The two ear theorem on matching-covered graphs

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Abstract

We give a simple and short proof for the two ear theorem on matching-covered graphs which is a well-known result of Lovász and Plummer. The proof relies only on the classical results of Tutte and Hall on the existence of perfect matching in (bipartite) graphs.

1 Introduction

A set M of edges is called *matching* if no two edges in M have a common end vertex. A matching M of a graph G is *perfect* if M covers all the vertices of G. We shall denote the number of perfect matchings of a graph G by $\Phi(G)$. Let M be a matching of G. A path or cycle P is said to be *alternating* if the edges of P are alternately in and not in M. For a subgraph F of G, the subset of M contained in F is denoted by $M(F)$.

Let G be a graph having a perfect matching. G is called *elementary* if the edges which belong to some perfect matching of G form a connected subgraph. Note that if G is elementary, then after adding some edges to G the resulting graph remains elementary. G is *matching-covered* if it is connected and each edge belongs to a perfect matching of G. Of course, if G is matching-covered then it is elementary.

Let G be an arbitrary graph. A subgraph H of G is *nice* if $G - V(H)$ has a perfect matching. A sequence of subgraphs of G, $(G_0, G_1, ..., G_m)$ is a *graded ear-decomposition* of G if G_0 is an even cycle, $G_m = G$, every G_i for $i = 0, 1, ..., m$ is a nice matching-covered subgraph of G and G_{i+1} is obtained from G_i by adding at most two disjoint odd paths which are openly disjoint from G_i but their end-vertices belong to G_i. Clearly, if G possesses a graded ear-decomposition, then it is matching-covered. Lovász and Plummer [6], [7] proved the following important result on matching-covered graphs.

Theorem 1 *Every matching-covered graph with at least four vertices has a graded ear-decomposition.*

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The proof of this theorem relies on the following theorem. For the sake of completeness we shall repeat the implication from [7] in Section 3.

Theorem 2 *Let G be an elementary graph and let $e_1, ..., e_k$ be edges not in G but having both end-vertices in $V(G)$. Suppose that $\Phi(G + e_1 + ... + e_k) > \Phi(G)$. Then there exist i and j, $1 \leq i \leq j \leq k$ such that $\Phi(G + e_i + e_j) > \Phi(G)$.*

The original proof of Theorem 2 in [7] is involved and it is far from being simple. Here we shall derive it by a standard method from the following theorem. The main contribution of this note is a new proof of Theorem 3 which relies only on Tutte's theorem and Hall's theorem.

Theorem 3 *Let G be an elementary graph and let e_1, e_2, e_3 be edges not in G so that $G + e_1 + e_2 + e_3$ has a perfect matching M containing e_1, e_2, e_3. Suppose that for each e_i $(1 \leq i \leq 3)$, no perfect matching of $G + e_i$ contains e_i. Then for each e_i $(1 \leq i \leq 3)$ there exists an e_j $(1 \leq j \leq 3)$ $i \neq j$ such that $G + e_i + e_j$ has a perfect matching containing e_i and e_j.*

However, we mention that the obvious generalization of Theorem 3 for $k \geq 4$ is not true, here is a counter-example. Let G be the cycle $(1, 2, ..., 8)$ on eight vertices and let $15, 24, 37, 68$ be the four new edges. Then for the edge 15 the generalization of Theorem 3 does not hold.

Little and Rendl [8] have given a shorter proof for Theorem 1 than the original one, but our proof is even shorter and simpler. Recently, Carvalho et al. [2] generalized Theorem 1 by showing that a matching-covered graph of maximum degree Δ has at least $\Delta!$ graded ear-decompositions.

2 Prelimineries

Let us recall the two classical and basic results on matching theory due to Hall [3] and Tutte [9].

Theorem 4 *[3] A bipartite graph $B = (U, V; E)$ possesses a perfect matching if and only if $|U| = |V|$ and $|\Gamma(X)| \geq |X|$ for all $X \subseteq U$, where $\Gamma(X)$ denotes the set of neighbors of X.*

Theorem 5 *[9] A graph G has a perfect matching if and only if for every $X \subseteq V(G)$, $c_0(G - X) \leq |X|$, where the number of odd components of the graph obtained from G by deleting a vertex set X is denoted by $c_0(G - X)$.*

In fact we shall use some well-known and easy corollaries of these theorems.

Claim 1 *[7] A bipartite graph $B = (U, V; E)$ is matching covered if and only if $|U| = |V|$ and $|\Gamma(X)| \geq |X| + 1$ for all $\emptyset \neq X \subset U$.*

For a graph G let $\mathrm{def}(G) := \max\{c_0(G-X) - |X| : X \subseteq V(G)\}$. A vertex set X of G is called *barrier* if X attains this maximum, that is if $G - X$ has exactly $|X|+\mathrm{def}(G)$ odd components. By a *maximal* barrier we mean one that is inclusionwise maximal. A graph G is called *factor-critical* if for each vertex v of G there exists a perfect matching in $G - v$. A barrier X is called *strong* if each odd component of $G - X$ is factor-critical. For more results on strong barriers see Király [4].

Claim 2 *[1] Let G be a graph so that it has an even number of vertices and it has no perfect matching. Let X be a maximal barrier of G. Then $c_0(G-X) \geq |X|+2$ and X is a strong barrier.*

Claim 3 *Let G be an elementary graph. Then for any barrier $X \neq \emptyset$ of G, $G - X$ has no even components.*

In fact, elementary graphs can be characterized this way. A graph having a perfect matching is elementary if and only if for any barrier $X \neq \emptyset$ of G, $G - X$ has no even components, see [7], but we shall not use this characterization. We mention that by Claim 3 the notion of maximal barriers and strong barriers coincide for elementary graphs.

Lovász [5] proved that for elementary graphs (i) the maximal barriers form a partition of the vertex set and (ii) an edge belongs to a perfect matching if and only if its end-vertices lie in different maximal barriers. We do not want to rely on these results, instead we prove the following claim. This claim will be applied frequently in our proof.

Claim 4 *Let X be a strong barrier of an elementary graph G. Then each edge leaving X belongs to some perfect matching of G.*

Proof. Since all the components of $G - X$ are factor-critical by Claim 3 it suffices to prove that each edge e of the bipartite graph B, obtained from G by deleting the edges spanned by X and contracting each component of $G-X$ into one vertex, belongs to a perfect matching of B, that is B is matching covered. Let us denote the colour class of B different from X by Y. Clearly $|X| = |Y|$. Furthermore, for any set $Z \subset Y$, $|\Gamma(Z)| \geq |Z|+1$, otherwise $\Gamma(Z)$ would violate in G either the Tutte's condition or Claim 3, both cases lead to contradiction. Then, by Claim 1, B is matching covered which was to be proved. $\square$

3 The proof

Proof. (of Theorem 6) Let us assume that there is no perfect matching of $G' := G + e_1 + e_2$ containing e_1 and e_2. We shall prove that there is a perfect matching of $G + e_1 + e_3$ containing e_1 and e_3. Let us denote the vertices of e_i by x_i, y_i.

(1) *There exists a strong barrier P in G' containing x_1 and y_1.*

$G' - x_1 - y_1$ has no perfect matching by assumption, thus by Claim 2 there exists a barrier of G' containing x_1 and y_1. Let P be a maximal barrier of G' containing x_1 and y_1. Then, by Claim 2, P is a strong barrier that is each component F_i of $G - P$ $(1 \leq i \leq |P|)$ is factor-critical.

(2) *e_2 is in one of the factor-critical components (say in F_1).*

Indeed, by Claim 4, e_2 does not enter P. Moreover, x_2 and y_2 can not be contained in P, otherwise $P - x_1 - y_1 - x_2 - y_2$ violates the Tutte's condition in $G+e_3-x_1-y_1-x_2-y_2$ contradicting the assumption that $G'' := G+e_1+e_2+e_3$ has the perfect matching M containing e_1, e_2 and e_3.

(3) *x_3 and y_3 are in different factor-critical components of $G' - P$.*

This follows from the fact that $G'-x_1-y_1+e_3$ contains the perfect matching $M - e_1$. It also follows that

(4) *for each F_i $(1 \leq i \leq |P|)$ exactly one edge m_i of M leaves F_i in G''.*

(5) *e_3 leaves the factor-critical component that contains e_2 in G'', that is $m_1 = e_3$.*

Suppose on the contrary that m_1 enters P. P is a strong barrier in $G + e_2$, thus, by Claim 4, m_1 belongs to a perfect matching M_1 of $G + e_2$. Then $(M_1 - M_1(F_1)) \cup M(F_1)$ is a perfect matching of $G + e_2$ containing e_2, a contradiction.

Assume without loss of generality that x_3 is in F_1. We know that $H := F_1 - x_3$ has a perfect matching, namely $M(H)$.

(6) *$H - e_2$ has a perfect matching M_2.*

Otherwise, for a maximal barrier X of $H - e_2$, we have by Claim 2, $c_0(H - e_2 - X) \geq |X| + 2$. Then, by Claim 2, $P' := P \cup X \cup x_3$ is a strong barrier in $G + e_3$, and e_3 enters P', thus by Claim 4, $G + e_3$ contains a perfect matching containing e_3, a contradiction.

(7) *$M(G'' - H) \cup M_2$ is a perfect matching of $G + e_1 + e_3$ containing e_1 and e_3,*

as we claimed. □

THEOREM 3 $\Longrightarrow$ THEOREM 2

Proof. We may suppose that $(*)$ no proper subset of $\{e_1, \ldots, e_k\}$ satisfies the conditions of the theorem. Then we claim that $k \leq 3$. Assume that $k \geq 4$ and let $G' := G + e_4 + \ldots + e_k$. Then by $(*)$ $\Phi(G') = \Phi(G)$ and $\Phi(G' + e_i) = \Phi(G')$ $i = 1, 2, 3$ but $\Phi(G' + e_1 + e_2 + e_3) > \Phi(G) = \Phi(G')$. Theorem 3 implies that for some $1 \leq i < j \leq 3$ $\Phi(G' + e_i + e_j) > \Phi(G')$, that is $\Phi(G + e_i + e_j + e_4 + \ldots + e_k) > \Phi(G)$, contradicting $(*)$. By applying Theorem 3 again Theorem 2 follows. □

THEOREM 2 $\Longrightarrow$ THEOREM 1

Proof. Let e and f be two incident edges of G. Let M_e and M_f be perfect matchings of G containing e and f. The symmetric difference of these two perfect matchings consists of vertex disjoint alternating cycles. Let G_0 be one of them. Then G_0 is a nice matching-covered subgraph of G. Assume that for some i the nice matching-covered subgraph G_i has already been contructed. If G_i does not span $V(G)$ then let e be an edge connecting $V(G_i)$ and $V(G) - V(G_i)$. Let M_i be a perfect matching of $G - V(G_i)$ and M_e a perfect matching of G containing e. The symmetric difference of M_i and M_e consists of vertex disjoint cycles and a set $(P_1, \dots, P_k)$ of alternating paths connecting vertices in $V(G_i)$. If G_i spans $V(G)$ but does not contain all the edges of G then the edges in $E(G) - E(G_i)$ are denoted by $(P_1, \dots, P_k)$. Clearly, after adding all these paths to G_i, the resulting graph is a nice matching-covered subgraph of G. We have to show that G_{i+1} can be constructed by adding at most two of these paths to G_i. We define an auxiliary graph $G_i' := G_i + e_1 + \ldots + e_k$, where e_i is the edge connecting the two end-vertices of P_i for $i = 1, \dots, k$. Clearly, for a subset $(P_{i_1}, \dots, P_{i_r})$ of $(P_1, \dots, P_k)$, $G_i + P_{i_1} + \ldots + P_{i_r}$ is matching-covered if and only if $G_i + e_{i_1} + \ldots + e_{i_r}$ is matching-covered. Thus Theorem 2 implies the theorem. $\square$

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