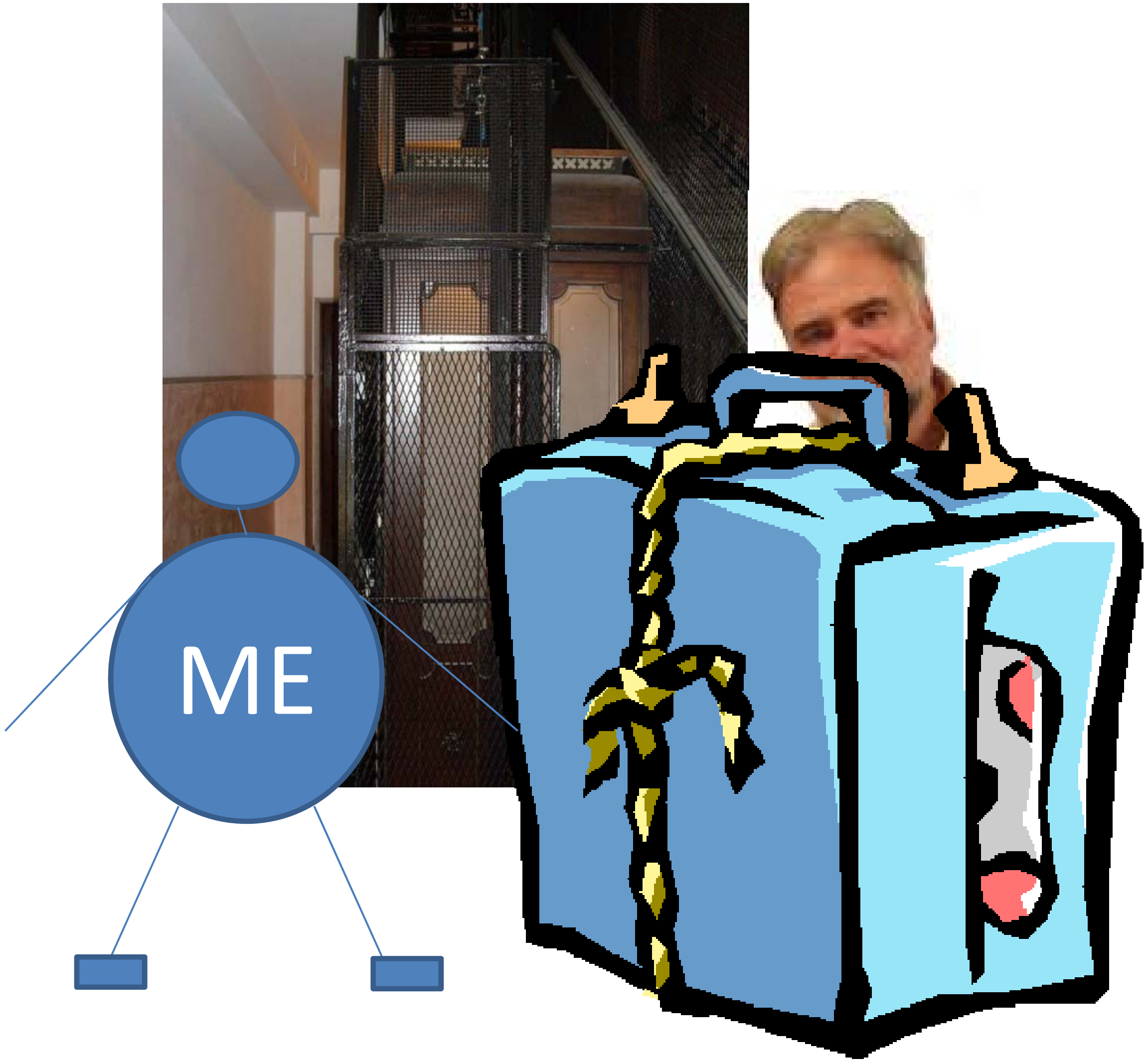
A preamble:
a concrete packing problem,
Apollonio and Sebo (2002).

Data



After several unsuccessful attempts…

Nicola: «shall we go by stairs?» (very proud of my elegant solution).

Andras starts thinking and after a while he provides a more refined solution.

Andras: «Yes! But put your giant bag into the elevator and then go by stairs»

Very, very disappointed Nicola thinks …that’s a pity I was almost there…..

A characterization of minimally unbalanced diamond-free graphs by forbidden subgraphs

N. Apollonio and A. Galluccio.

I.A.C.-C.N.R., Rome I.A.S.I.-C.N.R., Rome

CLIQUE-PERFECT GRAPHS

G is ***clique-perfect*** if for each induced subgraph G' of G

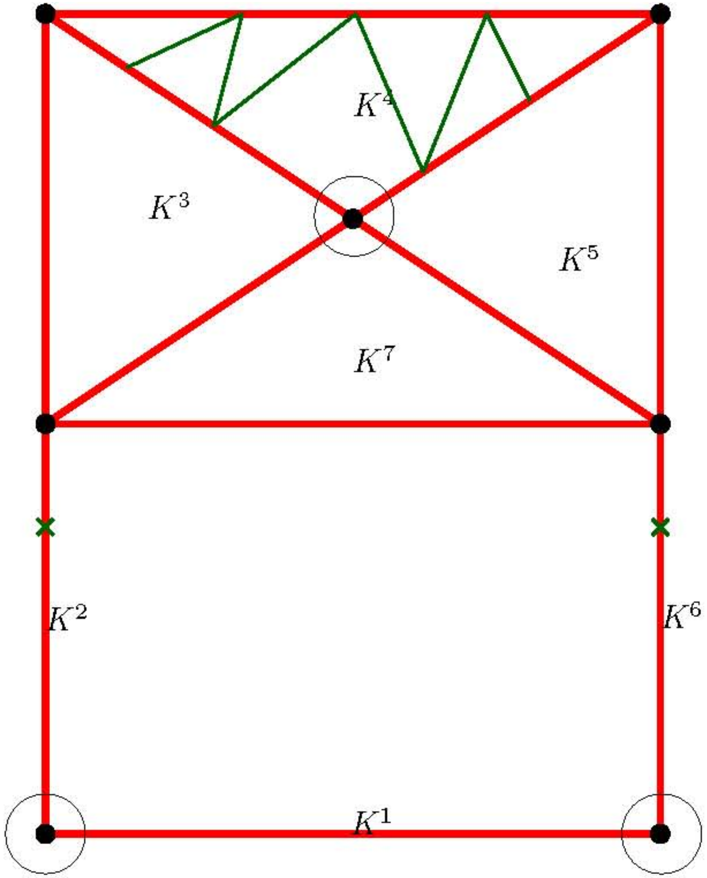
Max # vertex disjoint maximal cliques=min # of vertices that intersect all the maximal cliques

Otherwise stated: for each induced subgraph G' of G, **K**(G') packs (König property), namely,

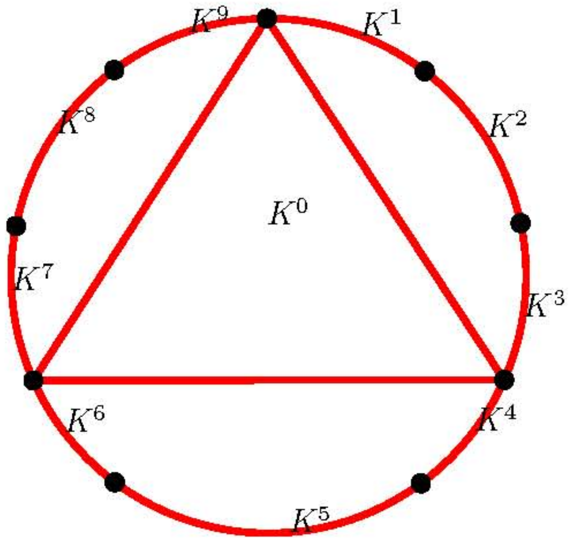
ν(**K**(G'))=τ(**K**(G')) for each induced subgraph G' of G.

We can think of **K**(G) either to as a clutter (clique-clutter) or to as a matrix (clique matrix)

Clique-perfect graphs



$$\nu(\mathbf{K}(G)) = \tau(\mathbf{K}(G)) = 3$$



$$\nu(\mathbf{K}(G)) = \nu(C_9) = 4 < \tau(\mathbf{K}(G)) = 5 = \tau(C_9)$$

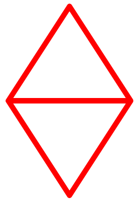
Clique-perfect graphs

The name *clique-perfect* is introduced in *V.Guruswami, C. Pandu Rangan, Algorithmic aspects of clique transversal and clique-independent sets, Discrete Applied Mathematics 100 (2000), 183--202.*

Our main references is: *F. Bonomo, M. Chudnovsky, G. Duran, Partial characterizations of clique-perfect graphs II: Diamond-free and Helly circular-arc graphs, Discrete Mathematics 309 (2009) 3485--3499.*

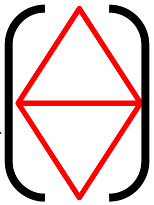
Clique-perfect diamond-free graphs

G is ***diamond-free***, no induced copy of $K_4 - e$



The cliques of G are edge-disjoint;

K(G) is ***linear***, namely, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is not a submatrix of **K**(G)

K () $= \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

A notational remark.

If G' is an induced subgraph of G, then **K**(G') is a *kind of minor* of **K**(G) (widely used for perfect matrices):

delete the columns whose index is in

V(G)-V(G') and retain the **nondominated** rows.

A notational remark.

This motivates the following:

An ***up-minor*** of $\mathbf{A} \in \{0,1\}^{M \times N}$ is a submatrix of the form $\mathbf{B}^{\uparrow}$ where: $\mathbf{B} \in \{0,1\}^{M \times L}$, $L \subseteq N$ and $\uparrow$ means deletion of dominated rows.

So: G' is an induced subgraph of G $\Leftrightarrow$ **K**(G') is an up-minor of **K**(G).

After the remark…..

G is clique perfect

⇕

K(G) **packs by up-minors**

…But the title says «Balanced….»

A 0,1-matrix **A** is ***balanced*** if **A** does not contain the adjacency matrix **C** of an odd polygon (contains means: up to permutations of rows and columns)

$$\begin{pmatrix} 1 & 1 & 0 & 0 & : & 0 \\ 0 & 1 & 1 & 0 & : & 0 \\ 0 & 0 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & 1 & : & 0 \\ : & : & : & : & : & : \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

…But the title says «Balanced….»

Theorem (Berge and Las Vergnas, 1970) *A 0,1-matrix **A** is balanced if and only if each submatrix of **A** packs.*

Hence, if G is such that K(G) is balanced, then G is clique-perfect (every up-minor is a submatrix)….look for the converse…**it fails in general but for linear matrices or diamond-free graphs**…

Here is why the title says «Balanced…»

Th. (N.A., Anna Galluccio) Let **A** be a linear matrix. The following statements are equivalent

- **A** is balanced
- $\mathbf{A}^{\uparrow}$ is balanced
- $\mathbf{A}^{\uparrow}$ is the clique matrix of some clique-perfect diamond-free graph

The same result was achieved by Safe in his Ph.D. thesis using a different approach.

Both the results answer an open question posed by Bonomo, Chudnovsky and Duran: ***recognize clique-perfect diamond-free graphs.***

Our proof is easy and elementary. It also follows by our characterization of minimally clique-imperfect (minimally non balanced) diamond free-graphs.

Otherwise stated

Th. Let **A** be a linear matrix. Then

ν(**A’**)=τ(**A’**) for each **submatrix** **A'** of **A**

$$\Updownarrow$$

ν(**A’**)=τ(**A’**) for each **up-minor** **A’** of **A**.

i.e., a discount

Hence

We can speak of ***balanced diamond-free graphs*** as those diamond-free graphs whose matrix is balanced.

A diamond-free graph is ***minimally nonbalanced*** if its clique matrix is not balanced but the clique matrix of every of its induced subgraph is.

A linear matrix is ***minimally nonbalanced*** if A is not balanced but every of its proper up-minor is balanced.

Sketch of the easy proof

A balanced $\Rightarrow$ König property holds for up-minors (trivially from Berge and Las Vergnas' Characterization)

Let us prove the contrapositive of the converse statement, namely,

<u>**A** not balanced $\Rightarrow$ $\exists$ up-minor **B** of **A** that does not pack.</u>

Sketch of the easy proof

Let $\mathbf{A} \epsilon \{0,1\}^{m \times n}$; g the least order of an odd cycle submatrix of **A**; **C** an odd cycle submatrix of **A** of order g. Thus $\mathbf{A} \sim \begin{pmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{F} & \mathbf{E} \end{pmatrix}$. We show that if $\mathbf{B} = \begin{pmatrix} \mathbf{C} \\ \mathbf{F} \end{pmatrix}^{\uparrow}$, then $\nu(\mathbf{B}) < \tau(\mathbf{B})$.

Sketch of the easy proof

If g=3, then the rows of **F** have ≤1 nonzero entries because **A** is linear

$$\mathbf{C}\quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

A row of **F**

Sketch of the easy proof

In any case the rows of **F** cannot have exactly two nonzero entries because

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

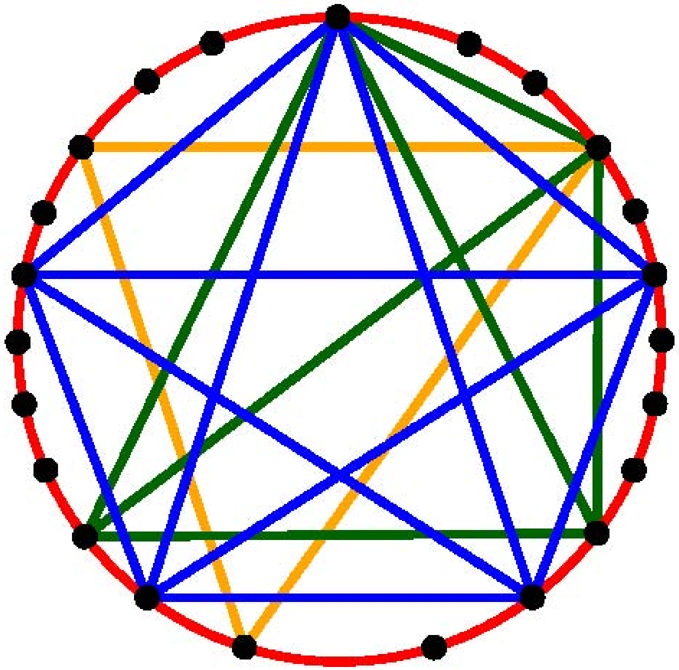
A row of **F**

Remark: rows of **F** induce stable sets in **C**

Sketch of the easy proof

Hence the rows of **F** have either 1 or at least 3 nonzero entries and

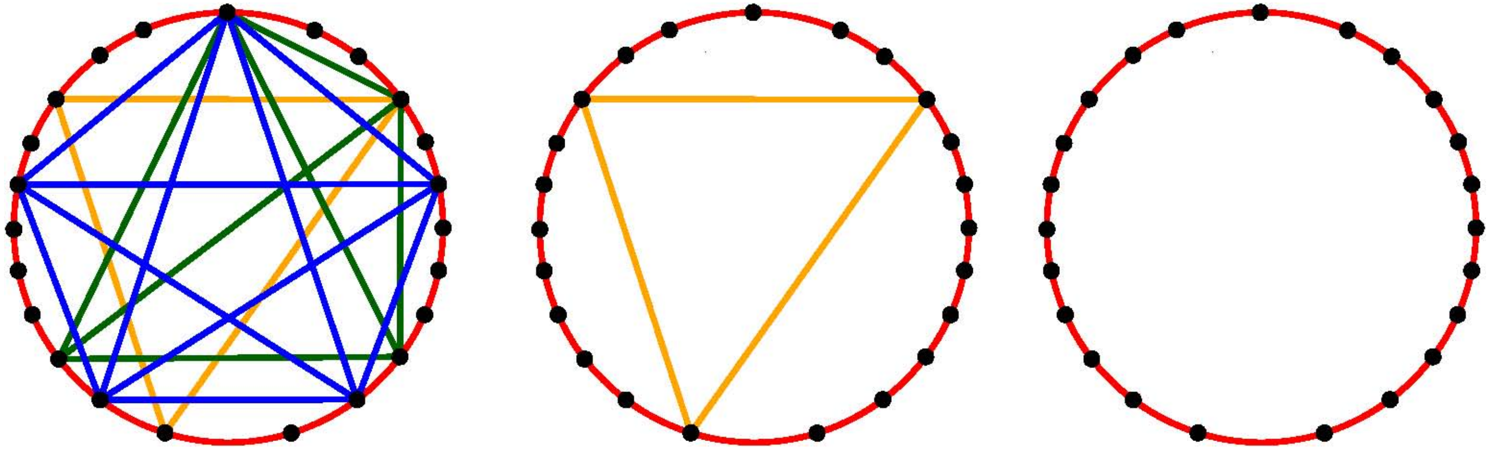
$$\mathbf{B} = \begin{pmatrix} \mathbf{C} \\ \mathbf{F} \end{pmatrix}^{\uparrow}$$ is the clique matrix of something like



Where **C** is the 'rim' and the rows of **F** are inscribed cliques.

Sketch of the easy proof

We are almost done: notice that the edges of the rim are maximal cliques, therefore we need at least $\tau(\mathbf{C}) \leq \tau(\mathbf{B})$ to cover all cliques. On the other hand, since the inscribed cliques have at least three vertices one has $\nu(\mathbf{C}) = \nu(\mathbf{B})$.



Hence **B** does not pack.

Sketch of the easy proof

- The proof above is rather easy: a few lines (some of which already encripted in Conforti and Rao, [M. Conforti, M. R. Rao. Structural properties and decomposition of
- linear balanced matrices. Mathematical Programming, 55:129--168, 1992.]) . However, it is strong enough to yield a polytime recognition algorithm for clique-perfect diamond-free graphs and, moreover, with some extra work, it leads to the following:

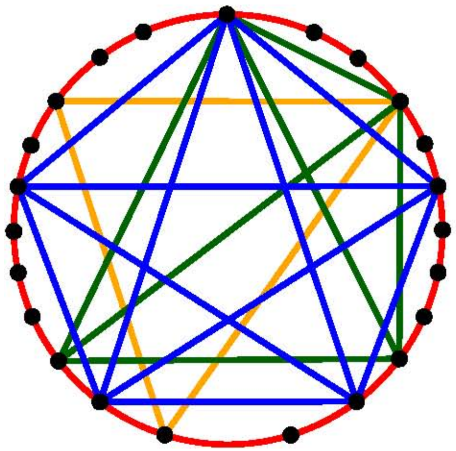
Th. (N.A., Anna Galluccio) A is a linear minimally non balanced matrix (by up-minors) if and only if it is the clique-matrix of a Hereditary odd hole free-multisun.

Hereditary Odd Hole (HOH)-free Multisuns

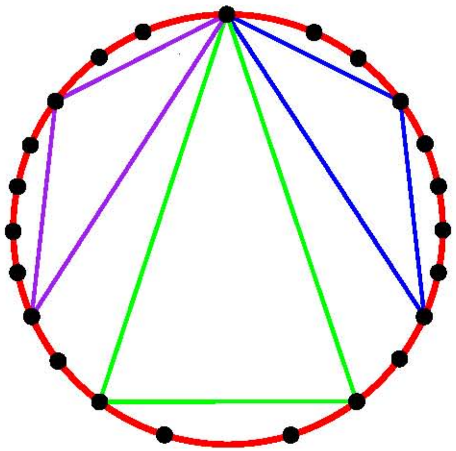
A **multisun** is a diamond-free graph G of odd order such that its maximal cliques of size 2 span a Hamiltonian cycle C of G (the ***rim*** of G). All the others maximal cliques have therefore at least three vertices and are ***inscribed*** in C.

A multisun is ***odd hole-free*** if it does not contain odd holes. It is ***hereditarily odd hole-free***, if so is the graph obtained after the removal of the edge-sets of some (but not all) arbitrarily chosen inscribed cliques

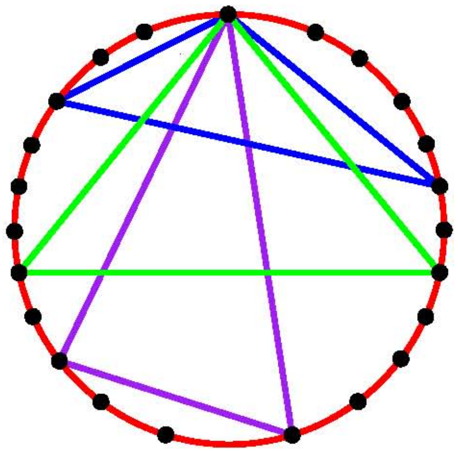
Hereditary Odd Hole (HOH)-free Multisuns



Multisun

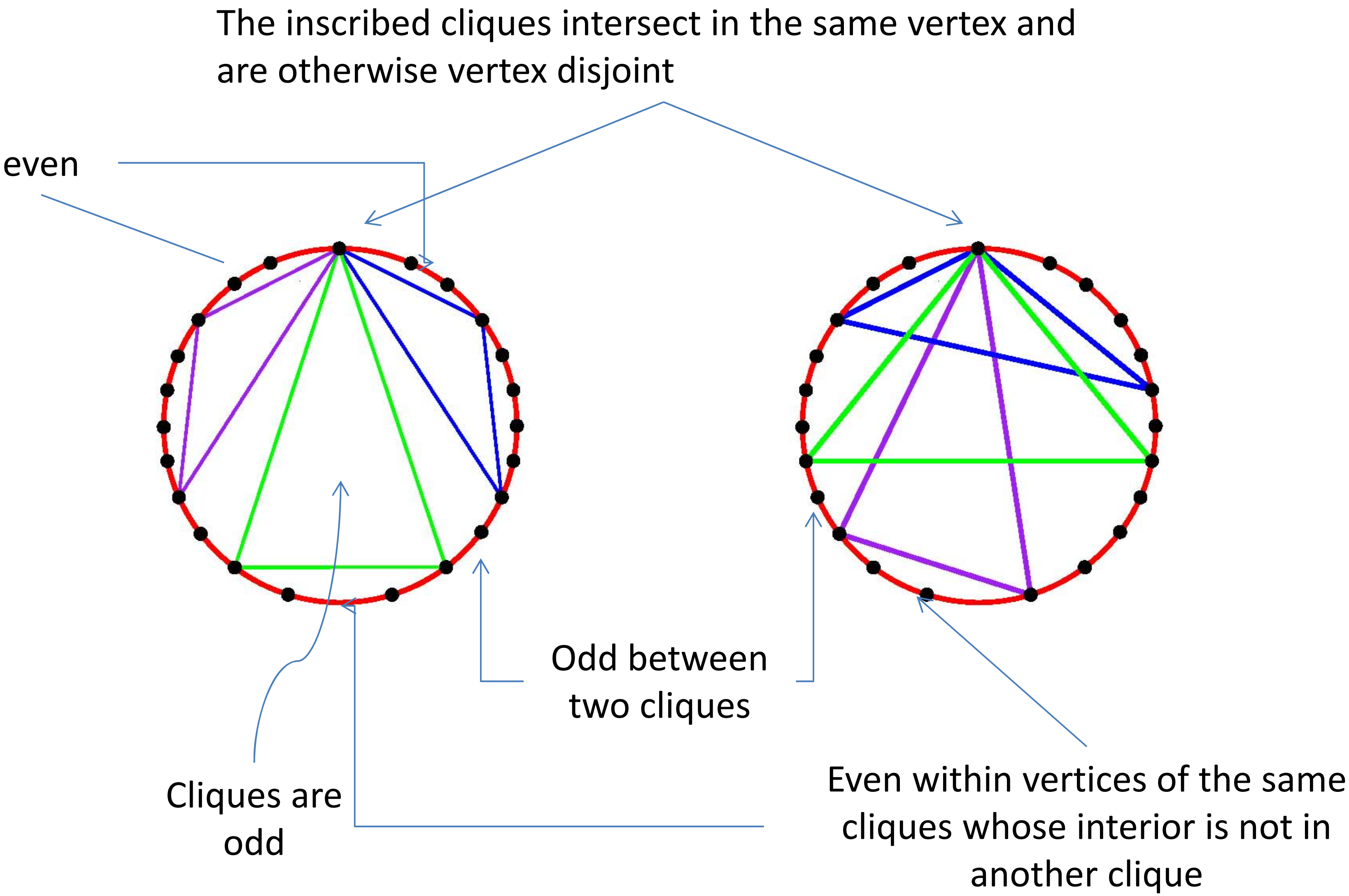


Multisun with no odd hole



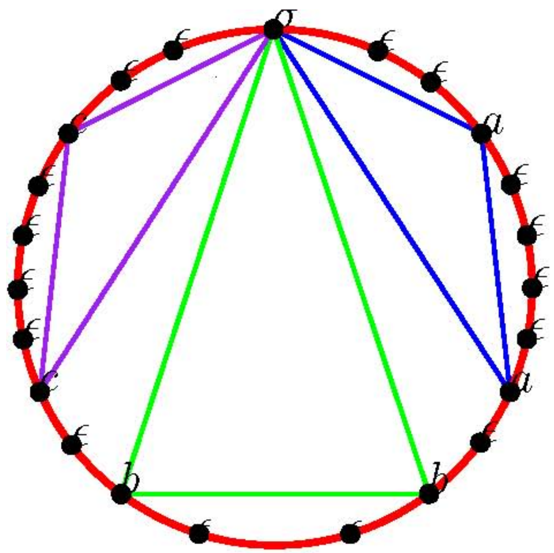
HOH-free multisun

HOH-free Multisuns: N-Conditions.

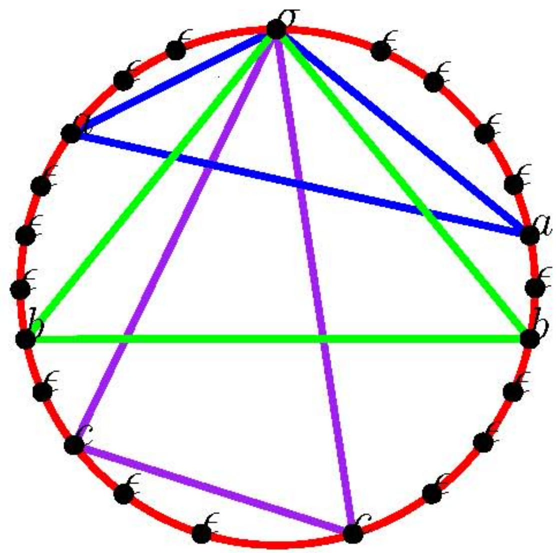


HOH-free Multisuns: sunwords.

After the N-Conditions we see that we can label multisuns as follows



$\sigma a^2 \epsilon b^2 \epsilon c^2$



$\sigma a \epsilon b \epsilon c^2 \epsilon b \epsilon a$

Working modulo the automorphism group of the rim and with the relations $\varepsilon^2 = \phi$, we can associate with a multisun G satisfying the N-conditions a (cyclic) word **w**(G) over the alphabet $\Sigma = \{\varepsilon, \sigma, a, b, c \ldots\}$.

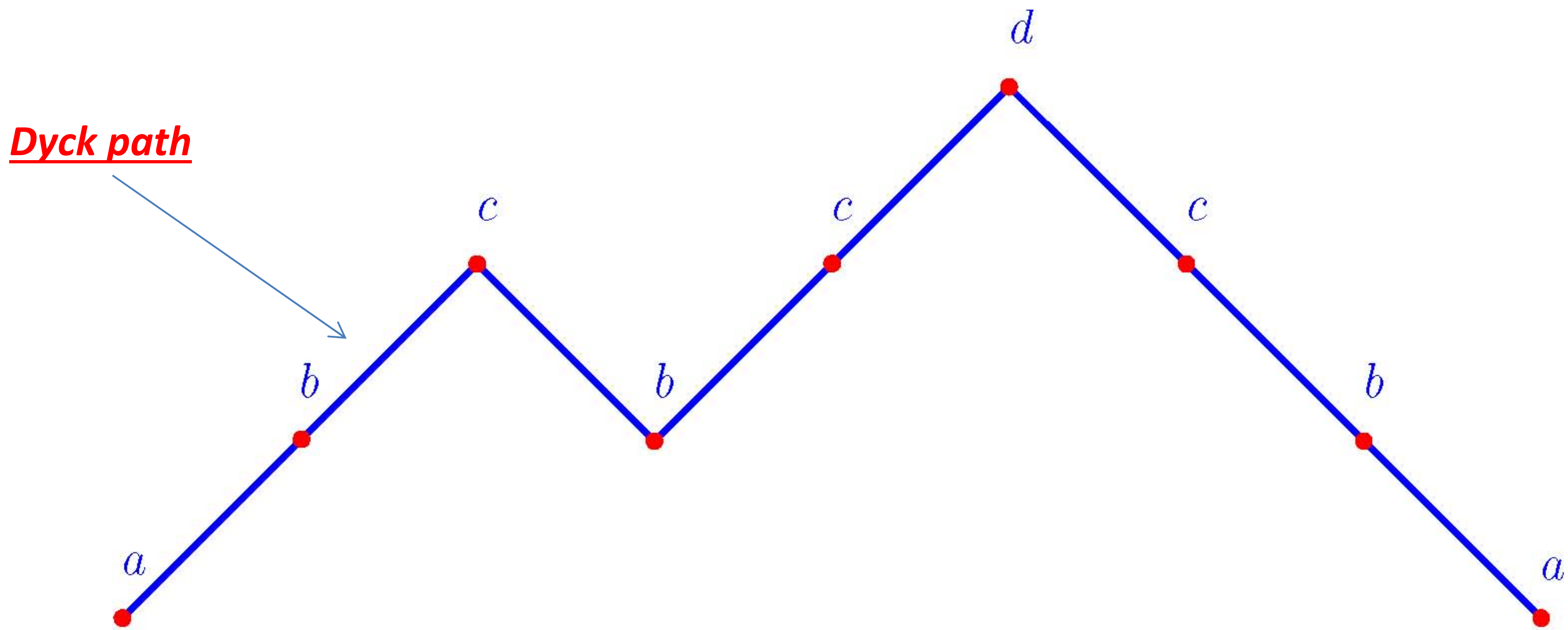
HOH-free Multisuns: sunwords.

Let Σ-ε be totally ordered by closeness to σ. A sunword contains a ***jump*** if two adjacent proper letters in the word are not consecutive in the order above. A sunword satisfies the ***parity conditions*** if the exponent of a proper letter x is even only whenever the latter is interlaced by the same letter y; odd otherwise.

Th. (N.A., Anna Galluccio). Let G be a diamond-free graph. The following statements are equivalent:
G is minimally non balanced (minimally clique-imperfect);
G ia a HOH-free multisun;
G is a multisun that satisfies the N-conditions hereditarily;
the sunword of G is jump-free and satifies the parity conditions.

HOH-free Multisuns: Dyck-paths.

Jump-freeness models a kind of continuity of the labels. Hence the following is now not surprising:



$$\sigma a^{\lambda_1}\epsilon b^{\lambda_2}\epsilon c^{\lambda_3}\epsilon b^{\lambda_4}\epsilon c^{\lambda_5}\epsilon d^{\lambda_6}\epsilon c^{\lambda_7}\epsilon b^{\lambda_8}\epsilon a^{\lambda_9}$$