempty $X \subseteq V, \mathsf{S}' \subseteq \pi^{-1}(X)$ and a vertex $v \in X$. By (s3) applied for S and for S', (s1) applied for all $\mathsf{s} \in \pi^{-1}(X) \setminus \mathsf{S}'$, $\mathsf{s} \notin \pi^{-1}(X), v \in V(T^{\mathsf{s},j})$ and $T^{\mathsf{s},j}$ is an $\pi(\mathsf{s})$-arborescence and (s2), we have

$$\begin{aligned} r(\mathsf{S}) &= \sum \{\lambda^{\mathsf{s},j} \mid v \in V(T^{\mathsf{s},j})\} \\ &= \sum_{\mathsf{s} \in \pi^{-1}(X) \cap \mathsf{S}', v \in V(T^{\mathsf{s},j})} \lambda^{\mathsf{s},j} + \sum_{\mathsf{s} \in \pi^{-1}(X) \setminus \mathsf{S}', v \in V(T^{\mathsf{s},j})} \lambda^{\mathsf{s},j} + \sum_{\mathsf{s} \notin \pi^{-1}(X), v \in V(T^{\mathsf{s},j})} \lambda^{\mathsf{s},j} \\ &\leq r(\mathsf{S}') + d(\pi^{-1}(X) \setminus \mathsf{S}') + \sum_{a \in \Delta^-(X) \cap A(T^{\mathsf{s},j})} \lambda^{\mathsf{s},j} \\ &\leq r(\mathsf{S}') + d(\pi^{-1}(X) \setminus \mathsf{S}') + c(\Delta^-(X)), \end{aligned}$$

thus (17) holds.

Next, we prove the sufficiency.

Claim 2. *The following hold:*

$$\hat{r}(\mathsf{s}) = d(\mathsf{s}) \qquad \textit{for all } \mathsf{s} \in \mathsf{S}, \tag{18}$$

$$\hat{r}(\mathsf{S}) = r(\mathsf{S}), \tag{19}$$

$$\boldsymbol{S}^v \in \mathrm{P}(\hat{r}) \qquad \textit{for all } v \in V, \tag{20}$$

$$c(\Delta^-(X)) \geq \hat{r}(\mathsf{S}) - \hat{r}(\pi^{-1}(X)) \qquad \textit{for all nonempty } X \subseteq V. \tag{21}$$

Proof. By (15) and (16), $\hat{r}(\mathsf{s}) = \min\{r(\mathsf{s}), d(\mathsf{s})\} = d(\mathsf{s})$ for all $\mathsf{s} \in \mathsf{S}$, so (18) is satisfied.

Let $\mathsf{S}' \subseteq \mathsf{S}$ such that $\hat{r}(\mathsf{S}) = r(\mathsf{S}') + d(\mathsf{S} \setminus \mathsf{S}')$. By (15) and (17) applied for $X = V$ and S', we have $\hat{r}(\mathsf{S}) \leq r(\mathsf{S}) \leq c(\Delta^-(V)) + r(\mathsf{S}') + d(\mathsf{S} \setminus \mathsf{S}') = \hat{r}(\mathsf{S})$, hence equality holds everywhere, so (19) is satisfied.

By (18), we have $\boldsymbol{S}^v = \boldsymbol{L}^v$, and then, by $\mathrm{P}(\hat{r}) = \mathrm{P}(r) \cap \times_1^{|\mathsf{S}|}[0, d(\mathsf{s})]$ and (16), (20) is satisfied.

By (15) applied for $\pi^{-1}(X)$, there exists $\mathsf{S}' \subseteq \pi^{-1}(X)$ such that $\hat{r}(\pi^{-1}(X)) = r(\mathsf{S}') + d(\pi^{-1}(X) \setminus \mathsf{S}')$. By (17) applied for $\pi^{-1}(X)$ and S' and by (19), it follows that (21) holds. ☐

By Claim 2 and Theorem 7, Problem 5 with the input $(D, c, \hat{\mathcal{P}}, d, \pi)$ has a solution, that is there exist $\pi(\mathsf{s})$-arborescences $\{T^{\mathsf{s},j}\}$ with values $\{\lambda^{\mathsf{s},j}\}$ satisfying (s1), (s2) and (s3). By $\mathrm{B}(\hat{r}) \subseteq \mathrm{B}(r)$ and (19), this solution is also a solution of Problem 5 with the input $(D, c, \mathcal{P}, d, \pi)$. □

The proofs presented in this subsection provide a pseudo-polynomial algorithm to solve Problem 5. Indeed, the construction in the proof of Theorem 7 is polynomial in $n := |V|, m := |A|$ and $C := \max\{c(a) : a \in A\}$ and hence the polynomial algorithm given in [1] solves Problem 5 in pseudo-polynomial time in the special case when $d(\mathsf{s}) = r(\mathsf{s})$ for $\mathsf{s} \in \mathsf{S}$. The proof of Theorem 9 shows how to reduce the general case to the above special case. This can be done in polynomial time by Theorem 8.

4.2 Strongly polynomial time algorithm

Now we improve the running time from pseudo-polynomial time to strongly polynomial time. We mimic the proof of Schrijver of Theorem 53.9 in [11]. For convenience, we deal with the following problem, which is a generalization of Problem 5. The difference is that we deal with packings of branchings in the following problem.

Problem 8. *Given a directed graph $D = (V, A)$, a capacity function $c : A \to \mathbb{Z}_{\geq 0}$, an integral polymatroid $\mathcal{P} = (\mathsf{S}, r)$, a demand function $d : \mathsf{S} \to \mathbb{Z}_{\geq 0}$ and a map $\pi : \mathsf{S} \to 2^V$, find $\pi(\mathsf{s})$-branchings $\{B^{\mathsf{s},j}\}$ and positive integers $\{\lambda^{\mathsf{s},j}\}$ for all $\mathsf{s} \in \mathsf{S}$ such that*

- *(b1)* $\sum \lambda^{\mathsf{s},j} = d(\mathsf{s})$ *for all* $\mathsf{s} \in \mathsf{S}$,
- *(b2)* $\sum\{\lambda^{\mathsf{s},j} \mid a \in A(B^{\mathsf{s},j})\} \leq c(a)$ *for all* $a \in A$,
- *(b3)* $\boldsymbol{\lambda}^v \in \mathrm{B}(r)$ *for all* $v \in V$, *where* $\boldsymbol{\lambda}^v \in \mathbb{Z}_{\geq 0}^{|\mathsf{S}|}$ *is defined by* $\boldsymbol{\lambda}^v(\mathsf{s}) := \sum\{\lambda^{\mathsf{s},j} \mid v \in V(B^{\mathsf{s},j})\}$ *for all* $\mathsf{s} \in \mathsf{S}$.

We can obtain a theorem for Problem 8 similar to Theorem 9. For this problem, let for all $X \subseteq V$,

$$\pi^\dagger(X) := \{\mathsf{s} \in \mathsf{S} \mid \pi(\mathsf{s}) \cap X \neq \emptyset\}.$$

We change the notation from that in the previous problem setting because now π is a map from S to 2^V.

Theorem 10. *Problem 8 has a solution if and only if the following conditions hold.*

$$\boldsymbol{K}^v \in \mathrm{P}(r) \qquad \text{for all } v \in V, \tag{22}$$

$$c(\Delta^-(X)) \geq r(\mathsf{S}) - r(\mathsf{S}') - d(\pi^\dagger(X) \setminus \mathsf{S}') \qquad \text{for all nonempty } X \subseteq V, \mathsf{S}' \subseteq \pi^\dagger(X), \tag{23}$$

where $\boldsymbol{K}^v \in \mathbb{Z}_{\geq 0}^{|\mathsf{S}|}$ *is defined by* $\boldsymbol{K}^v(\mathsf{s}) := \begin{cases} d(\mathsf{s}) & (v \in \pi(\mathsf{s})), \\ 0 & (v \notin \pi(\mathsf{s})) \end{cases}$ *for all* $\mathsf{s} \in \mathsf{S}$.

Proof. The necessity can be proved as in Theorem 9. The sufficiency will be proved by the algorithm presented below. □

4.2.1 The algorithm

In this subsection we provide our algorithm to solve Problem 8. The algorithm consists of 2 phases. In Phase 1, the idea is to reduce an instance of Problem 8 to a smaller one in the sense of capacity sum (the sum of capacities of the arcs). In order to have a strongly polynomial time algorithm the value by which the capacity sum decreases is maximized. For the reduction of the capacity sum we pay by the growth of the size of the ground set of the polymatroid: in each step a new element is added. The size of the ground set of the final polymatroid will be shown to be polynomial in the size of underlying graph and the size of the original polymatroid. When the reduction can not be applied any more Phase 2 starts. First, a solution for the last instance of Problem 8 is easily provided. Then, for all previous instances, one by one in the reverse order, a solution is constructed, hence at the end a solution for the original instance is obtained.

We will apply the following procedure Reduction (see Figure 3). Our algorithm is given in Figure 4.

Procedure (Reduction) :
Input : An instance $(D, c_i, \mathcal{P}_i = (\mathsf{S}_i, r_i), d_i, \pi_i)$ of Problem 8, $a_i = u_i v_i \in A$, $\mathsf{s}_i \in \mathsf{S}_i$ with $a_i \in \Delta^+_D(\pi_i(\mathsf{s}_i))$.
Output: An instance $(D, c_{i+1}, \mathcal{P}_{i+1} = (\mathsf{S}_{i+1}, r_{i+1}), d_{i+1}, \pi_{i+1})$ of Problem 8.

$$\mu_i := \min\{$$
$$c_i(a_i), \tag{24}$$
$$d_i(\mathsf{s}_i), \tag{25}$$
$$\min\{c_i(\Delta^-_D(X)) + \hat{r}_i(\pi^\dagger_i(X) \cup \mathsf{s}_i) - r_i(\mathsf{S}_i) \mid a_i \in \Delta^-_D(X)\}, \tag{26}$$
$$\min\{r_i(\tilde{\mathsf{S}} \cup \mathsf{s}_i) - d_i(\tilde{\mathsf{S}}) \mid \tilde{\mathsf{S}} \subseteq \pi^\dagger_i(v_i)\} \tag{27}$$
$$\}.$$
$$c_{i+1}(a_i) := c_i(a_i) - \mu_i \text{ and } c_{i+1}(a) := c_i(a) \text{ for all } a \in A \setminus a_i, \tag{28}$$
$$\mathsf{S}_{i+1} := \mathsf{S}_i \cup \bar{\mathsf{s}}_i, \text{where } \bar{\mathsf{s}}_i \text{ is a new element}, \tag{29}$$
$$r_{i+1}(\mathsf{S}') := r_i(\mathsf{S}') \text{ and } r_{i+1}(\mathsf{S}' \cup \bar{\mathsf{s}}_i) := r_i(\mathsf{S}' \cup \mathsf{s}_i) \text{ for all } \mathsf{S}' \subseteq \mathsf{S}_i, \tag{30}$$
$$d_{i+1}(\mathsf{s}_i) := d_i(\mathsf{s}_i) - \mu_i, d_{i+1}(\bar{\mathsf{s}}_i) := \mu_i \text{ and } d_{i+1}(\mathsf{s}) := d_i(\mathsf{s}) \text{ for all } \mathsf{s} \in \mathsf{S}_i \setminus \mathsf{s}_i, \tag{31}$$
$$\pi_{i+1}(\bar{\mathsf{s}}_i) := \pi_i(\mathsf{s}_i) \cup v_i \text{ and } \pi_{i+1}(\mathsf{s}) := \pi_i(\mathsf{s}) \text{ for all } \mathsf{s} \in \mathsf{S}_i. \tag{32}$$

Figure 3: Procedure Reduction.

4.2.2 Preliminaries

Let us introduce the following function g_i and *i-tight* sets for which it takes 0.

$$\begin{aligned} g_i(X) &:= c_i(\Delta^-_D(X)) + \hat{r}_i(\pi^\dagger_i(X)) - r_i(\mathsf{S}_i), \\ \mathcal{C}_i &:= \{\emptyset \neq X \subseteq V \mid g_i(X) = 0\}. \end{aligned}$$

Note that, by Theorem 8 and the facts that $\pi^\dagger_i(X \cap Y) \subseteq \pi^\dagger_i(X) \cap \pi^\dagger_i(Y)$ and $\pi^\dagger_i(X \cup Y) = \pi^\dagger_i(X) \cup \pi^\dagger_i(Y)$, we have that $\hat{r}_i(\pi^\dagger_i(X))$ is submodular (and hence so is the function g_i).

Lemma 1. *The following hold for $\mathcal{C}_i$:*

(a) If $X, Y \in \mathcal{C}_i$ are intersecting, then $X \cup Y, X \cap Y \in \mathcal{C}_i$. Moreover, if $\mathsf{s} \in \mathsf{Span}_{\hat{r}_i}(\pi^\dagger_i(X)) \cap \mathsf{Span}_{\hat{r}_i}(\pi^\dagger_i(Y))$, then $\mathsf{s} \in \mathsf{Span}_{\hat{r}_i}(\pi^\dagger_i(X \cap Y))$.

(b) Let $X_i \subseteq V$ such that $c_i(\Delta^-_D(X_i)) + \hat{r}_i(\pi^\dagger_i(X_i) \cup \mathsf{s}_i) - r_i(\mathsf{S}_i) = \mu_i$ and $a_i \in \Delta^-_D(X_i)$. Then $X_i \in \mathcal{C}_{i+1}$.

(c) $\mathcal{C}_i \subseteq \mathcal{C}_{i+1}$.

Proof. (a) Let $X, Y \in \mathcal{C}_i$ such that $X \cap Y \neq \emptyset$. If we have s, then let $\sigma = \mathsf{s}$, otherwise let $\sigma = \emptyset$. Then, by $X, Y \in \mathcal{C}_i$, the submodularity of c_i and $\hat{r}_i(\pi^\dagger_i(X) \cup \sigma)$, $X \cap Y \neq \emptyset$ and, by (23), the non-negativity of g_i on $2^V \setminus \emptyset$, we have

$$\begin{aligned} 2r_i(\mathsf{S}_i) &= g_i(X) + g_i(Y) + 2r_i(\mathsf{S}_i) \\ &= c_i(\Delta^-_D(X)) + \hat{r}_i(\pi^\dagger_i(X)) + c_i(\Delta^-_D(Y)) + \hat{r}_i(\pi^\dagger_i(Y)) \\ &= c_i(\Delta^-_D(X)) + \hat{r}_i(\pi^\dagger_i(X) \cup \sigma) + c_i(\Delta^-_D(Y)) + \hat{r}_i(\pi^\dagger_i(Y) \cup \sigma) \\ &\geq c_i(\Delta^-_D(X \cap Y)) + \hat{r}_i(\pi^\dagger_i(X \cap Y) \cup \sigma) + c_i(\Delta^-_D(X \cup Y)) + \hat{r}_i(\pi^\dagger_i(X \cup Y) \cup \sigma) \\ &\geq c_i(\Delta^-_D(X \cap Y)) + \hat{r}_i(\pi^\dagger_i(X \cap Y)) + c_i(\Delta^-_D(X \cup Y)) + \hat{r}_i(\pi^\dagger_i(X \cup Y)) \\ &= g_i(X \cap Y) + g_i(X \cup Y) + 2r_i(\mathsf{S}_i) \\ &\geq 2r_i(\mathsf{S}_i), \end{aligned}$$

ALGORITHM (INTEGRAL POLYMATROID-BASED CAPACITATED PACKING OF BRANCHINGS) :
INPUT : An instance $(D, c, \mathcal{P} = (\mathsf{S}, r), d, \pi)$ of Problem 8 satisfying conditions (22) and (23).
OUTPUT: A solution $\{(B^{\mathsf{s},j}, \lambda^{\mathsf{s},j})\}$ for $\mathsf{s} \in \mathsf{S}$ of Problem 8.

1. $(D, c_1, \mathcal{P}_1 = (\mathsf{S}_1, r_1), d_1, \pi_1) := (D, c, \mathcal{P} = (\mathsf{S}, r), d, \pi)$.
2. While there exists $u_iv_i \in A$ and $\mathsf{s}_i \in \mathsf{S}_i$ with $u_iv_i \in \Delta^+_D(\pi_i(\mathsf{s}_i))$ so that the above defined $\mu_i > 0$ execute Procedure REDUCTION. (Let τ be the last index.)
3. $\{(B^{\mathsf{s}1}_\tau, \lambda^{\mathsf{s}1}_\tau) := ((\pi_\tau(\mathsf{s}), \emptyset), d_\tau(\mathsf{s})) \mid \mathsf{s} \in \mathsf{S}_\tau\}$.
4. For $i+1$ from τ to 2,
   (a) $(B^{\mathsf{s},j}_i, \lambda^{\mathsf{s},j}_i) := (B^{\mathsf{s},j}_{i+1}, \lambda^{\mathsf{s},j}_{i+1})$ for $\mathsf{s} \in \mathsf{S}_i$ and $j \in \{1, \dots, \ell_\mathsf{s}\}$,
   (b) $(B^{\mathsf{s}_i\ell_{\mathsf{s}_i}+j}_i, \lambda^{\mathsf{s}_i\ell_{\mathsf{s}_i}+j}_i) := (B^{\bar{\mathsf{s}}_i,j}_{i+1} + a_i, \lambda^{\bar{\mathsf{s}}_i,j}_{i+1})$ for $j \in \{1, \dots, \ell_{\bar{\mathsf{s}}_i}\}$.
   (c) If $B^{\mathsf{s}_i\ell_{\mathsf{s}_i}+j}_i = B^{\mathsf{s}_i,j'}_i$ for some $j \in \{1, \dots, \ell_{\bar{\mathsf{s}}_i}\}$ and $j' \in \{1, \dots, \ell_{\mathsf{s}_i}\}$, then replace $(B^{\mathsf{s}_i\ell_{\mathsf{s}_i}+j}_i, \lambda^{\mathsf{s}_i\ell_{\mathsf{s}_i}+j}_i)$ and $(B^{\mathsf{s}_i\ell_{j'}}_i, \lambda^{\mathsf{s}_i\ell_{j'}}_i)$ by $(B^{\mathsf{s}_i\ell_{j'}}_i, \lambda^{\mathsf{s}_i\ell_{j'}}_i + \lambda^{\mathsf{s}_i\ell_{\mathsf{s}_i}+j}_i)$.
5. STOP with $\{(B^{\mathsf{s},j}_1, \lambda^{\mathsf{s},j}_1)\}$ for $s \in \mathsf{S}$.

Figure 4: The algorithm.

hence equality holds everywhere, in particular, $g_i(X \cap Y) = 0 = g_i(X \cup Y)$, that is $X \cup Y, X \cap Y \in \mathcal{C}_i$. Moreover, $\sigma \in \mathsf{Span}_{\hat{r}_i}(\pi^\dagger_i(X \cap Y))$, and (a) follows.

(b) By the definition of g_{i+1}, $a_i \in \Delta^-_D(X_i)$, (28), (32), (30) and the definition of X_i, we have

$$
\begin{aligned}
g_{i+1}(X_i) &= c_{i+1}(\Delta^-_D(X_i)) + \hat{r}_{i+1}(\pi^\dagger_{i+1}(X_i)) - r_{i+1}(\mathsf{S}_{i+1}) \\
&= c_i(\Delta^-_D(X_i)) - \mu_i + \hat{r}_{i+1}(\pi^\dagger_i(X_i) \cup \bar{\mathsf{s}}_i) - r_i(\mathsf{S}_i) \\
&= c_i(\Delta^-_D(X_i)) - \mu_i + \hat{r}_i(\pi^\dagger_i(X_i) \cup \mathsf{s}_i) - r_i(\mathsf{S}_i) \\
&= 0,
\end{aligned}
$$

that is (b) holds.

(c) Let $X \in \mathcal{C}_i$ and $\mathsf{S}' \subseteq \pi^\dagger_i(X)$ such that $\hat{r}_i(\pi^\dagger_i(X)) = r_i(\mathsf{S}') + d_i(\pi^\dagger_i(X) \setminus \mathsf{S}')$.

(I) If $a_i \in \Delta^-_D(X)$, then $\bar{\mathsf{s}}_i \in \pi^\dagger_{i+1}(X)$ and $c_{i+1}(\Delta^-(X)) = c_i(\Delta^-(X)) - \mu_i$. By $X \in \mathcal{C}_i$, $a_i \in \Delta^-_D(X)$, (28), (23), (15), $\bar{\mathsf{s}}_i \in \pi^\dagger_{i+1}(X)$, (31) and the definition of S', we have

$$
\begin{aligned}
\hat{r}_i(\pi^\dagger_i(X)) + \mu_i &= r_i(\mathsf{S}_i) - c_i(\Delta^-_D(X)) + \mu_i \\
&= r_i(\mathsf{S}_i) - c_{i+1}(\Delta^-_D(X)) \\
&\le \hat{r}_{i+1}(\pi^\dagger_{i+1}(X)) \\
&\le r_{i+1}(\mathsf{S}') + d_{i+1}(\pi^\dagger_{i+1}(X) \setminus \mathsf{S}') \\
&= r_{i+1}(\mathsf{S}') + d_{i+1}(\pi^\dagger_i(X) \setminus \mathsf{S}') + d_{i+1}(\bar{\mathsf{s}}_i) \\
&\le r_i(\mathsf{S}') + d_i(\pi^\dagger_i(X) \setminus \mathsf{S}') + \mu_i \\
&= \hat{r}_i(\pi^\dagger_i(X)) + \mu_i.
\end{aligned}
$$

Hence equality holds everywhere, in particular, $c_{i+1}(\Delta^-_D(X)) = r_{i+1}(\mathsf{S}_{i+1}) - \hat{r}_{i+1}(\pi^\dagger_{i+1}(X))$, that is $X \in \mathcal{C}_{i+1}$.

(II) If $a_i \notin \Delta_D^-(X)$, then, by $X \in \mathcal{C}_i$, $a_i \notin \Delta_D^-(X)$, (28) and (23), we have

$$\begin{aligned} \hat{r}_i(\pi_i^\dagger(X)) &= r_i(\mathsf{S}_i) - c_i(\Delta_D^-(X)) \\ &= r_{i+1}(\mathsf{S}_i) - c_{i+1}(\Delta_D^-(X)) \\ &\leq \hat{r}_{i+1}(\pi_{i+1}^\dagger(X)). \end{aligned} \tag{33}$$

(i) If $\bar{\mathsf{s}}_i \notin \pi_{i+1}^\dagger(X) \setminus \mathsf{S}'$, then $\pi_{i+1}^\dagger(X) = \pi_i^\dagger(X)$, so, by (15), (30), (31) and the definition of S', we have

$$\begin{aligned} \hat{r}_{i+1}(\pi_{i+1}^\dagger(X)) &\leq r_{i+1}(\mathsf{S}') + d_{i+1}(\pi_{i+1}^\dagger(X) \setminus \mathsf{S}') \\ &\leq r_i(\mathsf{S}') + d_i(\pi_i^\dagger(X) \setminus \mathsf{S}') \\ &= \hat{r}_i(\pi_i^\dagger(X)). \end{aligned} \tag{34}$$

(ii) If $\bar{\mathsf{s}}_i \in \pi_{i+1}^\dagger(X) \setminus \mathsf{S}'$, then, by $a_i \notin \Delta_D^-(X)$, $\mathsf{s}_i \in \pi_i^\dagger(X)$.

1. If $\mathsf{s}_i \notin \mathsf{S}'$, then, by (15), (31), (30) and the definition of S', we have

$$\begin{aligned} \hat{r}_{i+1}(\pi_{i+1}^\dagger(X)) &\leq r_{i+1}(\mathsf{S}') + d_{i+1}(\pi_{i+1}^\dagger(X) \setminus \mathsf{S}') \\ &= r_{i+1}(\mathsf{S}') + d_{i+1}(\pi_i^\dagger(X) \setminus (\mathsf{S}' \cup \mathsf{s}_i)) + d_{i+1}(\mathsf{s}_i) + d_{i+1}(\bar{\mathsf{s}}_i) \\ &= r_{i+1}(\mathsf{S}') + d_i(\pi_i^\dagger(X) \setminus (\mathsf{S}' \cup \mathsf{s}_i)) + d_i(\mathsf{s}_i) - \mu_i + \mu_i \\ &= r_i(\mathsf{S}') + d_i(\pi_i^\dagger(X) \setminus \mathsf{S}') \\ &= \hat{r}_i(\pi_i^\dagger(X)). \end{aligned} \tag{35}$$

2. If $\mathsf{s}_i \in \mathsf{S}'$, then, by (15), (30), (31) and the definition of S', we have

$$\begin{aligned} \hat{r}_{i+1}(\pi_{i+1}^\dagger(X)) &\leq r_{i+1}(\mathsf{S}' \cup \bar{\mathsf{s}}_i) + d_{i+1}(\pi_{i+1}^\dagger(X) \setminus (\mathsf{S}' \cup \bar{\mathsf{s}}_i)) \\ &= r_i(\mathsf{S}') + d_i(\pi_i^\dagger(X) \setminus \mathsf{S}') \\ &= \hat{r}_i(\pi_i^\dagger(X)). \end{aligned} \tag{36}$$

Hence, by (33) and (34) or (35) or (36), equality holds in (33), so $c_{i+1}(\Delta_D^-(X)) = r_{i+1}(\mathsf{S}_{i+1}) - \hat{r}_{i+1}(\pi_{i+1}^\dagger(X))$, that is $X \in \mathcal{C}_{i+1}$.

□

Let

$$\begin{aligned} A_i^> &:= \{a \in A \mid c_i(a) > 0\} \text{ and} \\ \mathsf{S}_i^> &:= \{\mathsf{s} \in \mathsf{S}_i \mid d_i(\mathsf{s}) > 0\}. \end{aligned}$$

A pair (uv, s) is called *good* if $uv \in A_i^>$, $\mathsf{s} \in \mathsf{S}_i^>$, $uv \in \Delta_D^+(\pi_i(\mathsf{s}))$ and $\min\{r_i(\tilde{\mathsf{S}} \cup \mathsf{s}) - d_i(\tilde{\mathsf{S}}) \mid \tilde{\mathsf{S}} \subseteq \pi_i^\dagger(v)\} > 0$. An arc uv is called *good* if there exists s such that (uv, s) is good. For a set $X \subseteq V$, let

$$\pi_i^>(X) := \pi_i^\dagger(X) \cap \mathsf{S}_i^>.$$

For sets $X, Y \subseteq V$, we say that X *dominates* Y if $\pi_i^>(Y) \subseteq \mathsf{Span}_{\hat{r}_i}(\pi_i^\dagger(X))$, that is $\hat{r}_i(\pi_i^\dagger(X) \cup \pi_i^>(Y)) = \hat{r}_i(\pi_i^\dagger(X))$.

Claim 3. *For* $X \subseteq V$, $\hat{r}_i(\pi_i^>(X)) = \hat{r}_i(\pi_i^\dagger(X))$.

Proof. Let $X \subseteq V$. By (15), there exists $\mathsf{S}' \subseteq \pi_i^{>}(X)$ such that $\hat{r}_i(\pi_i^{>}(X)) = r_i(\mathsf{S}') + d_i(\pi_i^{>}(X) \setminus \mathsf{S}')$. Then, by (r3) for $\hat{r}_i$, the definition of S' and $\pi_i^{>}$, and (15), we have

$$\begin{aligned} \hat{r}_i(\pi_i^{\dagger}(X)) &\geq \hat{r}_i(\pi_i^{>}(X)) \\ &= r_i(\mathsf{S}') + d_i(\pi_i^{>}(X) \setminus \mathsf{S}') \\ &= r_i(\mathsf{S}') + d_i(\pi_i^{\dagger}(X) \setminus \mathsf{S}') \\ &\geq \hat{r}_i(\pi_i^{\dagger}(X)). \end{aligned}$$

Hence equality holds everywhere and the claim follows. □

Claim 4. *For $w \in V$, $\hat{r}_i(\pi_i^{\dagger}(w)) = d_i(\pi_i^{\dagger}(w))$.*

Proof. By (15), there exists $\mathsf{S}' \subseteq \pi_i^{\dagger}(w)$ such that $r_i(\mathsf{S}') + d_i(\pi_i^{\dagger}(w) \setminus \mathsf{S}') = \hat{r}_i(\pi_i^{\dagger}(w))$. Then, by (22), the definition of S', (r3) for $\hat{r}_i$ and (15), we have

$$\begin{aligned} d_i(\pi_i^{\dagger}(w)) &= d_i(\pi_i^{\dagger}(w) \setminus \mathsf{S}') + d_i(\mathsf{S}') \\ &\leq d_i(\pi_i^{\dagger}(w) \setminus \mathsf{S}') + r_i(\mathsf{S}') \\ &= \hat{r}_i(\pi_i^{\dagger}(w)) \\ &\leq d_i(\pi_i^{\dagger}(w)). \end{aligned}$$

Hence equality holds everywhere and the claim follows. □

Claim 5. *For $w \in V$ and $\mathsf{s} \in \mathsf{S}_i^{>} \setminus \pi_i^{\dagger}(w)$, $\mathsf{s} \in \mathsf{Span}_{\hat{r}_i}(\pi_i^{\dagger}(w))$ if and only if there exists $\tilde{\mathsf{S}} \subseteq \pi_i^{\dagger}(w)$ such that $r_i(\tilde{\mathsf{S}} \cup \mathsf{s}) = d_i(\tilde{\mathsf{S}})$.*

Proof. Suppose first that $r_i(\tilde{\mathsf{S}} \cup \mathsf{s}) = d_i(\tilde{\mathsf{S}})$ for some $\tilde{\mathsf{S}} \subseteq \pi_i^{\dagger}(w)$. Then, by (r3) for $\hat{r}_i$, (15), the assumption and Claim 4, we have

$$\begin{aligned} \hat{r}_i(\pi_i^{\dagger}(w)) &\leq \hat{r}_i(\pi_i^{\dagger}(w) \cup \mathsf{s}) \\ &\leq r_i(\tilde{\mathsf{S}} \cup \mathsf{s}) + d_i((\pi_i^{\dagger}(w) \cup \mathsf{s}) \setminus (\tilde{\mathsf{S}} \cup \mathsf{s})) \\ &= d_i(\tilde{\mathsf{S}}) + d_i(\pi_i^{\dagger}(w) \setminus \tilde{\mathsf{S}}) \\ &= d_i(\pi_i^{\dagger}(w)) \\ &= \hat{r}_i(\pi_i^{\dagger}(w)). \end{aligned}$$

Hence equality holds everywhere and the sufficiency follows.

Suppose now that $\mathsf{s} \in \mathsf{Span}_{\hat{r}_i}(\pi_i^{\dagger}(w))$. By (15), there exists $\tilde{\mathsf{S}}' \subseteq \mathsf{S}_i \cup \mathsf{s}$ such that $r_i(\tilde{\mathsf{S}}') + d_i((\pi_i^{\dagger}(w) \cup \mathsf{s}) \setminus \tilde{\mathsf{S}}') = \hat{r}_i(\pi_i^{\dagger}(w) \cup \mathsf{s})$. Let $\tilde{\mathsf{S}} := \tilde{\mathsf{S}}' \setminus \mathsf{s}$. Then, by $\mathsf{s} \in \mathsf{Span}_{\hat{r}_i}(\pi_i^{\dagger}(w))$, the definition of $\tilde{\mathsf{S}}'$, (r3) for r_i, (22), $\mathsf{s} \notin \pi_i^{\dagger}(w)$, non-negativity of d_i and Claim 4, we have

$$\begin{aligned} \hat{r}_i(\pi_i^{\dagger}(w)) &= \hat{r}_i(\pi_i^{\dagger}(w) \cup \mathsf{s}) \\ &= r_i(\tilde{\mathsf{S}}') + d_i((\pi_i^{\dagger}(w) \cup \mathsf{s}) \setminus \tilde{\mathsf{S}}') \\ &\geq r_i(\tilde{\mathsf{S}}) + d_i(\pi_i^{\dagger}(w) \setminus \tilde{\mathsf{S}}) + d_i(\mathsf{s} \setminus \tilde{\mathsf{S}}') \\ &\geq d_i(\tilde{\mathsf{S}}) + d_i(\pi_i^{\dagger}(w) \setminus \tilde{\mathsf{S}}) \\ &= d_i(\pi_i^{\dagger}(w)) \\ &= \hat{r}_i(\pi_i^{\dagger}(w)). \end{aligned}$$

Hence equality holds everywhere. Then, $d_i(\mathsf{s} \setminus \tilde{\mathsf{S}}') = 0$ and hence, by $\mathsf{s} \in \mathsf{S}_i^{>}$, we have $\mathsf{s} \in \tilde{\mathsf{S}}'$, that is $\tilde{\mathsf{S}}' = \tilde{\mathsf{S}} \cup \mathsf{s}$, and $d_i(\tilde{\mathsf{S}}) = r_i(\tilde{\mathsf{S}}) = r_i(\tilde{\mathsf{S}}') = r_i(\tilde{\mathsf{S}} \cup \mathsf{s})$ and the necessity follows. □

Corollary 2. *$uv \in A_i^>$ is a good arc if and only if v does not dominate u.*

Proof. Suppose first that uv is a good arc. Then there exists $\mathsf{s} \in \mathsf{S}_i^>$ such that $uv \in \Delta_D^+(\pi_i(\mathsf{s}))$ and $r_i(\tilde{\mathsf{S}} \cup \mathsf{s}) > d_i(\tilde{\mathsf{S}})$ for all $\tilde{\mathsf{S}} \subseteq \pi_i^\dagger(v)$. Then, by Claims 5, $\mathsf{s} \notin \mathsf{Span}_{\hat{r}_i}(\pi_i^\dagger(v))$, that is v does not dominate u.

Suppose now that v does not dominate u, that is there exists $\mathsf{s} \in \pi_i^>(u) \setminus \mathsf{Span}_{\hat{r}_i}(\pi_i^\dagger(v))$. Then $\mathsf{s} \in \mathsf{S}_i^> \cap (\pi_i^\dagger(u) \setminus \pi_i^\dagger(v))$ and so $uv \in \Delta_D^+(\pi_i(\mathsf{s}))$. Moreover, by Claim 5, (r3) for r_i and (22), $\min\{r_i(\tilde{\mathsf{S}} \cup \mathsf{s}) - d_i(\tilde{\mathsf{S}}) \mid \tilde{\mathsf{S}} \subseteq \pi_i^\dagger(v)\} > 0$. Thus uv is a good arc. □

Claim 6. *Let X be an i-tight set and v a vertex of X. If no good arc exists in $D[X]$, then v dominates X.*

Proof. Let us denote by Y the set of vertices from which v is reachable by a path using vertices in $D' := (X, A(D[X]) \cap A_i^>)$. We show that (a) Y dominates X and (b) v dominates Y and then, by Claim 3, we have $\pi_i^>(X) \subseteq \mathsf{Span}_{r_i}(\pi^\dagger(Y)) = \mathsf{Span}_{r_i}(\pi^>(Y)) \subseteq \mathsf{Span}_{r_i}(\pi^\dagger(v))$, and the claim follows.

(a) By the definition of Y, every arc of $A_i^>$ that enters Y enters X as well. Then, by the i-tightness of X, (r3) for r_i and (23) for Y, we have

$$\begin{aligned} c_i(\Delta_D^-(Y)) &\leq c_i(\Delta_D^-(X)) \\ &= r_i(\mathsf{S}_i) - \hat{r}_i(\pi_i^\dagger(X)) \\ &\leq r_i(\mathsf{S}_i) - \hat{r}_i(\pi_i^\dagger(Y)) \\ &\leq c_i(\Delta_D^-(Y)). \end{aligned}$$

Thus equality holds everywhere and Y dominates X.

(b) For all $y \in Y$, there exists a directed path $y = v_\ell, \ldots, v_1 = v$ from y to v in D'. Since no good arc exists in $D[X]$, by Corollary 2 and Claim 3, $\pi_i^>(y) = \pi_i^>(v_\ell) \subseteq \mathsf{Span}_{r_i}(\pi_i^\dagger(v_{\ell-1})) = \mathsf{Span}_{r_i}(\pi_i^>(v_{\ell-1})) \cdots \subseteq \mathsf{Span}_{r_i}(\pi_i^\dagger(v_1)) = \mathsf{Span}_{r_i}(\pi_i^\dagger(v))$. Hence $\pi_i^>(Y) = \bigcup_{y \in Y} \pi_i^>(y) \subseteq \mathsf{Span}_{r_i}(\pi_i^\dagger(v))$ and v dominates Y. □

4.2.3 Correctness of the algorithm

In this section we prove that our algorithm runs correctly. This will be shown through the following lemmas.

Lemma 2. *If $(D, c_i, \mathcal{P}_i = (\mathsf{S}_i, r_i), d_i, \pi_i)$ is an instance of Problem 8 satisfying conditions* (22) *and* (23), *then so is $(D, c_{i+1}, \mathcal{P}_{i+1} = (\mathsf{S}_{i+1}, r_{i+1}), d_{i+1}, \pi_{i+1})$.*

Proof. Let us start by mentioning that, by (28), (24) and $c_i(a) \geq 0$, we have $c_{i+1}(a) \geq 0$ for all $a \in A$, by (31), (24) and $d_i(\mathsf{s}) \geq 0$, we have $d_{i+1}(\mathsf{s}) \geq 0$ for all $\mathsf{s} \in \mathsf{S}_{i+1}$, and that $\mathcal{P}_{i+1} = (\mathsf{S}_{i+1}, r_{i+1})$ is clearly an integral polymatroid.

Now, suppose that (22) is satisfied for the instance $(D, c_i, \mathcal{P}_i = (\mathsf{S}_i, r_i), d_i, \pi_i)$, that is $\boldsymbol{K}_i^v \in \mathrm{P}(r_i)$ for all $v \in V$ ($\boldsymbol{K}_i^v$ denotes $\boldsymbol{K}^v$ for $(D, c_i, \mathcal{P}_i, d_i, \pi_i)$ and we use the same kind of notation in the sequel), which means that $d_i(\tilde{\mathsf{S}}) \leq r_i(\tilde{\mathsf{S}})$ for all $\tilde{\mathsf{S}} \subseteq \pi_i^\dagger(v)$. Let $v \in V$ and $\tilde{\mathsf{S}} \subseteq \pi_{i+1}^\dagger(v)$.

1. If $\bar{\mathsf{s}}_i \notin \tilde{\mathsf{S}}$, then, by $\bar{\mathsf{s}}_i \notin \tilde{\mathsf{S}}$, (31), $d_i(\tilde{\mathsf{S}}) \leq r_i(\tilde{\mathsf{S}})$ and (30), we have

$$\begin{aligned} d_{i+1}(\tilde{\mathsf{S}}) &\leq d_i(\tilde{\mathsf{S}}) \\ &\leq r_i(\tilde{\mathsf{S}}) \\ &= r_{i+1}(\tilde{\mathsf{S}}). \end{aligned}$$

2. If $\bar{\mathsf{s}}_i \in \tilde{\mathsf{S}}$ and $v \neq v_i$, then, $\mathsf{s}_i \in \pi_i^\dagger(v)$. Thus, by (25), (31), $d_i((\tilde{\mathsf{S}} \setminus \bar{\mathsf{s}}_i) \cup \mathsf{s}_i) \leq r_i((\tilde{\mathsf{S}} \setminus \bar{\mathsf{s}}_i) \cup \mathsf{s}_i)$, (30) and $\bar{\mathsf{s}}_i \in \tilde{\mathsf{S}}$, we have

$$\begin{aligned}
d_{i+1}(\tilde{\mathsf{S}}) &\leq d_{i+1}((\tilde{\mathsf{S}} \setminus \{\mathsf{s}_i, \bar{\mathsf{s}}_i\}) \cup \{\mathsf{s}_i, \bar{\mathsf{s}}_i\}) \\
&= d_{i+1}(\tilde{\mathsf{S}} \setminus \{\mathsf{s}_i, \bar{\mathsf{s}}_i\}) + d_{i+1}(\bar{\mathsf{s}}_i) + d_{i+1}(\mathsf{s}_i) \\
&= d_i(\tilde{\mathsf{S}} \setminus \{\mathsf{s}_i, \bar{\mathsf{s}}_i\}) + \mu_i + (d_i(\mathsf{s}_i) - \mu_i) \\
&= d_i((\tilde{\mathsf{S}} \setminus \bar{\mathsf{s}}_i) \cup \mathsf{s}_i) \\
&\leq r_i((\tilde{\mathsf{S}} \setminus \bar{\mathsf{s}}_i) \cup \mathsf{s}_i) \\
&= r_{i+1}(\tilde{\mathsf{S}} \cup \bar{\mathsf{s}}_i) \\
&= r_{i+1}(\tilde{\mathsf{S}}).
\end{aligned}$$

3. If $\bar{\mathsf{s}}_i \in \tilde{\mathsf{S}}$ and $v = v_i$, then, by $\bar{\mathsf{s}}_i \in \tilde{\mathsf{S}}$, $\mathsf{s}_i \notin \pi_i^\dagger(v_i)$, (31), (27) and (30), we have

$$\begin{aligned}
d_{i+1}(\tilde{\mathsf{S}}) &= d_{i+1}(\tilde{\mathsf{S}} \setminus \bar{\mathsf{s}}_i) + d_{i+1}(\bar{\mathsf{s}}_i) \\
&= d_i(\tilde{\mathsf{S}} \setminus \bar{\mathsf{s}}_i) + \mu_i \\
&\leq r_i((\tilde{\mathsf{S}} \setminus \bar{\mathsf{s}}_i) \cup \mathsf{s}_i) \\
&= r_{i+1}((\tilde{\mathsf{S}} \setminus \bar{\mathsf{s}}_i) \cup \bar{\mathsf{s}}_i) \\
&= r_{i+1}(\tilde{\mathsf{S}}).
\end{aligned}$$

The above arguments imply that $\boldsymbol{K}_{i+1}^v \in \mathrm{P}(r_{i+1})$ for all $v \in V$, that is (22) is satisfied for the instance $(D, c_{i+1}, \mathcal{P}_{i+1} = (\mathsf{S}_{i+1}, r_{i+1}), d_{i+1}, \pi_{i+1})$.

Finally, suppose that (23) is satisfied for the instance $(D, c_i, \mathcal{P}_i = (\mathsf{S}_i, r_i), d_i, \pi_i)$. Let $X \subseteq V$ and $\mathsf{S}' \subseteq \pi_{i+1}^\dagger(X)$. We distinguish some cases. Let $\varepsilon_i := \mu_i$ if $a_i \in \Delta^-(X)$ and 0 otherwise. Then

$$c_{i+1}(\Delta^-(X)) = c_i(\Delta^-(X)) - \varepsilon_i. \tag{37}$$

(i) If $\mathsf{s}_i \notin \pi_{i+1}^\dagger(X)$,

(a) If $\bar{\mathsf{s}}_i \notin \pi_{i+1}^\dagger(X)$, then, $a_i \notin \Delta^-(X)$ and by (37), (23), (30) and (31), we have

$$\begin{aligned}
c_{i+1}(\Delta^-(X)) &= c_i(\Delta^-(X)) \\
&\geq r_i(\mathsf{S}_i) - r_i(\mathsf{S}') - d_i(\pi_i^\dagger(X) \setminus \mathsf{S}') \\
&= r_{i+1}(\mathsf{S}_{i+1}) - r_{i+1}(\mathsf{S}') - d_{i+1}(\pi_{i+1}^\dagger(X) \setminus \mathsf{S}').
\end{aligned}$$

(b) If $\bar{\mathsf{s}}_i \in \pi_{i+1}^\dagger(X)$, then, $a_i \in \Delta^-(X)$ and by (37),

1. If $\bar{\mathsf{s}}_i \notin \mathsf{S}'$, then, by (28), (23), (30), (31) and (32), we have

$$\begin{aligned}
c_{i+1}(\Delta^-(X)) &= c_i(\Delta^-(X)) - \mu_i \\
&\geq r_i(\mathsf{S}_i) - r_i(\mathsf{S}') - d_i(\pi_i^\dagger(X) \setminus \mathsf{S}') - \mu_i \\
&= r_{i+1}(\mathsf{S}_{i+1}) - r_{i+1}(\mathsf{S}') - d_i(\pi_i^\dagger(X) \setminus \mathsf{S}') - d_{i+1}(\bar{\mathsf{s}}_i) \\
&= r_{i+1}(\mathsf{S}_{i+1}) - r_{i+1}(\mathsf{S}') - d_{i+1}(\pi_{i+1}^\dagger(X) \setminus \mathsf{S}').
\end{aligned}$$

2. If $\bar{\mathsf{s}}_i \in \mathsf{S}'$, then, by (28), (26), (15), (30), (31) and (32), we have

$$\begin{aligned}
c_{i+1}(\Delta^-(X)) &= c_i(\Delta^-(X)) - \mu_i \\
&\geq c_i(\Delta^-(X)) - (c_i(\Delta^-(X)) - r_i(\mathsf{S}_i) + r_i((\mathsf{S}' \setminus \bar{\mathsf{s}}_i) \cup \mathsf{s}_i) + d_i(\pi_i^\dagger(X) \setminus (\mathsf{S}' \setminus \bar{\mathsf{s}}_i))) \\
&= r_i(\mathsf{S}_i) - r_i((\mathsf{S}' \setminus \bar{\mathsf{s}}_i) \cup \mathsf{s}_i) - d_i(\pi_i^\dagger(X) \setminus (\mathsf{S}' \setminus \bar{\mathsf{s}}_i))) \\
&= r_{i+1}(\mathsf{S}_{i+1}) - r_{i+1}(\mathsf{S}') - d_{i+1}(\pi_{i+1}^\dagger(X) \setminus \mathsf{S}').
\end{aligned}$$

(ii) If $\mathsf{s}_i \in \pi^\dagger_{i+1}(X)$,

(a) If $\mathsf{s}_i, \bar{\mathsf{s}}_i \notin \mathsf{S}'$, then, by (37), (23) if $a_i \notin \Delta^-(X)$ or (26), (15) if $a_i \notin \Delta^-(X)$, (30), (31) and (32), we have

$$\begin{aligned} c_{i+1}(\Delta^-(X)) &= c_i(\Delta^-(X)) - \varepsilon_i \\ &\geq r_i(\mathsf{S}_i) - r_i(\mathsf{S}') - d_i(\pi^\dagger_i(X) \setminus \mathsf{S}') \\ &= r_i(\mathsf{S}_i) - r_i(\mathsf{S}') - d_i(\pi^\dagger_i(X) \setminus (\mathsf{S}' \cup \mathsf{s}_i)) - d_i(\mathsf{s}_i) + \mu_i - \mu_i \\ &= r_{i+1}(\mathsf{S}_{i+1}) - r_{i+1}(\mathsf{S}') - d_{i+1}(\pi^\dagger_{i+1}(X) \setminus (\mathsf{S}' \cup \mathsf{s}_i \cup \bar{\mathsf{s}}_i)) - d_{i+1}(\mathsf{s}_i) - d_{i+1}(\bar{\mathsf{s}}_i) \\ &= r_{i+1}(\mathsf{S}_{i+1}) - r_{i+1}(\mathsf{S}') - d_{i+1}(\pi^\dagger_{i+1}(X) \setminus \mathsf{S}'). \end{aligned}$$

(b) If $\mathsf{s}_i, \bar{\mathsf{s}}_i \in \mathsf{S}'$, then, by (37), (23) if $a_i \notin \Delta^-(X)$ or (26), (15) if $a_i \notin \Delta^-(X)$, (30), (31) and (32), we have

$$\begin{aligned} c_{i+1}(\Delta^-(X)) &= c_i(\Delta^-(X)) - \varepsilon_i \\ &\geq r_i(\mathsf{S}_i) - r_i(\mathsf{S}' \setminus \bar{\mathsf{s}}_i) - d_i(\pi^\dagger_i(X) \setminus (\mathsf{S}' \setminus \bar{\mathsf{s}}_i)) \\ &= r_{i+1}(\mathsf{S}_{i+1}) - r_{i+1}(\mathsf{S}') - d_{i+1}(\pi^\dagger_{i+1}(X) \setminus \mathsf{S}'). \end{aligned}$$

(c) If $|\{\mathsf{s}_i, \bar{\mathsf{s}}_i\} \cap \mathsf{S}'| = 1$, then, by the previous case, (30), (25), and (31), we have

$$\begin{aligned} c_{i+1}(\Delta^-(X)) &\geq r_{i+1}(\mathsf{S}_{i+1}) - r_{i+1}(\mathsf{S}' \cup \mathsf{s}_i \cup \bar{\mathsf{s}}_i) - d_{i+1}(\pi^\dagger_{i+1}(X) \setminus (\mathsf{S}' \cup \mathsf{s}_i \cup \bar{\mathsf{s}}_i)) \\ &\geq r_{i+1}(\mathsf{S}_{i+1}) - r_{i+1}(\mathsf{S}') - d_{i+1}(\pi^\dagger_{i+1}(X) \setminus \mathsf{S}'). \end{aligned}$$

The above arguments show that (23) is satisfied for the instance $(D, c_{i+1}, \mathcal{P}_{i+1} = (\mathsf{S}_{i+1}, r_{i+1}), d_{i+1}, \pi_{i+1})$. ☐

Lemma 3. *If there exists a good arc, then there exists a good pair (u_iv_i, s_i) for which $\mu_i > 0$.*

Proof. Suppose that for all good pair (uv, s), $\mu = 0$, that is there exists $X^{\mathsf{s}}_{uv} \subseteq V$ such that $uv \in \Delta^-_D(X^{\mathsf{s}}_{uv})$ and $c_i(\Delta^-_D(X^{\mathsf{s}}_{uv})) + \hat{r}_i(\pi^\dagger_i(X^{\mathsf{s}}_{uv}) \cup \mathsf{s}) - r_i(\mathsf{S}_i) = 0$. Then, by (23), (15) and (r3) for $\hat{r}_i$, we have

$$\hat{r}_i(\pi^\dagger_i(X^{\mathsf{s}}_{uv}) \cup \mathsf{s}) = r_i(\mathsf{S}_i) - c_i(\Delta^-_D(X^{\mathsf{s}}_{uv})) \leq \hat{r}_i(\pi^\dagger_i(X^{\mathsf{s}}_{uv})) \leq \hat{r}_i(\pi^\dagger_i(X^{\mathsf{s}}_{uv}) \cup \mathsf{s}),$$

so equality holds everywhere, that is X^{s}_{uv} is i-tight and $\mathsf{s} \in \mathsf{Span}_{\hat{r}_i}(\pi^\dagger_i(X^{\mathsf{s}}_{uv}))$.

Claim 7. *Each good arc uv enters an i-tight set X_u that dominates u.*

Proof. Let uv be a good arc. By the above argument, for all $\mathsf{s} \in \pi^>_i(u) \setminus \pi^\dagger_i(v)$, there exists an i-tight set X^{s}_{uv} such that $uv \in \Delta^-_D(X^{\mathsf{s}}_{uv})$ and $\mathsf{s} \in \mathsf{Span}_{\hat{r}_i}(\pi^\dagger_i(X^{\mathsf{s}}_{uv}))$. Let $X_u := \bigcup_{\mathsf{s} \in \pi^>_i(u) \setminus \pi^\dagger_i(v)} X^{\mathsf{s}}_{uv}$. Since $v \in X^{\mathsf{s}}_{uv}$ for all $\mathsf{s} \in \mathsf{Span}_{\hat{r}_i}(\pi^\dagger_i(X^{\mathsf{s}}_{uv}))$, Claim 1 implies that X_u is i-tight. Moreover, $\pi^>_i(u) = (\pi^>_i(u) \setminus \pi^\dagger_i(v)) \cup \pi^\dagger_i(v) \subseteq \mathsf{Span}_{\hat{r}_i}(\pi^\dagger_i(X_u)) \cup \pi^\dagger_i(v) = \mathsf{Span}_{\hat{r}_i}(\pi^\dagger_i(X_u))$, that is X_u that dominates u. ☐

Among all pairs of a good arc uv and an i-tight set X such that uv enters X and X dominates u choose one with X minimal.

Since X dominates u but, by Corollary 2, v does not dominate u, v does not dominate X. Then, by Claim 6, there exists a good arc $u'v'$ in $D[X]$. Then, by Claim 7, $u'v'$ enters a tight set Y that dominates u'. By $v' \in X \cap Y$, the i-tightness of X and Y, $u' \in X$, $\pi^>_i(u') \subseteq \mathsf{Span}_{r_i}(\pi^\dagger_i(Y))$, Lemma 1, we have that $X \cap Y$ is i-tight and, by Lemma 1(a), $\pi^>_i(u') \subseteq \mathsf{Span}_{r_i}(\pi^\dagger_i(X \cap Y))$. Since the good arc $u'v'$ enters the i-tight set $X \cap Y$ that dominates u' and $X \cap Y$ is a proper subset of X (since $u' \in X \setminus Y$), this contradicts the minimality of X. ☐

Lemma 4. *If no good arc exists, then $\{(B^{\mathsf{s},1}_\tau, \lambda^{\mathsf{s},1}_\tau) := ((\pi_\tau(\mathsf{s}), \emptyset), d_\tau(\mathsf{s})) \mid \mathsf{s} \in \mathsf{S}_\tau\}$ (provided in Step 3 in Figure 4) is a solution of Problem 8 for the instance $(D, c_\tau, \mathcal{P}_\tau, d_\tau, \pi_\tau)$.*

Proof. Note that $B_\tau^{\mathsf{s},1}$ is a $\pi_\tau(\mathsf{s})$-branching for all $\mathsf{s} \in \mathsf{S}_\tau$.

(b1) is satisfied by $\lambda_\tau^{\mathsf{s},1} = d_\tau(\mathsf{s})$ for all $\mathsf{s} \in \mathsf{S}_\tau$.

(b2) is satisfied since $A(B_\tau^{\mathsf{s},1}) = \emptyset$ for all $\mathsf{s} \in \mathsf{S}_\tau$ and $c_\tau(a) \geq 0$ for all $a \in A$.

(b3) By Lemma 2, the instance $(D, c_\tau, \mathcal{P}_\tau = (\mathsf{S}_\tau, r_\tau), d_\tau, \pi_\tau)$ of Problem 8 satisfies conditions (22) and (23). By (15) and (23), we have $0 \geq \hat{r}_\tau(S_\tau) - r_\tau(\mathsf{S}_\tau) = c_\tau(\Delta_D^-(V)) + \hat{r}_\tau(\pi_\tau^\dagger(V)) - r_\tau(\mathsf{S}_\tau) \geq 0$. It follows that V is a τ-tight set and $\hat{r}_\tau(\pi_\tau^\dagger(V)) = r_\tau(\mathsf{S}_\tau)$. Then, by Claim 6, any vertex v dominates V, that is $\hat{r}_\tau(\pi_\tau^\dagger(v)) = \hat{r}_\tau(\pi_\tau^\dagger(V))$. Moreover, by Claim 6, $d_\tau(\pi_\tau^\dagger(v)) = \hat{r}_\tau(\pi_\tau^\dagger(v)) = r_\tau(\mathsf{S}_\tau)$. Since $\boldsymbol{\lambda}^v = \boldsymbol{K}_\tau^v \in \mathrm{P}(r_\tau)$ and $\boldsymbol{\lambda}^v(\mathsf{S}_\tau) = d_\tau(\pi_\tau^\dagger(v)) = r_\tau(\mathsf{S}_\tau)$ for all $v \in V$, (b3) is satisfied. □

Lemma 5. *$\{(B_i^{\mathsf{s},j}, \lambda_i^{\mathsf{s},j})\}$ for $\mathsf{s} \in \mathsf{S}_i$ (provided in Step 4 in Figure 4) is a solution of Problem 8 for the instance $(D, c_i, \mathcal{P}_i, d_i, \pi_i)$.*

Proof. (b1) For all $\mathsf{s} \in \mathsf{S}_i \setminus \mathsf{s}_i$, by the definition of $\lambda_i^{\mathsf{s},j}$, (31) and (b1) for $i+1$ and s, we have,

$$\sum_j \lambda_i^{\mathsf{s},j} = \sum_j \lambda_{i+1}^{\mathsf{s},j} = d_{i+1}(\mathsf{s}) = d_i(\mathsf{s}).$$

For s_i, by the definition of $\lambda_i^{\mathsf{s}_i,j}$, (b1) for $i+1$ and s_i and for $i+1$ and $\bar{\mathsf{s}}_i$ and (31), we have

$$\begin{aligned}\sum_j \lambda_i^{\mathsf{s}_i,j} &= \sum_j \lambda_{i+1}^{\mathsf{s}_i,j} + \sum_j \lambda_{i+1}^{\bar{\mathsf{s}}_i,j}\\ &= d_{i+1}(\mathsf{s}_i) + d_{i+1}(\bar{\mathsf{s}}_i)\\ &= d_i(\mathsf{s}_i) - \mu_i + \mu_i\\ &= d_i(\mathsf{s}_i).\end{aligned}$$

Hence (b1) is satisfied.

(b2) For all $a \in A \setminus a_i$, by the definition of $\lambda_i^{\mathsf{s},j}$, (b2) for $i+1$ and (28), we have,

$$\begin{aligned}\sum\{\lambda_i^{\mathsf{s},j} \mid a \in A(B_i^{\mathsf{s},j})\} &= \sum\{\lambda_{i+1}^{\mathsf{s},j} \mid a \in A(B_{i+1}^{\mathsf{s},j})\}\\ &\leq c_{i+1}(a)\\ &= c_i(a).\end{aligned}$$

For a_i, by the definition of $\lambda_i^{\mathsf{s}_i,j}$, (b2) for $i+1$ and a_i, (b1) for $i+1$ and $\bar{\mathsf{s}}_i$, (31) and (28), we have

$$\begin{aligned}\sum\{\lambda_i^{\mathsf{s},j} \mid a_i \in A(B_i^{\mathsf{s},j})\} &= \sum\{\lambda_{i+1}^{\mathsf{s},j} \mid a_i \in A(B_{i+1}^{\mathsf{s},j})\} + \sum \lambda_{i+1}^{\bar{\mathsf{s}}_i,j}\\ &\leq c_{i+1}(a_i) + d_{i+1}(\bar{\mathsf{s}}_i)\\ &= c_{i+1}(a_i) + \mu_i\\ &= c_i(a_i).\end{aligned}$$

Hence (b2) is satisfied.

(b3) Let $v \in V$ and $\mathsf{S}' \subseteq \mathsf{S}_i$. By definition, $\boldsymbol{\lambda}_i^v(\mathsf{s}) = \boldsymbol{\lambda}_{i+1}^v(\mathsf{s})$ for all $\mathsf{s} \in \mathsf{S}_i \setminus \mathsf{s}_i$ and $\boldsymbol{\lambda}_i^v(\mathsf{s}_i) = \boldsymbol{\lambda}_{i+1}^v(\mathsf{s}_i) + \boldsymbol{\lambda}_{i+1}^v(\bar{\mathsf{s}}_i)$. If $\mathsf{s}_i \notin \mathsf{S}'$, then, by (b3) for v and (30), we have $\sum_{\mathsf{s} \in \mathsf{S}'} \boldsymbol{\lambda}_i^v(\mathsf{s}) = \sum_{\mathsf{s} \in \mathsf{S}'} \boldsymbol{\lambda}_{i+1}^v(\mathsf{s}) \leq r_{i+1}(\mathsf{S}') = r_i(\mathsf{S}')$.

If $\mathsf{s}_i \in \mathsf{S}'$, then, by (29), (b3) for v and (30), we have

$$\begin{aligned}\sum_{\mathsf{s} \in \mathsf{S}'} \boldsymbol{\lambda}_i^v(\mathsf{s}) &= \boldsymbol{\lambda}_i^v(\mathsf{s}_i) + \sum_{\mathsf{s} \in \mathsf{S}' \setminus \mathsf{s}_i} \boldsymbol{\lambda}_i^v(\mathsf{s})\\ &= \boldsymbol{\lambda}_{i+1}^v(\mathsf{s}_i) + \boldsymbol{\lambda}_{i+1}^v(\bar{\mathsf{s}}_i) + \sum_{\mathsf{s} \in \mathsf{S}' \setminus \mathsf{s}_i} \boldsymbol{\lambda}_{i+1}^v(\mathsf{s})\\ &= \sum_{\mathsf{s} \in \mathsf{S}' \cup \bar{\mathsf{s}}_i} \boldsymbol{\lambda}_{i+1}^v(\mathsf{s})\\ &\leq r_{i+1}(\mathsf{S}' \cup \bar{\mathsf{s}}_i)\\ &= r_i(\mathsf{S}').\end{aligned}$$

Moreover, if $\mathsf{S}' = \mathsf{S}_i$, then, by (b3) for v and S_{i+1}, the above inequality holds with equality, and hence $\boldsymbol{\lambda}_i^v \in \mathrm{B}(r_i)$, so (b3) is satisfied.

□

By the above lemmas, our algorithm runs correctly.

Theorem 11. *If the instance* $(D, c, \mathcal{P} = (\mathsf{S}, r), d, \pi)$ *of Problem 8 satisfies the conditions* (22) *and* (23)*, then Algorithm* Integral Polymatroid-based Capacitated Packing of Branchings *provides a solution for Problem 8.*

Proof. By Lemma 2, the instance $(D, c_i, \mathcal{P}_i = (\mathsf{S}_i, r_i), d_i, \pi_i)$ of Problem 8 satisfies the conditions (22) and (23) for all $i \in \{1, \ldots, \tau\}$. By Lemma 3, no good arc exists in $(D, c_\tau, \mathcal{P}_\tau, d_\tau, \pi_\tau)$. Then, by Lemma 4, the packing provided in Step 3 in Figure 4 is a solution of Problem 8 for the instance $(D, c_\tau, \mathcal{P}_\tau, d_\tau, \pi_\tau)$. By Lemma 5, the algorithm provides a solution for the instance $(D, c, \mathcal{P} = (\mathsf{S}, r), d, \pi)$ of Problem 8. □

4.2.4 Complexity of the algorithm

In this subsection we show that our algorithm runs in strongly polynomial time. First we check that the number of the executions of the procedure is strongly polynomial and then that each procedure can be executed in strongly polynomial time.

Theorem 12. *The total number of the executions of Procedure* Reduction *is strongly polynomial in* $|V|, |A|$ *and* $|\mathsf{S}|$.

Proof. By definition, μ_i is equal to one of the four terms in (24), (25), (26) and (27). For each of the four terms, we evaluate the maximum number of procedures when μ_i is equal to that term.

Term (24)**:** In this case $\mu_i = c_i(a_i)$. Then, by (28), $c_{i+1}(a_i) = c_i(a_i) - \mu_i(a_i) = c_i(a_i) - c_i(a_i) = 0$, that is the capacity of an arc becomes zero. Since the capacity of an arc never increases during the procedures, this can happen at most $|A|$ times.

Term (26)**:** In this case $\mu_i = c_i(\Delta_D^-(X_i)) + \hat{r}_i(\pi_i^\dagger(X_i) \cup \mathsf{s}_i) - r_i(\mathsf{S}_i)$ for some $X_i \subseteq V$ with $a_i \in \Delta_D^-(X_i)$. Since, by Lemma 1(a), $\mathcal{C}_i^v = \{V' \in \mathcal{C}_i \mid v \in V'\}$ is a lattice for each $v \in V$, $|\mathcal{C}_i^v|$ increases at most $|V|^2$ times, and hence $|\mathcal{C}_i|$ increases at most $|V|^3$ times. (See p. 921 in [11].) By Lemma 1(b), μ_i is attained by term (26) at most $|V|^3$ times.

Term (27)**:** Suppose that, for a fixed vertex v and for $j \in \{1, \ldots, \ell\}$,

$$\mu_{i_j} = r_{i_j}(\tilde{\mathsf{S}}_{i_j} \cup \mathsf{s}_{i_j}) - d_{i_j}(\pi_{i_j}^\dagger(v) \cap \tilde{\mathsf{S}}_{i_j}). \tag{38}$$

We need the following lemma.

Lemma 6. $\ell \leq |\mathsf{S}|$.

Proof. The demand function d_i on S_i satisfies (22) which is equivalent to the following condition (39). $\tilde{\mathsf{S}}$ is called (v,i)-*tight* if $d_i(\pi_i^\dagger(v) \cap \tilde{\mathsf{S}}) = r_i(\tilde{\mathsf{S}})$.

$$d_i(\pi_i^\dagger(v) \cap \tilde{\mathsf{S}}) \leq r_i(\tilde{\mathsf{S}}) \quad \text{for all } \tilde{\mathsf{S}} \subseteq \mathsf{S}_i. \tag{39}$$

Claim 8. *If* $\tilde{\mathsf{S}}$ *and* $\tilde{\mathsf{S}}'$ *are* (v,i)*-tight then* $\tilde{\mathsf{S}} \cup \tilde{\mathsf{S}}'$ *and* $\tilde{\mathsf{S}} \cap \tilde{\mathsf{S}}'$ *are also* (v,i)*-tight.*

Proof. By the submodularity of r_i, (39), the modularity of d_i and the (v,i)-tightness of $\tilde{\mathsf{S}}$ and $\tilde{\mathsf{S}}'$, we have

$$\begin{aligned}
r_i(\tilde{\mathsf{S}}) + r_i(\tilde{\mathsf{S}}') &\geq r_i(\tilde{\mathsf{S}} \cap \tilde{\mathsf{S}}') + r_i(\tilde{\mathsf{S}} \cup \tilde{\mathsf{S}}') \\
&\geq d_i(\pi_i^\dagger(v) \cap (\tilde{\mathsf{S}} \cap \tilde{\mathsf{S}}')) + d_i(\pi_i^\dagger(v) \cap (\tilde{\mathsf{S}} \cup \tilde{\mathsf{S}}')) \\
&= d_i(\pi_i^\dagger(v) \cap \tilde{\mathsf{S}}) + d_i(\pi_i^\dagger(v) \cap \tilde{\mathsf{S}}') \\
&= r_i(\tilde{\mathsf{S}}) + r_i(\tilde{\mathsf{S}}').
\end{aligned}$$

Hence equality holds everywhere and the claim follows. □

Claim 9. *$\tilde{\mathsf{S}}_{i_j} \cup \bar{\mathsf{s}}_{i_j} \subseteq \mathsf{S}_{i_j+1}$ is (v, i_j+1)-tight for all $j \in \{1, \dots, \ell\}$.*

Proof. By the definition of r_{i_j+1} and $\bar{\mathsf{s}}_{i_j}$, condition (38), $d_{i_j+1}(\bar{\mathsf{s}}_{i_j}) = \mu_{i_j}$ and $v \in \pi_{i_j+1}(\bar{\mathsf{s}}_{i_j})$,

$$\begin{aligned} r_{i_j+1}(\tilde{\mathsf{S}}_{i_j} \cup \bar{\mathsf{s}}_{i_j}) &= r_{i_j}(\tilde{\mathsf{S}}_{i_j} \cup \mathsf{s}_{i_j}) \\ &= d_{i_j}(\pi^{\dagger}_{i_j}(v) \cap \tilde{\mathsf{S}}_{i_j}) + \mu_{i_j} \\ &= d_{i_j}(\pi^{\dagger}_{i_j}(v) \cap \tilde{\mathsf{S}}_{i_j}) + d_{i_j+1}(\bar{\mathsf{s}}_{i_j}) \\ &= d_{i_j+1}(\pi^{\dagger}_{i_j+1}(v) \cap (\tilde{\mathsf{S}}_{i_j} \cup \bar{\mathsf{s}}_{i_j})). \end{aligned}$$

□

For $j \in \{1, \dots, \ell\}$, let $\tilde{\mathsf{S}}^*_{i_j} \subseteq \mathsf{S}_{i_j+1}$ be the maximum (v, i_j+1)-tight set containing $\tilde{\mathsf{S}}_{i_j} \cup \bar{\mathsf{s}}_{i_j}$. By Claims 9 and 8, it exists.

Claim 10. *$r_{i_j+1}(\tilde{\mathsf{S}}^*_{i_j}) = r_{i_j+1}(\tilde{\mathsf{S}}^*_{i_j} \cap \mathsf{S})$.*

Proof. Let Q_{i_j} be the set of elements of S that are "parallel" to the elements of $\tilde{\mathsf{S}}^*_{i_j}$. Since each element is "parallel" to an element of S,

$$r_{i_j+1}(\tilde{\mathsf{S}}^*_{i_j}) = r_{i_j+1}(Q_{i_j}), \tag{40}$$

and, by condition (39), the modularity of d_{i_j+1}, the (v, i_j+1)-tightness of $\tilde{\mathsf{S}}^*_{i_j}$ and the nonnegativity of d_{i_j+1},

$$\begin{aligned} r_{i_j+1}(\tilde{\mathsf{S}}^*_{i_j}) &= r_{i_j+1}(Q_{i_j} \cup \tilde{\mathsf{S}}^*_{i_j}) \\ &\geq d_{i_j+1}(\pi^{\dagger}_{i_j+1}(v) \cap (Q_{i_j} \cup \tilde{\mathsf{S}}^*_{i_j})) \\ &= d_{i_j+1}(\pi^{\dagger}_{i_j+1}(v) \cap \tilde{\mathsf{S}}^*_{i_j}) + d_{i_j+1}(\pi^{\dagger}_{i_j+1}(v) \cap (Q_{i_j} \setminus \tilde{\mathsf{S}}^*_{i_j})) \\ &\geq r_{i_j+1}(\tilde{\mathsf{S}}^*_{i_j}) + 0. \end{aligned}$$

Thus $Q_{i_j} \cup \tilde{\mathsf{S}}^*_{i_j}$ is a (v, i_j+1)-tight set containing $\tilde{\mathsf{S}}^*_{i_j}$ and so $\tilde{\mathsf{S}}_{i_j} \cup \bar{\mathsf{s}}_{i_j}$. Then, by the maximality of $\tilde{\mathsf{S}}^*_{i_j}$, $Q_{i_j} \subseteq \tilde{\mathsf{S}}^*_{i_j}$, so $Q_{i_j} = \tilde{\mathsf{S}}^*_{i_j} \cap \mathsf{S}$ and hence $r_{i_j+1}(Q_{i_j}) = r_{i_j+1}(\tilde{\mathsf{S}}^*_{i_j} \cap \mathsf{S})$ which, by (40), implies the claim. □

For a set $\tilde{\mathsf{S}}_k \subseteq \mathsf{S}_{k+1}$, let us define $(\tilde{\mathsf{S}}_k \subseteq)\tilde{\mathsf{S}}'_{k+1}(\subseteq \mathsf{S}_{k+2})$, as follows: $\tilde{\mathsf{S}}'_{k+1} = \tilde{\mathsf{S}}_k$ if $\mathsf{s}_{k+1} \notin \pi^{\dagger}_{k+1}(v) \cap \tilde{\mathsf{S}}_k$ and $\tilde{\mathsf{S}}'_{k+1} = \tilde{\mathsf{S}}_k \cup \bar{\mathsf{s}}_{k+1}$ otherwise.

Claim 11. *If $\tilde{\mathsf{S}}_k(\subseteq \mathsf{S}_{k+1})$ is $(v, k+1)$-tight, then $(\tilde{\mathsf{S}}_k \subseteq)\tilde{\mathsf{S}}'_{k+1}(\subseteq \mathsf{S}_{k+2})$ is $(v, k+2)$-tight.*

Proof. If $\mathsf{s}_{k+1} \notin \pi^{\dagger}_{k+1}(v) \cap \tilde{\mathsf{S}}_k$, then, by $\tilde{\mathsf{S}}'_{k+1} = \tilde{\mathsf{S}}_k$, condition (39), $\mathsf{s}_{k+1}, \bar{\mathsf{s}}_{k+1} \notin \tilde{\mathsf{S}}_k = \tilde{\mathsf{S}}'_{k+1}$ and the $(v, k+1)$-tightness $\tilde{\mathsf{S}}_k$, we have

$$\begin{aligned} r_{k+1}(\tilde{\mathsf{S}}_k) &= r_{k+2}(\tilde{\mathsf{S}}'_{k+1}) \\ &\geq d_{k+2}(\pi^{\dagger}_{k+2}(v) \cap \tilde{\mathsf{S}}'_{k+1}) \\ &= d_{k+1}(\pi^{\dagger}_{k+1}(v) \cap \tilde{\mathsf{S}}_k) \\ &= r_{k+1}(\tilde{\mathsf{S}}_k). \end{aligned}$$

Hence equality holds everywhere and $\tilde{\mathsf{S}}'_{k+1}$ is $(v, k+2)$-tight.

If $\mathsf{s}_{k+1} \in \pi^\dagger_{k+1}(v) \cap \tilde{\mathsf{S}}_k$, then, by $\tilde{\mathsf{S}}'_{k+1} = \tilde{\mathsf{S}}_k \cup \bar{\mathsf{s}}_{k+1}$, the definition of r_{k+2}, $\mathsf{s}_{k+1} \in \tilde{\mathsf{S}}_k$, the $(v, k+1)$-tightness $\tilde{\mathsf{S}}_k$, the modularity of d_{k+1}, the definition of d_{k+2}, the modularity of d_{k+2}, we have

$$\begin{aligned}
r_{k+2}(\tilde{\mathsf{S}}'_{k+1}) &= r_{k+2}(\tilde{\mathsf{S}}_k \cup \bar{\mathsf{s}}_{k+1}) \\
&= r_{k+1}(\tilde{\mathsf{S}}_k \cup \mathsf{s}_{k+1}) \\
&= r_{k+1}(\tilde{\mathsf{S}}_k) \\
&= d_{k+1}(\pi^\dagger_{k+1}(v) \cap \tilde{\mathsf{S}}_k) \\
&= d_{k+1}(\pi^\dagger_{k+1}(v) \cap (\tilde{\mathsf{S}}_k \setminus \mathsf{s}_{k+1})) + d_{k+1}(\mathsf{s}_{k+1}) - \mu_{k+1} + \mu_{k+1} \\
&= d_{k+2}(\pi^\dagger_{k+2}(v) \cap (\tilde{\mathsf{S}}_k \setminus \mathsf{s}_{k+1})) + d_{k+2}(\mathsf{s}_{k+1}) + d_{k+2}(\bar{\mathsf{s}}_{k+1}) \\
&= d_{k+2}(\pi^\dagger_{k+2}(v) \cap (\tilde{\mathsf{S}}_k \cup \bar{\mathsf{s}}_{k+1})) \\
&= d_{k+2}(\pi^\dagger_{k+2}(v) \cap \tilde{\mathsf{S}}'_{k+1}),
\end{aligned}$$

that is $\tilde{\mathsf{S}}'_{k+1}$ is $(v, k+2)$-tight. □

For a fixed $j \in \{1, \ldots, \ell\}$, by applying repeatedly Claim 11 starting by $\tilde{\mathsf{S}}^*_{i_j}$, we have that $\tilde{\mathsf{S}}^*_{i_j} \subseteq \tilde{\mathsf{S}}'_{i_{j+1}} \subseteq \mathsf{S}_{i_{j+1}+1}$ is a $(v, i_{j+1}+1)$-tight set.

Claim 12. $r_{i_j+1}(\tilde{\mathsf{S}}^*_{i_j} \cap \mathsf{S}) < r_{i_{j+1}+1}(\tilde{\mathsf{S}}^*_{i_{j+1}} \cap \mathsf{S})$.

Proof. First we remark that $\bar{\mathsf{s}}_{i_{j+1}} \notin \tilde{\mathsf{S}}'_{i_{j+1}}$. Indeed, since the arc $a_{i_{j+1}}$ enters v and leaves $\pi_{i_{j+1}}(\mathsf{s}_{i_{j+1}})$, $\mathsf{s}_{i_{j+1}} \notin \pi^\dagger_{i_{j+1}}(v)$, and then, by definition of $\tilde{\mathsf{S}}'_{i_{j+1}}$, the remark follows.

Note that, by the $(v, i_{j+1}+1)$-tightness of $\tilde{\mathsf{S}}^*_{i_{j+1}}$ and $\tilde{\mathsf{S}}'_{i_{j+1}}$, Claim 8, the maximality of $\tilde{\mathsf{S}}^*_{i_{j+1}}$ and $\bar{\mathsf{s}}_{i_{j+1}} \notin \tilde{\mathsf{S}}'_{i_{j+1}}$, we have

$$\tilde{\mathsf{S}}'_{i_{j+1}} \subseteq \tilde{\mathsf{S}}^*_{i_{j+1}} \setminus \bar{\mathsf{s}}_{i_{j+1}}. \tag{41}$$

By Claim 10, the $(v, i_{j+1}+1)$-tightness of $\tilde{\mathsf{S}}^*_{i_{j+1}}$, the modularity of $d_{i_{j+1}+1}$, the $(v, i_{j+1}+1)$-tightness of $\tilde{\mathsf{S}}'_{i_{j+1}}$, $\bar{\mathsf{s}}_{i_{j+1}} \in \tilde{\mathsf{S}}^*_{i_{j+1}}$, (41), non-negativity of $d_{i_{j+1}+1}$, $\tilde{\mathsf{S}}^*_{i_j} \subseteq \tilde{\mathsf{S}}'_{i_{j+1}}$, $d_{i_{j+1}+1}(\bar{\mathsf{s}}_{i_{j+1}}) = \mu_{i_{j+1}+1} > 0$, we have

$$\begin{aligned}
r_{i_{j+1}+1}(\tilde{\mathsf{S}}^*_{i_{j+1}} \cap \mathsf{S}) &= r_{i_{j+1}+1}(\tilde{\mathsf{S}}^*_{i_{j+1}}) \\
&= d_{i_{j+1}+1}(\pi^\dagger_{i_{j+1}+1}(v) \cap \tilde{\mathsf{S}}^*_{i_{j+1}}) \\
&= d_{i_{j+1}+1}(\pi^\dagger_{i_{j+1}+1}(v) \cap \tilde{\mathsf{S}}'_{i_{j+1}}) + d_{i_{j+1}+1}(\pi^\dagger_{i_{j+1}+1}(v) \cap (\tilde{\mathsf{S}}^*_{i_{j+1}} \setminus \tilde{\mathsf{S}}'_{i_{j+1}})) \\
&\geq r_{i_{j+1}+1}(\tilde{\mathsf{S}}'_{i_{j+1}}) + d_{i_{j+1}+1}(\bar{\mathsf{s}}_{i_{j+1}}) \\
&\geq r_{i_j+1}(\tilde{\mathsf{S}}^*_{i_j}) + \mu_{i_{j+1}+1} \\
&> r_{i_j+1}(\tilde{\mathsf{S}}^*_{i_j} \cap \mathsf{S}).
\end{aligned}$$

□

Claim 12 implies the lemma. □

By Lemma 6, μ_i is attained by term (27) at most $|V||\mathsf{S}|$ times.

Term (25)**:** Now we know that the first, third and fourth terms are attained at most $|A|$, $|V|^3$ and $|V||\mathsf{S}|$ times respectively. Based on this, we estimate the number of times the second term is attained. When one of the above three terms is attained, the cardinality of the ground set of the polymatroid increases by one. Each element of the ground set can be chosen at most $|V|$ times for the second term; indeed in this case $\bar{\mathsf{s}}_i$ can be chosen as s_i and $\pi_{i+1}(\bar{\mathsf{s}}_i) = \pi_i(\mathsf{s}_i) \cup v_i$. Therefore, μ_i can be attained by the second term at most $|V|(|\mathsf{S}| + |A| + |V|^3 + |V||\mathsf{S}|)$ times.

Summing up, the total number of the executions of Procedure REDUCTION is strongly polynomial in $|V|, |A|$ and $|\mathsf{S}|$. □

Corollary 3. $|\mathsf{S}_i|$ *is polynomial in* $|V|, |A|$ *and* $|\mathsf{S}|$ *for all* $i \in \{1, \ldots, \tau\}$.

Theorem 13. *Each Procedure* Reduction *can be executed in strongly polynomial time.*

Proof. We only have to show that, for a given arc a_i and element $\mathsf{s}_i \in \mathsf{S}_i$ such that $a_i \in \Delta_D^+(\pi_i(\mathsf{s}_i))$, the following lemma holds.

Lemma 7. μ_i *can be calculated in strongly polynomial time.*

Proof. The first and the second terms, (24) and (25), can be obviously obtained in constant time.

(26) We want to minimize

$$f_i(X) := c_i(\Delta_D^-(X)) + \hat{r}_i(\pi_i^\dagger(X) \cup \mathsf{s}_i) - r_i(\mathsf{S}_i)$$

over $X \subseteq V$ such that $a_i \in \Delta_D^-(X)$. We can get rid of the last condition by introducing for all $X \subseteq V$,

$$\delta_i(X) := \begin{cases} 0 & a_i \in \Delta_D^-(X) \\ +\infty & \text{otherwise.} \end{cases}$$

Then

$$\min\{f_i(X) \mid X \subseteq V, a_i \in \Delta_D^-(X)\} = \min\{(f_i + \delta_i)(X) \mid X \subseteq V\}. \tag{42}$$

It is well-known that $c_i(\Delta_D^-(X))$ is a submodular function. By Theorem 8 and the facts that $\pi_i^\dagger(X \cap Y) \subseteq \pi_i^\dagger(X) \cap \pi_i^\dagger(Y)$ and $\pi_i^\dagger(X \cup Y) = \pi_i^\dagger(X) \cup \pi_i^\dagger(Y)$, the function $\hat{r}_i(\pi_i^\dagger(X) \cup \mathsf{s}_i)$ is submodular. Moreover, by Theorem 8 and since $|\mathsf{S}_i|$ is polynomial in $|V|, |A|, |\mathsf{S}|$, the value $\hat{r}_i(\pi_i^\dagger(X) \cup \mathsf{s}_i)$ (and hence $(f_i + \delta_i)(X)$) can be computed in strongly polynomial time for all X. By (30), $r_i(\mathsf{S}_i) = r(\mathsf{S})$ is constant for all i. Note that $\delta_i(X)$ is a submodular function. It follows that $(f_i + \delta_i)(X)$ is a submodular function. Thus, by Corollary 3 and Theorem 5, (42) can be computed in strongly polynomial time.

(27) Since the function $r_i(\tilde{\mathsf{S}} \cup \mathsf{s}_i) - d_i(\tilde{\mathsf{S}})$ $(\tilde{\mathsf{S}} \subseteq \pi_i^\dagger(v_i))$ is submodular, we can minimize it in strongly polynomial time by Theorem 5. ☐

Lemma 7 immediately implies Theorem 13. ☐

We are ready to prove the main result of the paper.

Theorem 14. *Problem 8 can be solved in strongly polynomial time.*

Proof. It is clear that Steps 1, 3, and 5 can be executed in strongly polynomial time. By Theorem 13, in Step 2, for a given pair of $a \in A$ and $\mathsf{s} \in \mathsf{S}_i$, Procedure Reduction can be executed in strongly polynomial time. The number of such pairs in Step 2 is at most $|A||\mathsf{S}_i|$, which is, by Corollary 3, polynomial in $|V|, |A|, |\mathsf{S}|$. Thus Step 2 can be executed in strongly polynomial time. By Corollary 3, for $i+1$ from τ to 2, the solution of Problem 8 for each instance $(D, c_i, \mathcal{P}_i, d_i, \pi_i)$ provided in Steps 3 and 4 can be constructed in strongly polynomial time and the number of instances is equal to τ, which, by Theorem 12, is strongly polynomial in $|V|, |A|, |\mathsf{S}|$, and the theorem follows. ☐

From the above theorem, we obtain the following corollary.

Corollary 4. *Problem 5 can be solved in strongly polynomial time.*

Note that we can cope with Problem 8 (and hence Problem 5) in the case when the input is of rational numbers similarly. Let us consider the problem which is similar to Problem 8 with the difference that the capacity function, the demand function, the rank function of the input polymatroid and the output are rational numbers. The same kind of result as Theorem 10 holds for this problem since we can multiply all values with the smallest common multiple.

5 Concluding remarks

In this paper, we consider the integral polymatroid-based arborescence packing problem. We deal with two problem settings, the unsplittable version and the splittable version, which generalize the matroid-based arborescence packing [1] both. The unsplittable version is strongly NP-complete, which is shown by reduction from 3-Partition, whereas the splittable version can be solved in strongly polynomial time.

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