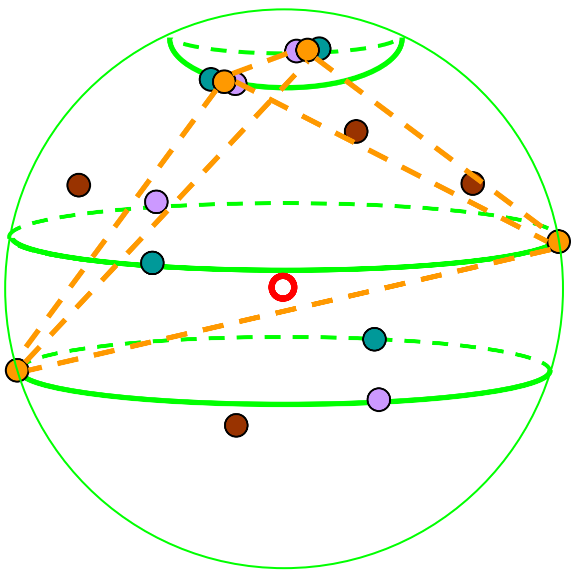
Combinatorial approaches to the colourful simplicial depth



Antoine Deza (McMaster)

based on joint work with

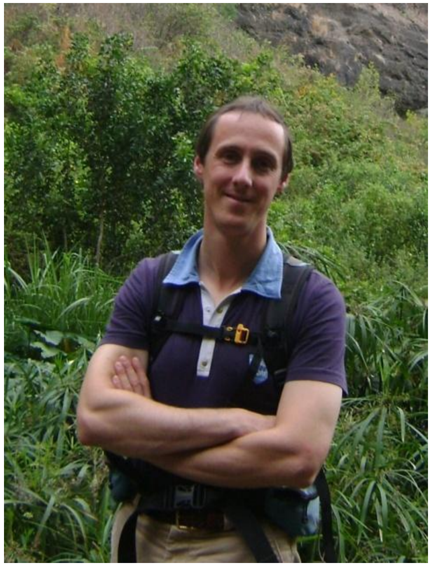
Frédéric Meunier (ENPC Paris)

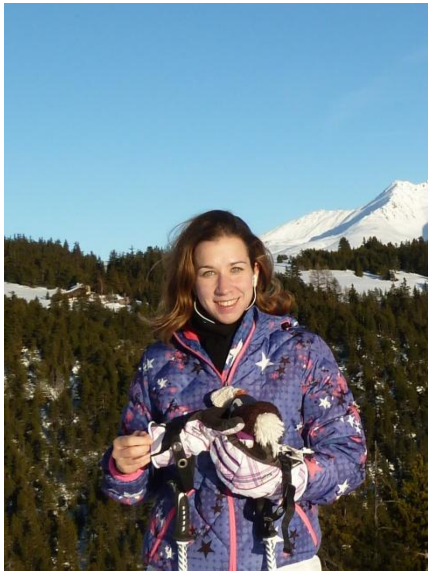
Pauline Sarrabezolles (ENPC Paris)

McMaster University

Advanced Optimization Laboratory

Combinatorial approaches to the colourful simplicial depth



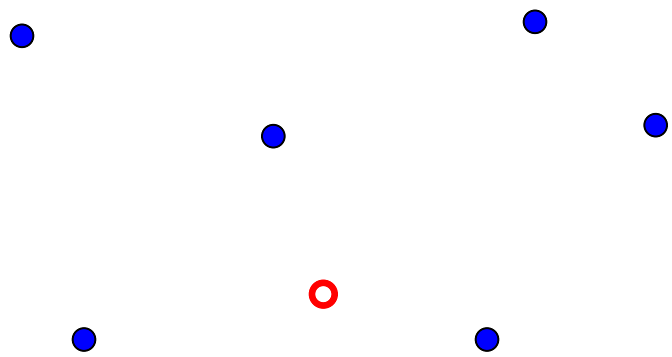


Frédéric Meunier
András academic son

Pauline Sarrabezolles
András academic granddaughter

András a few years ago

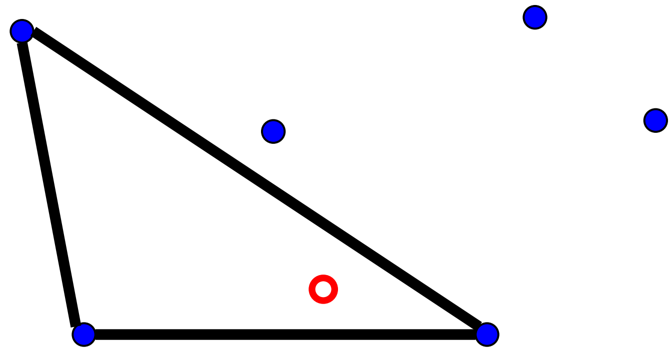
Carathéodory Theorem



Given a set S of n points in dimension d, then there exists an open simplex generated by points in S containing p

S, p general position

Carathéodory Theorem



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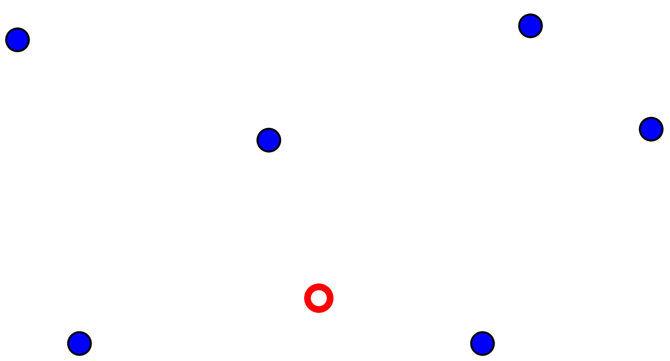
S, p general position

Simplicial Depth

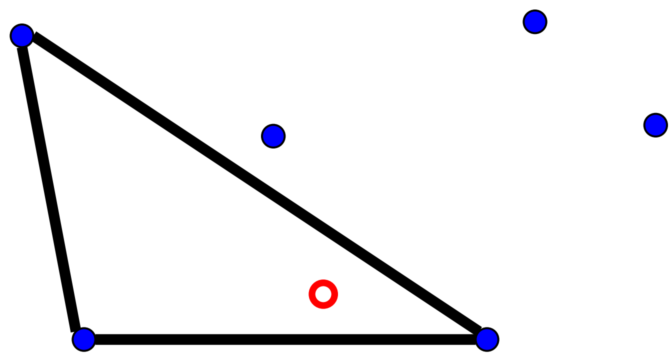


Given a set S of n points in dimension d, the *simplicial depth* of p is the number of open simplices generated by points in S containing p [Liu 1990]

S, p general position

Simplicial Depth

depth$_S$(*p*)=1

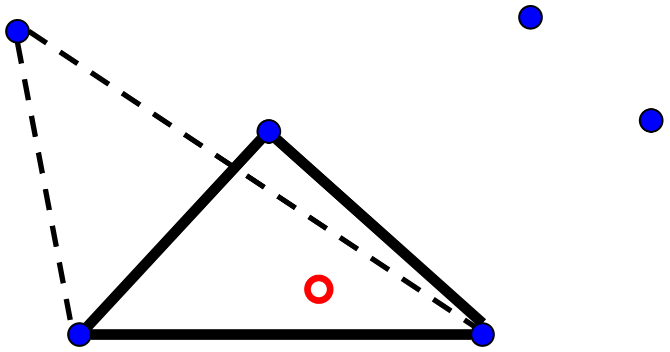


Given a set *S* of *n* points in dimension *d*, the *simplicial depth* of *p* is the number of open simplices generated by points in *S* containing *p* [Liu 1990]

S, *p general position*

Simplicial Depth

$depth_S(p)=2$

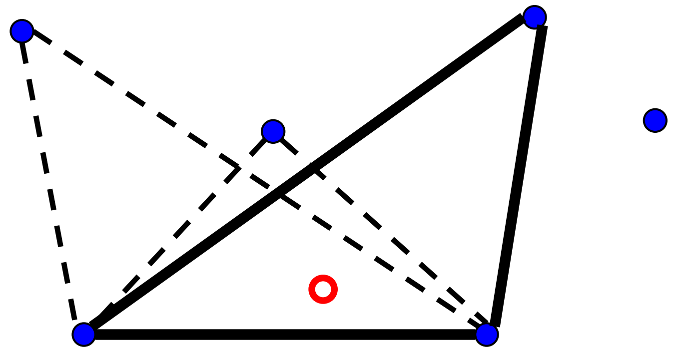


Given a set S of n points in dimension d, the *simplicial depth* of p is the number of open simplices generated by points in S containing p [Liu 1990]

S, p general position

Simplicial Depth

$depth_S(p)=3$

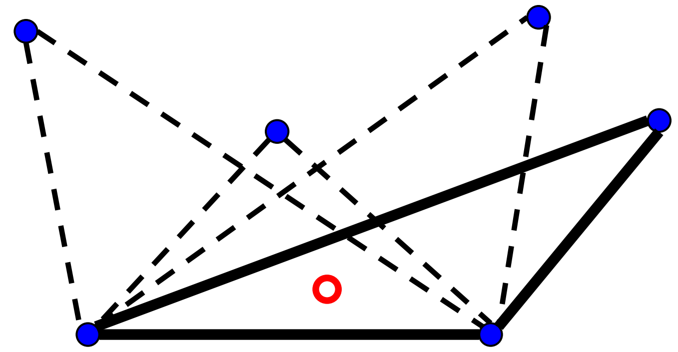


Given a set S of n points in dimension d, the *simplicial depth* of p is the number of open simplices generated by points in S containing p [Liu 1990]

S, p general position

Simplicial Depth

$depth_S(p)=4$



Given a set S of n points in dimension d, the *simplicial depth* of p is the number of open simplices generated by points in S containing p [Liu 1990]

S, p general position

Simplicial Depth

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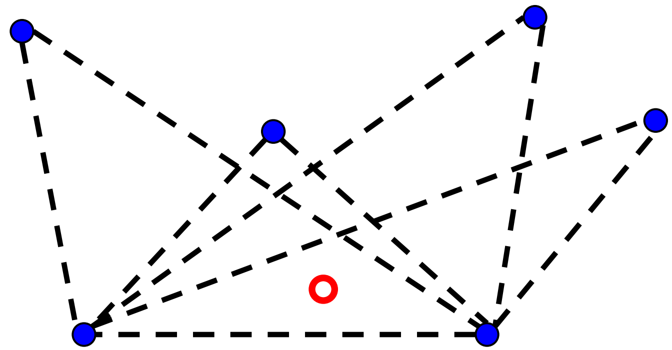
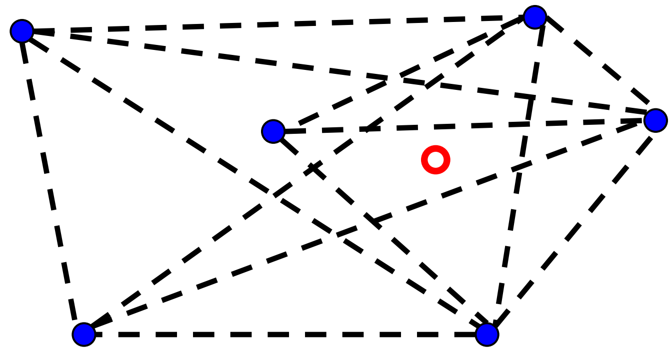


Given a set S of n points in dimension d, the *simplicial depth* of p is the number of open simplices generated by points in S containing p [Liu 1990]

S, p general position

Simplicial Depth

depth$_S$(p)=9



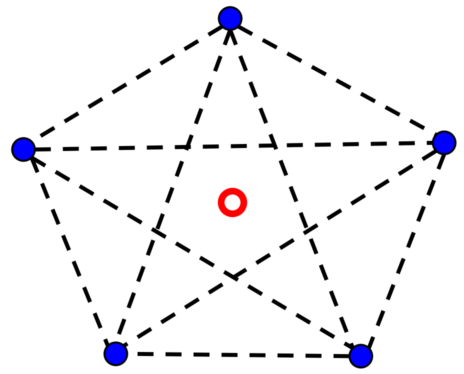
Given a set S of n points in dimension d, the *simplicial depth* of p is the number of open simplices generated by points in S containing p [Liu 1990]

S, p general position

Deepest Point in Dimension 2

Deepest point bounds in dimension 2 [Kárteszi 1955], [Boros, Füredi 1984], [Bukh, Matoušek, Nivasch 2010]

$$\frac{n^3}{27} + O(n^2) \le \max_p depth_S(p) \le \frac{n^3}{24} + O(n^2)$$



$depth_S(p) = 5$

S, p *general position*

Deepest Point in Dimension d

Deepest point bounds in dimension *d* [Bárány 1982]

$$\frac{1}{(d+1)^{d+1}}\binom{n}{d+1}+O(n^d)\leq \max_p depth_S(p)\leq \frac{1}{2^d(d+1)!}n^{d+1}+O(n^d)$$

S, *p* *general position*

Deepest Point in Dimension d

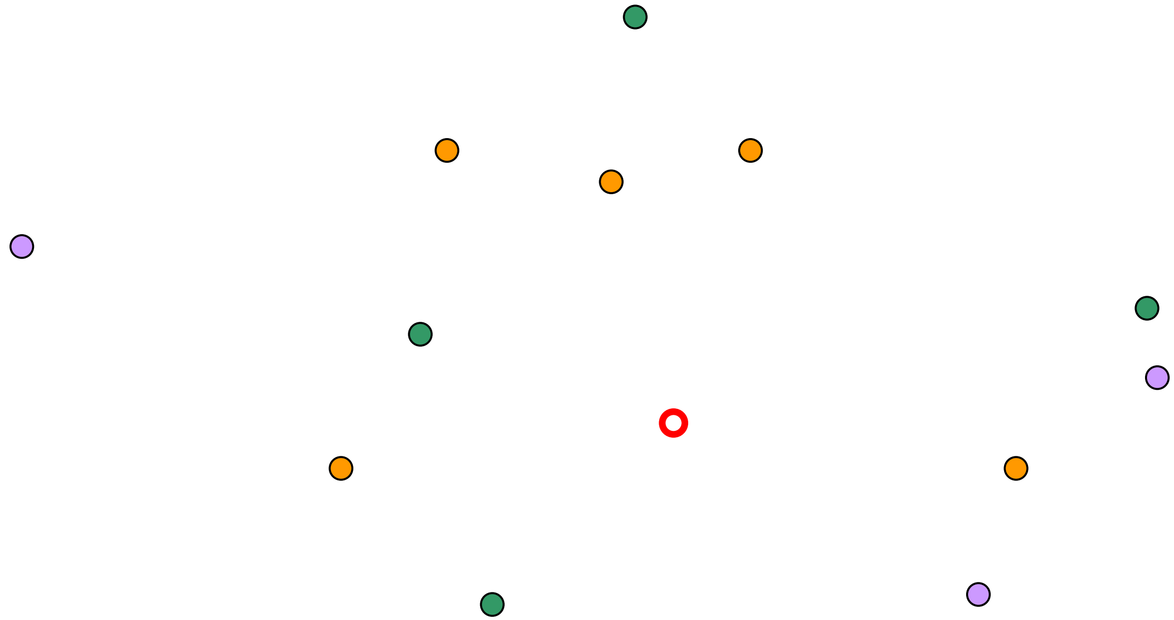
Deepest point bounds in dimension d [Bárány 1982]

$$\frac{1}{(d+1)^{d+1}}\binom{n}{d+1}+O(n^d)\leq \max_{p} depth_S(p) \leq \frac{1}{2^d(d+1)!}n^{d+1}+O(n^d)$$

- tight upper bound
- lower bound uses Colourful Carathéodory theorem
- breakthrough [Gromov 2010] & further improvements

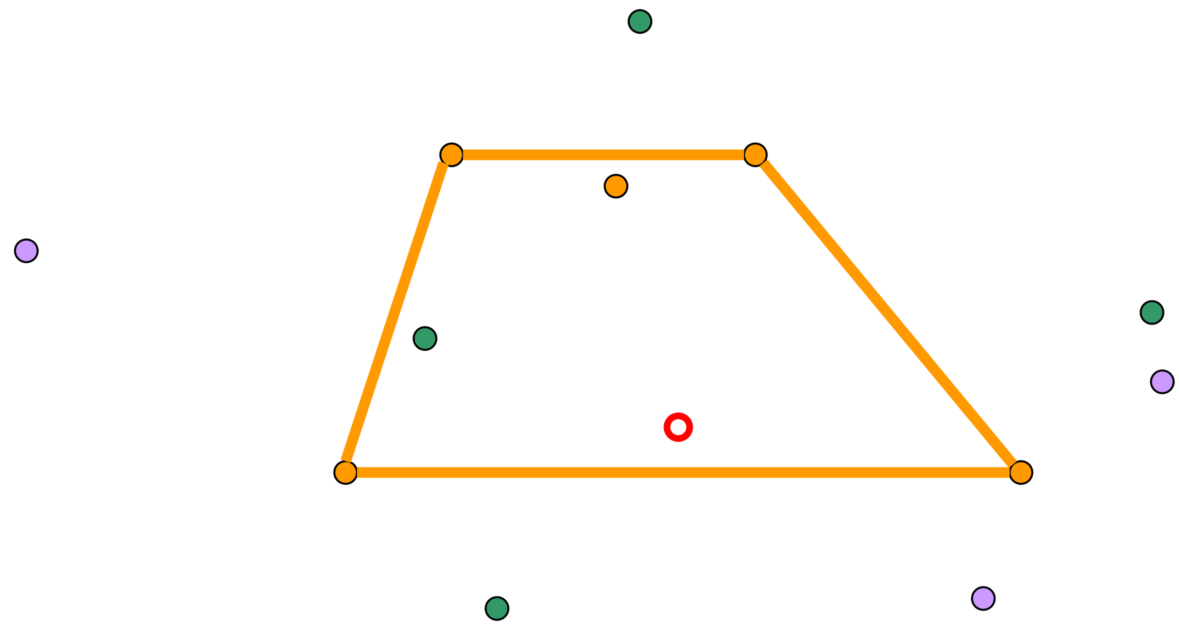
S, p general position

Colourful Carathéodory Theorem



Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension d, and $p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \ldots \cap \text{conv}(S_{d+1})$, there exists a colourful simplex containing p [Bárány 1982]

Colourful Carathéodory Theorem



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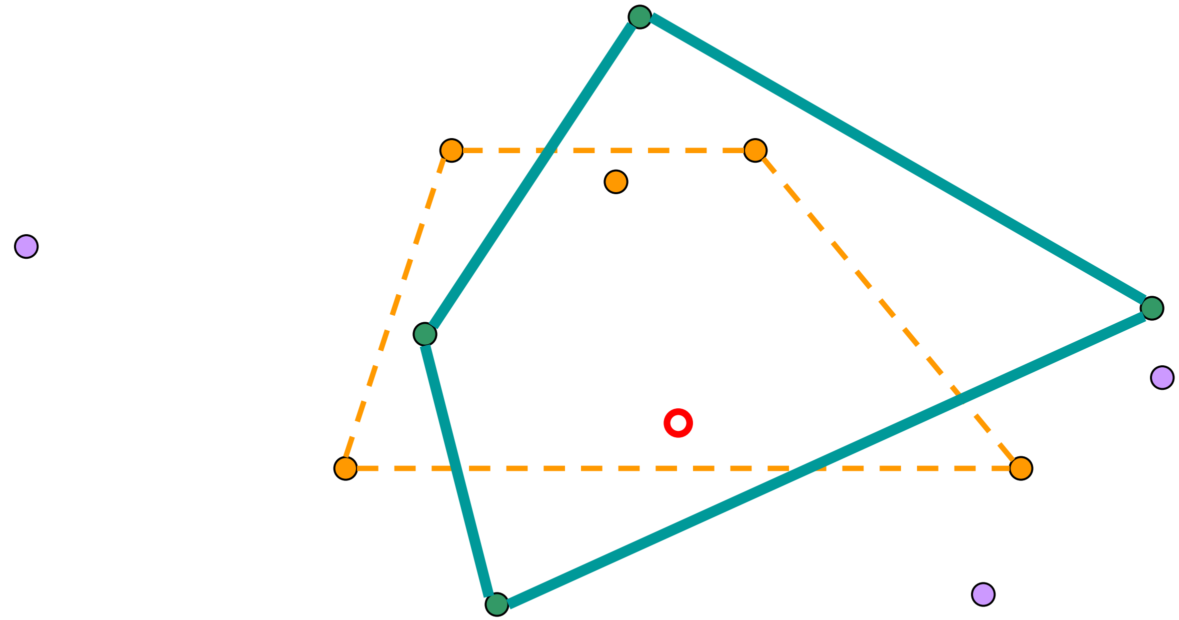

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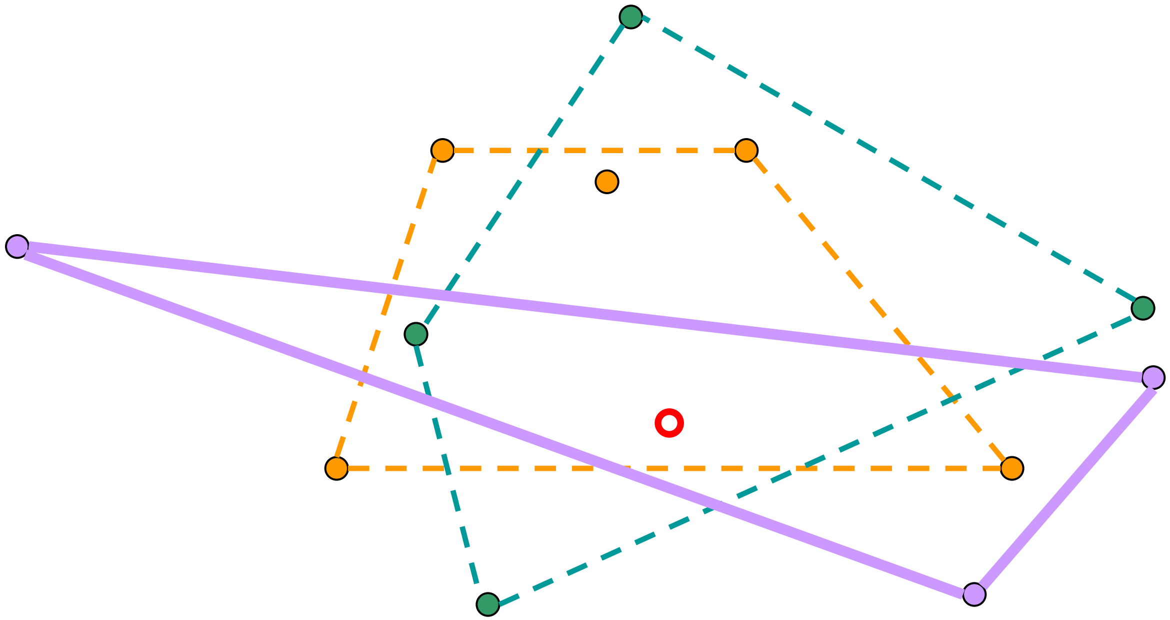

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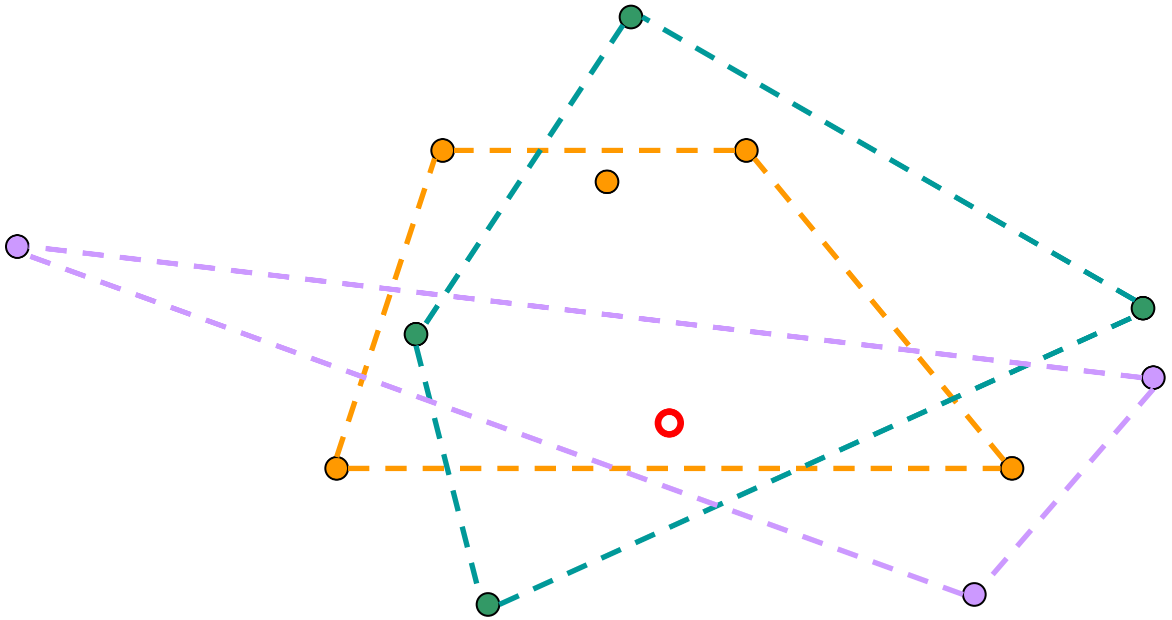

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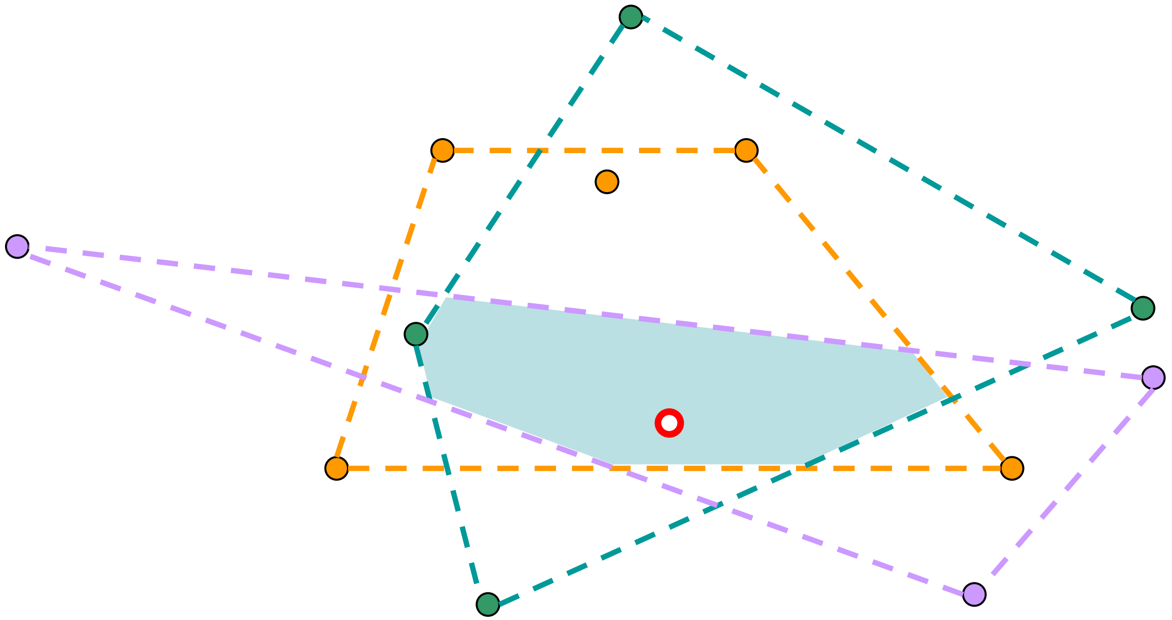

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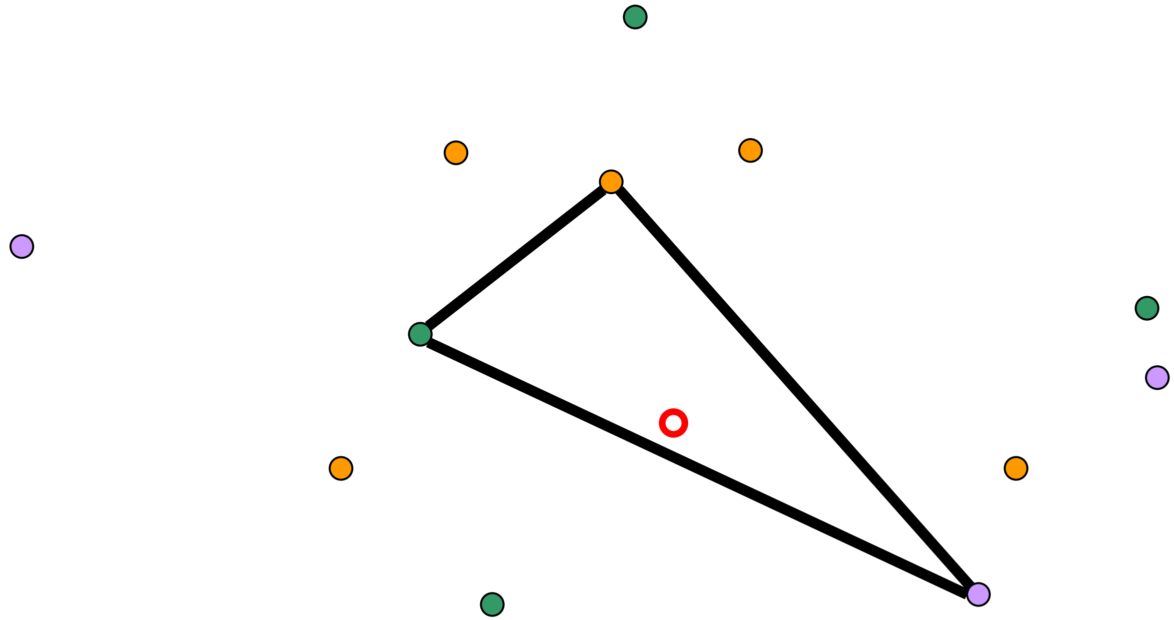
Colourful Simplicial Depth



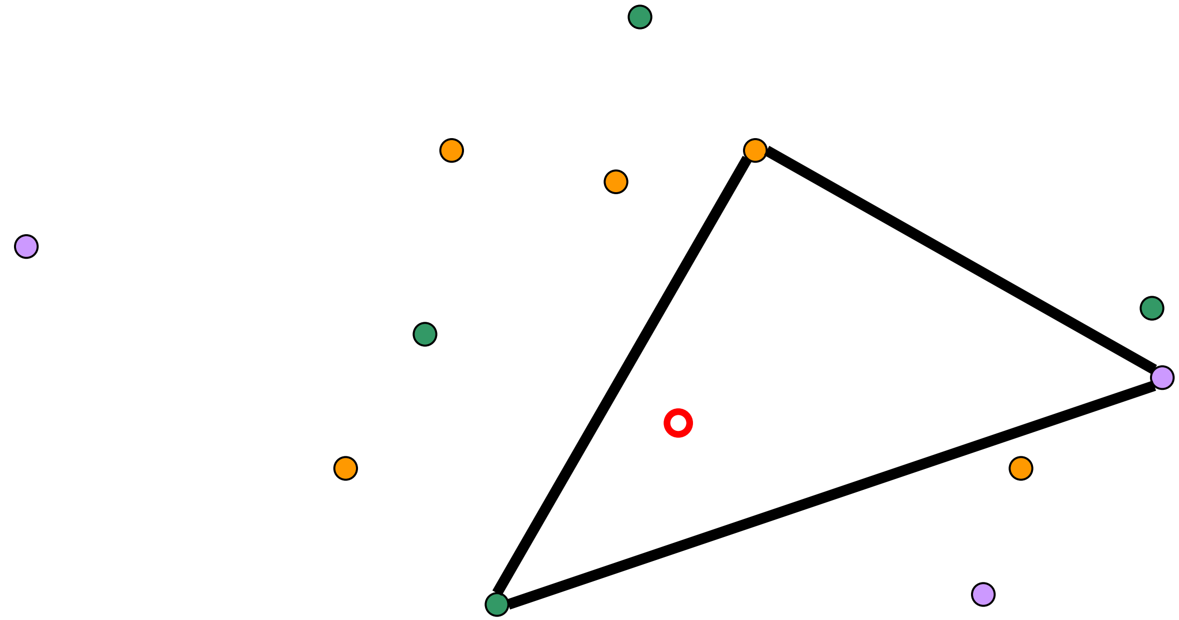
$depth_{\boldsymbol{S}}(p)=1$

Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension d, the *colourful simplicial depth* of p is the number of open colourful simplexes generated by points in $\boldsymbol{S}$ containing p

***S**, p general position*

$p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S_2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$

Colourful Simplicial Depth



$depth_S(p)=2$

Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension d, the *colourful simplicial* depth of p is the number of open colourful simplexes generated by points in $\boldsymbol{S}$ containing p

$\boldsymbol{S}$, p *general position*

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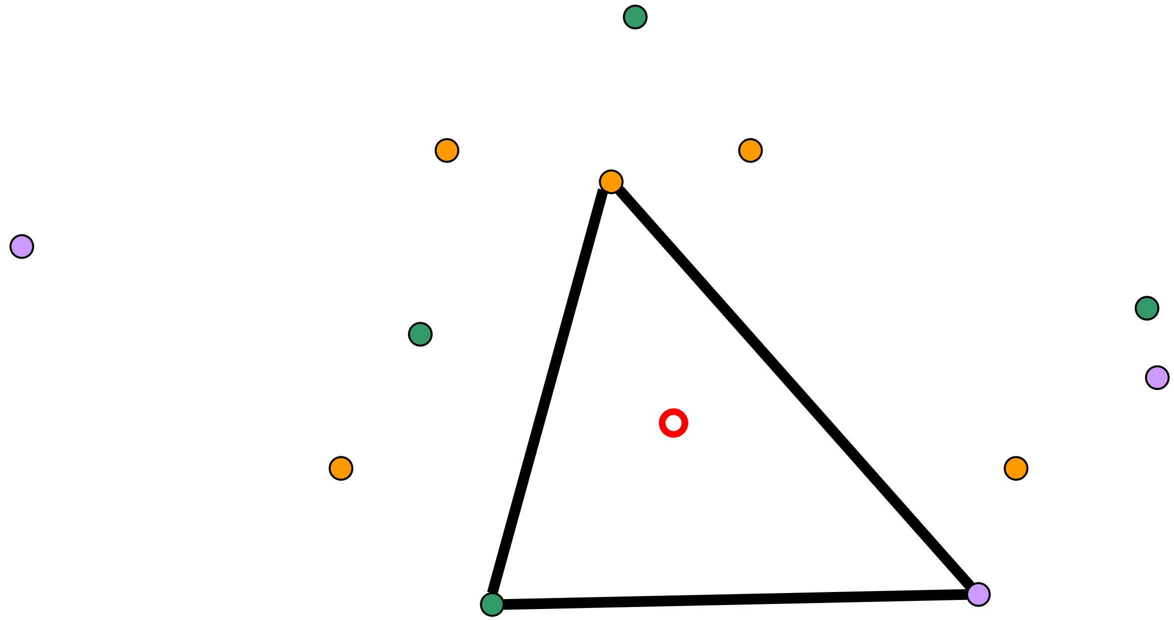
Colourful Simplicial Depth



$depth_{S}(p)=3$

Given colourful set $S = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension d, the *colourful simplicial depth* of p is the number of open colourful simplexes generated by points in S containing p

S, p general position

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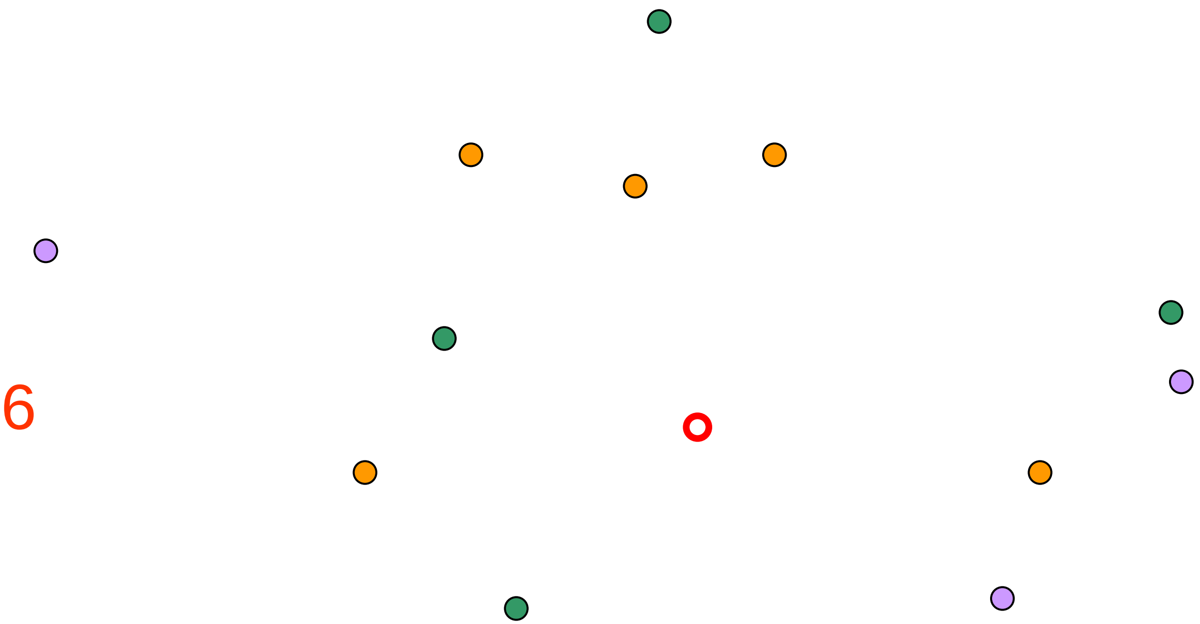
Colourful Simplicial Depth



depth$_{\boldsymbol{S}}$(p)=16

Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension d, the *colourful simplicial depth* of p is the number of open colourful simplexes generated by points in $\boldsymbol{S}$ containing p

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Deepest Point in Dimension d

Deepest point bounds in dimension *d* [Bárány 1982]

$$\frac{\mu(d)}{(d+1)^{d+1}}\binom{n}{d+1}+O(n^d)\le\max_p depth_S(p)\le\frac{1}{2^d(d+1)!}n^{d+1}+O(n^d)$$

with $\mu(d)=\min_{S,p} depth_S(p)$

[Bárány 1982]: $\mu(d)\ge 1$

S, p general position

Deepest Point in Dimension d

$$\max_{p} depth_S(p) \geq c_d \binom{n}{d+1}$$

[Bárány 1982] $c_d \geq \frac{d+1}{(d+1)^{(d+1)}}$

[Wagner 2003] $c_d \geq \frac{d^2+1}{(d+1)^{(d+1)}}$

[Gromov 2010] $c_d \geq \frac{2d}{(d+1)!(d+1)}$

simpler proofs: [Karazev 2012], [Matoušek, Wagner 2012]

d=3: [Kráľ, Mach, Sereni 2012]

S, p general position

Colourful Research Directions

- Generalize the sufficient condition of Bárány for the existence of a colourful simplex
- Determine $\mu(d) = \min_{S,p} depth_S(p)$
- Computational approaches for $\mu(d)$ for small *d*.
- Obtain an efficient algorithm to find a colourful simplex : *Colourful Linear Programming Feasibility* problem

$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \ldots \cap \text{conv}(S_{d+1})$

S, *p general position* and $|S_1|, |S_2|, \ldots, |S_{d+1}| \geq d+1$

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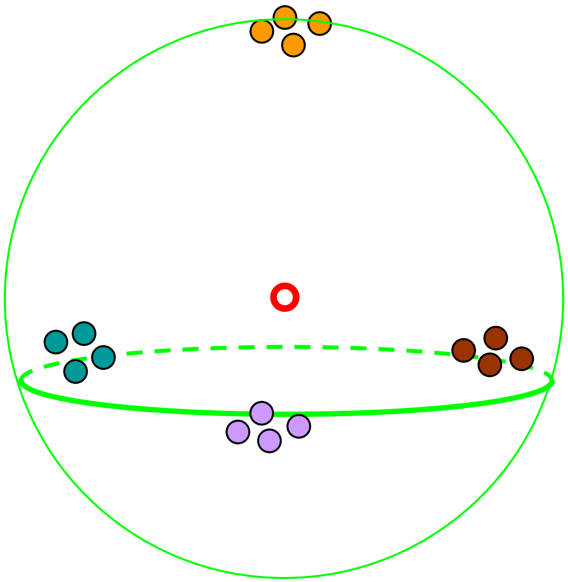
Colourful Carathéodory Theorems

[Bárány 1982] Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension d, and $p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \ldots \cap \text{conv}(S_{d+1})$, then there exists a colourful simplex containing p

[Holmsen, Pach, Tverberg 2008] and [Arocha, Bárány, Bracho, Fabila, Montejano 2009] Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension d, and $p \in \text{conv}(S_\text{i} \cup S_\text{j})$ for $1 \leq i < j \leq d+1$, then there exists a colourful simplex containing p

[Meunier, D. 2013] Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension d, if for $1 \leq i < j \leq d+1$ there exists $k \neq i$, $k \neq j$, such that for all $x_k \in S_k$ the ray $[x_k\, p)$ intersects $\text{conv}(S_\text{i} \cup S_\text{j})$ in a point distinct from x_k, then there exists a colourful simplex containing p

Colourful Carathéodory Theorems



[Meunier, D. 2013] Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension d, if for $i \neq j$ the open half-space containing p and defined by an i-facet of a colourful simplex intersects $S_i \cup S_j$, then there exists a colourful simplex containing p

❖ *further generalization in dimension 2*

Colourful Research Directions

- Generalize the sufficient condition of Bárány for the existence of a colourful simplex
- Determine $\mu(d) = \min_{S,p} depth_S(p)$
- Computational approaches for $\mu(d)$ for small *d*.
- Obtain an efficient algorithm to find a colourful simplex : *Colourful Linear Programming Feasibility* problem

$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \ldots \cap \text{conv}(S_{d+1})$

S, *p general position* and $|S_1|, |S_2|, \ldots, |S_{d+1}| \geq d+1$

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

[Bárány 1982] $$1 \leq \mu(d)$$

$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S_2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$

S, *p* *general position* and $|S_1|, |S_2|, \ldots, |S_{d+1}| \geq d+1$

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

[Bárány 1982]

$$d+1 \leq \mu(d)$$

$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

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Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

[Bárány 1982]

$$d+1 \le \mu(d)$$

[D.,Huang,Stephen,Terlaky 2006]

$$2d \le \mu(d) \le d^2+1$$

$$\mu(d) \text{ even for odd } d$$

$$\max_p depth_S(p) \ge \frac{\mu(d)}{(d+1)^{d+1}}\binom{n}{d+1} + O(n^d)$$

$p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S_2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$

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$$\mu(d) = \min_{S,p} depth_S(p)$$

[Bárány 1982] $d+1 \leq \mu(d)$

[D.,Huang,Stephen,Terlaky 2006] $2d \leq \mu(d) \leq d^2+1$

$\mu(d)$ even for odd d

[Bárány, Matoušek 2007] $\max(3d, \frac{d^2+d)}{5}) \leq \mu(d)$ for $d \geq 3$

$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S_2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$

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S, p *general position* and $|S_1|,|S_2|,\ldots,|S_{d+1}| \geq d+1$

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S, p} depth_S(p)$$

[Bárány 1982] $d + 1 \le \mu(d)$

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$\mu(d)$ even for odd d

[Bárány, Matoušek 2007] $3d \le \mu(d)$ for $d \ge 3$

[Stephen,Thomas 2008] $\left\lfloor \frac{(d+2)^2}{4} \right\rfloor \le \mu(d)$ for $d \ge 8$

$$\max_p depth_S(p) \ge \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S_2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$

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[D.,Stephen, Xie 2011] $\left\lceil \frac{(d+1)^2}{2} \right\rceil \le \mu(d) \text{ for } d \ge 4$

$$\max_p depth_S(p) \ge \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S_2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$

S, *p* *general position* and $|S_1|,|S_2|,\ldots,|S_{d+1}| \ge d+1$

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

$$\mu(1) = 2 \qquad \mu(2) = 5 \qquad \mu(3) = 10$$

$$\left\lceil \frac{(d+1)^2}{2} \right\rceil \leq \mu(d) \leq d^2 + 1 \quad \text{for } d \geq 4$$

$$\mu(d) \text{ even for odd } d$$

$$\text{conjecture: } \mu(d) = d^2 + 1$$

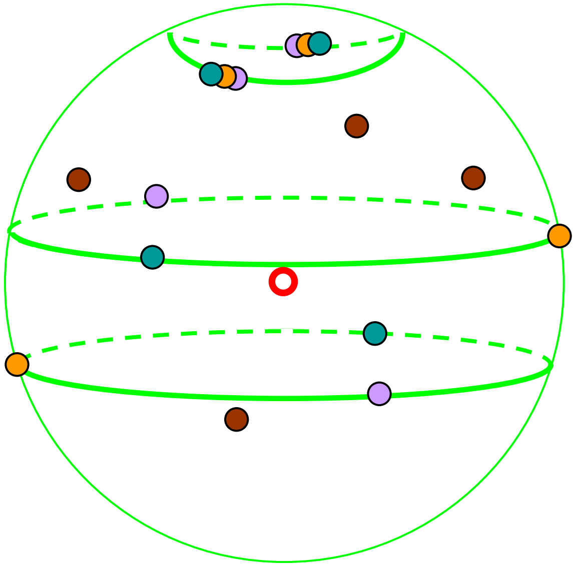
$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S_2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$

S, *p* *general position* and $|S_1|, |S_2|, \ldots, |S_{d+1}| \geq d+1$

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,\,p} depth_S(p)$$



$$\mu(3) = 10$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \ldots \cap \text{conv}(S_{d+1})$

S, *p general position* and $|S_1|, |S_2|, \ldots, |S_{d+1}| \geq d+1$

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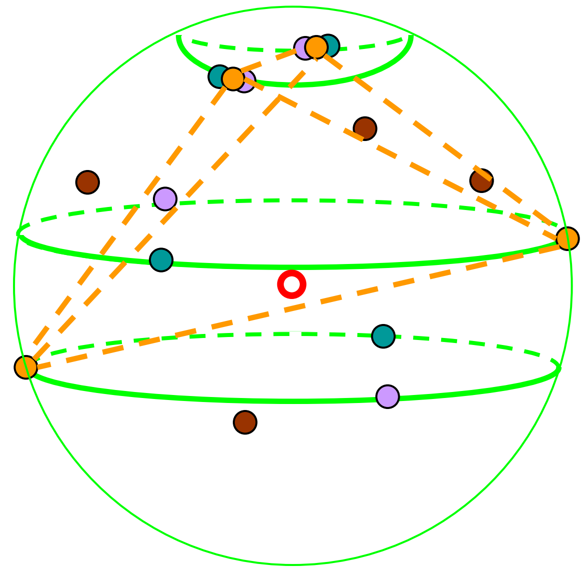

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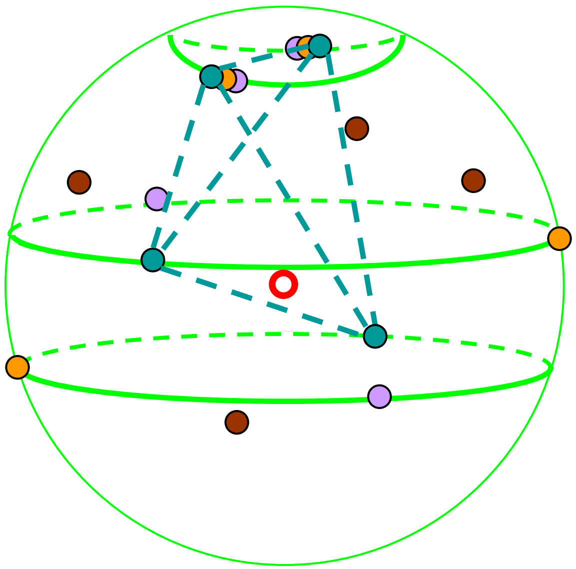

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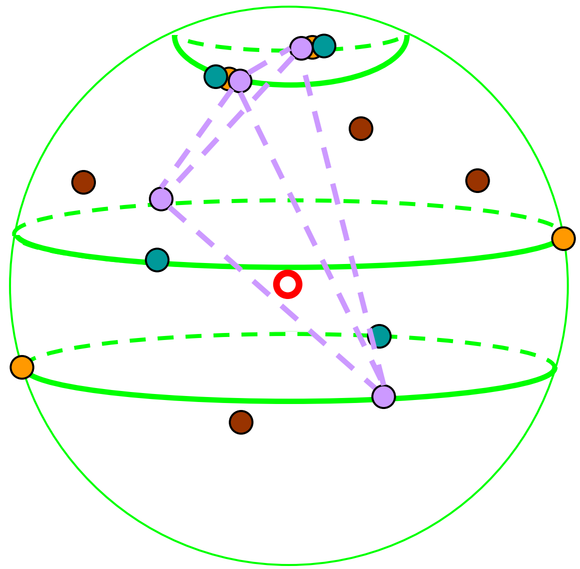

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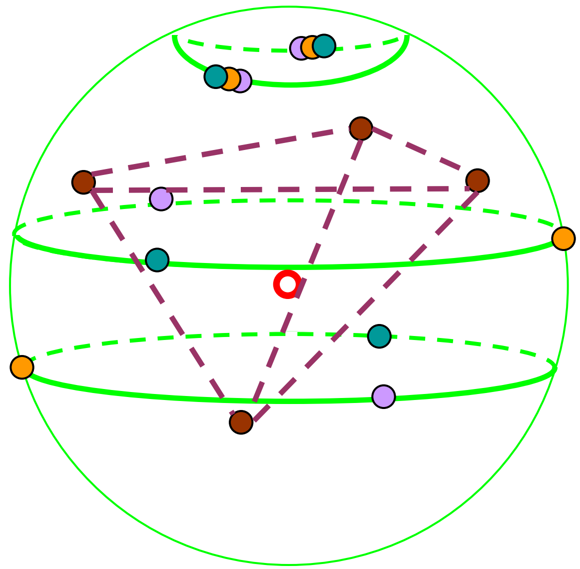

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Colourful Simplicial Depth Bounds

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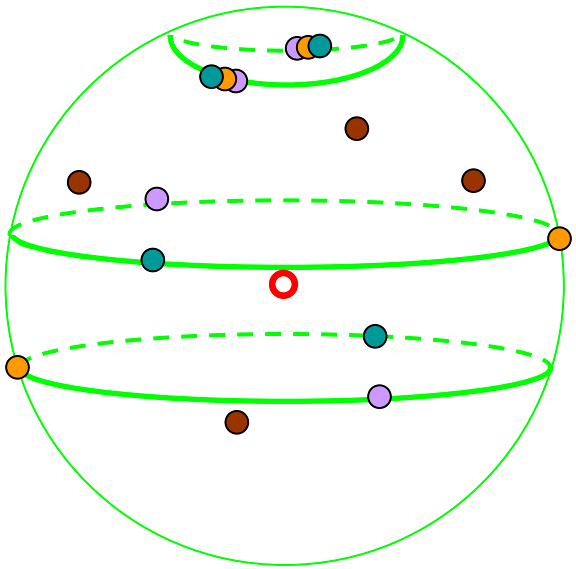
$depth_S(p)=10$



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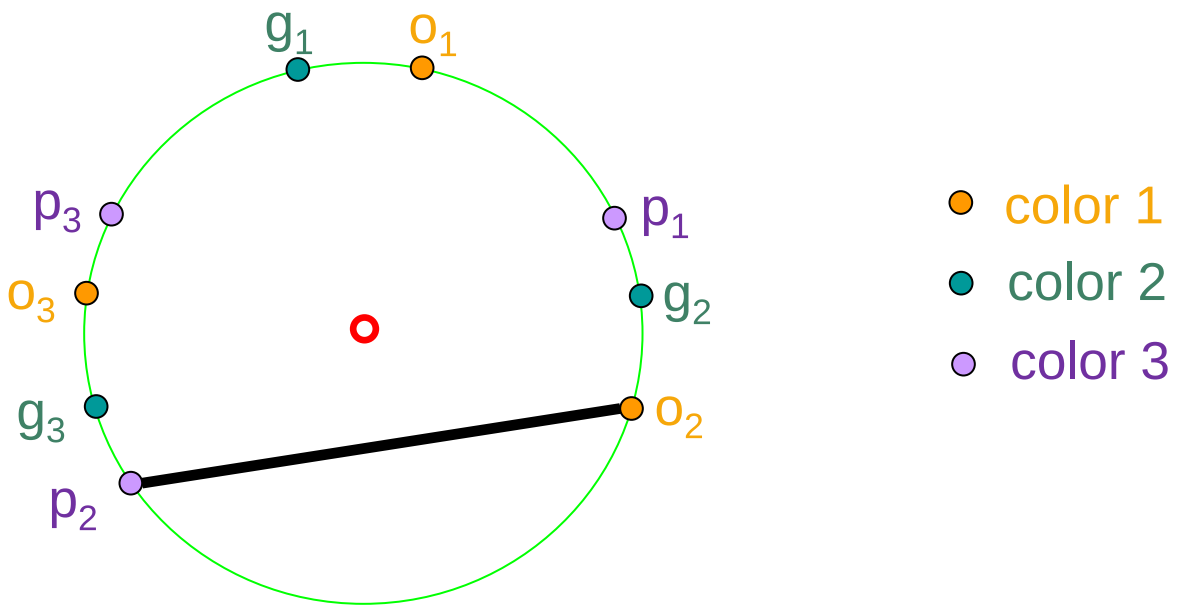
Transversal

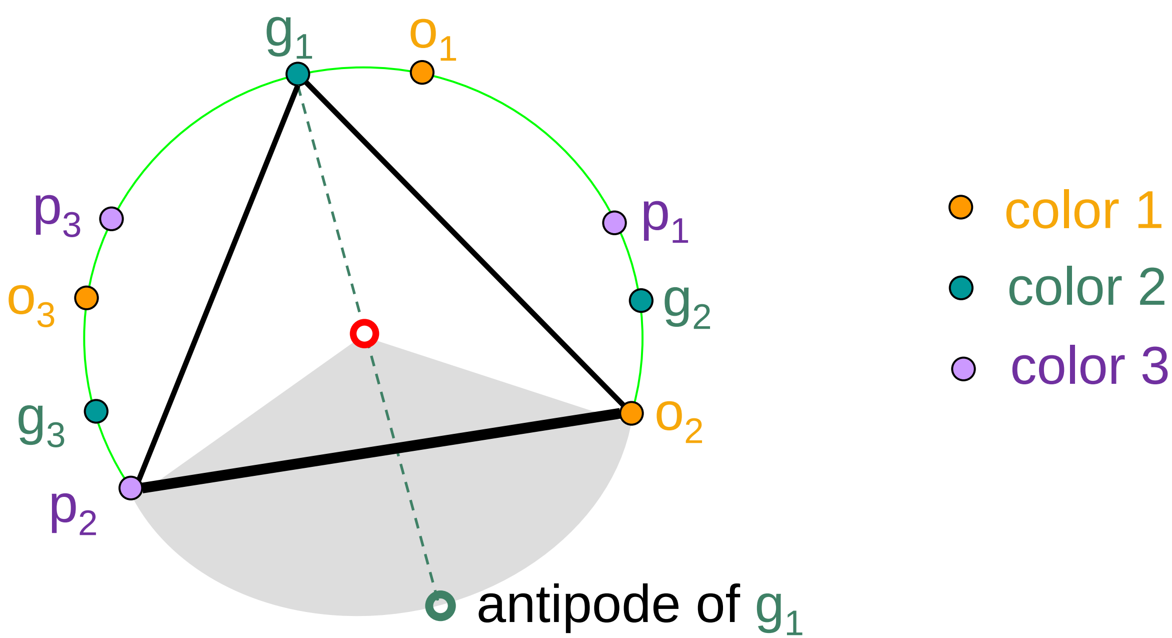
colourful set of d points (one colour missing)



$\hat{2}$ -transversal (o_2,p_2)

Transversal

colourful set of d points (one colour missing)

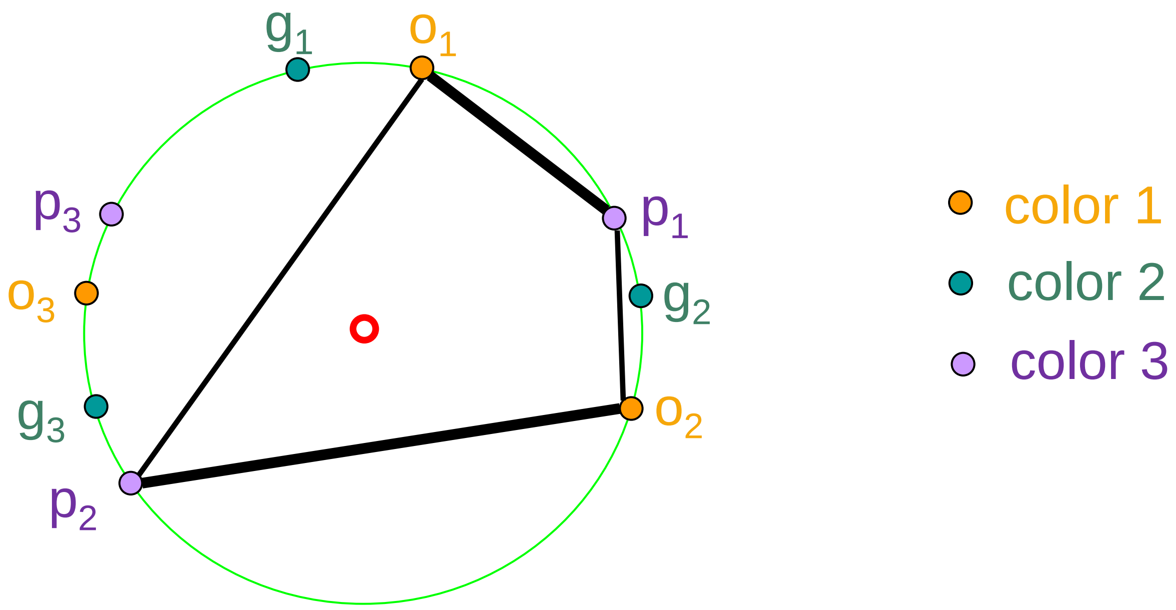


$\hat{2}$-transversal (o_2,p_2) *spans* the antipode of g_1

iff (o_2,p_2,g_1) is a colourful simplex

Combinatorial (topological) Octahedra

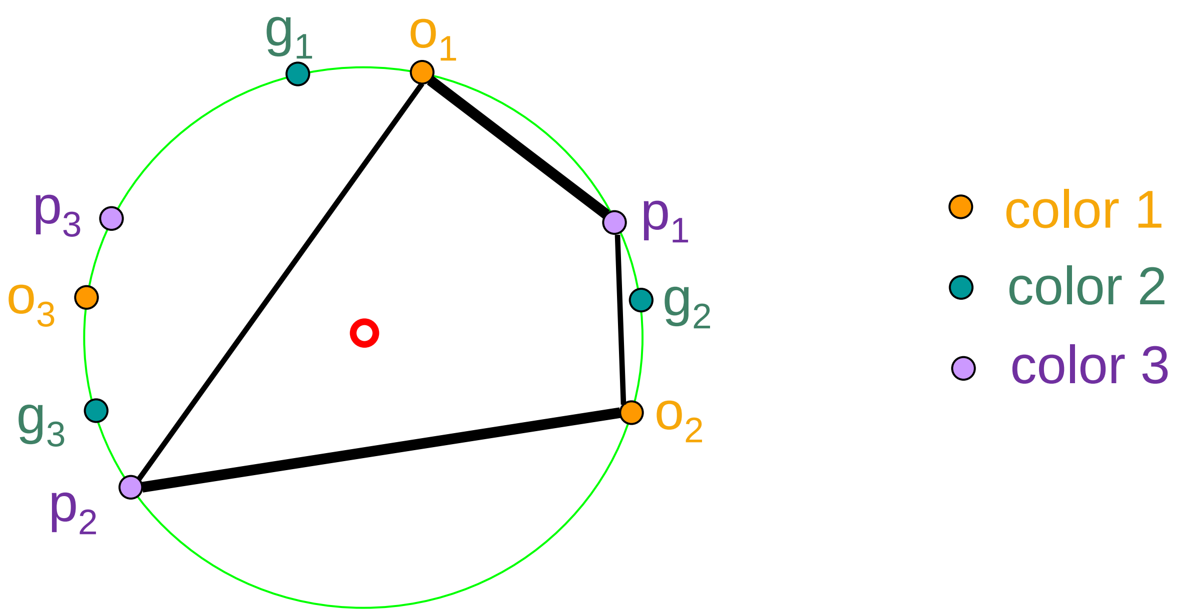
pair of disjoint $\hat{i}$-transversals



octahedron $[(o_1,p_1),(o_2,p_2)]$

Octahedron Lemma

origin-*containing* octahedra

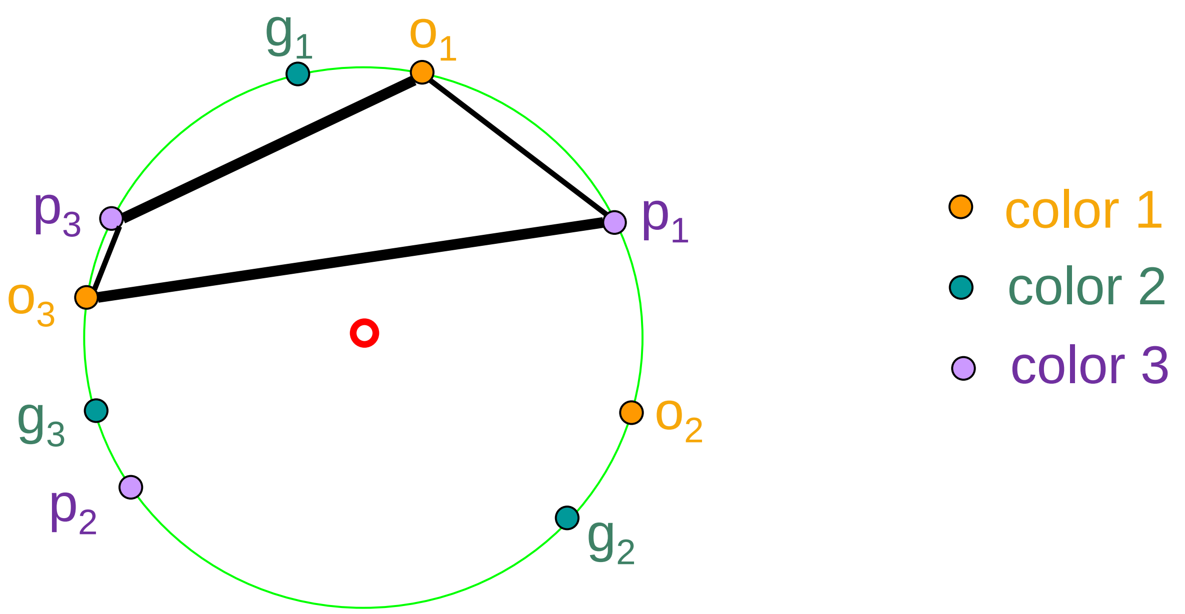


octahedron [(o_1,p_1),(o_2,p_2)]

2^d colourful faces span the whole sphere if it contains the origin (creating d+1 colourful simplexes)

Octahedron Lemma

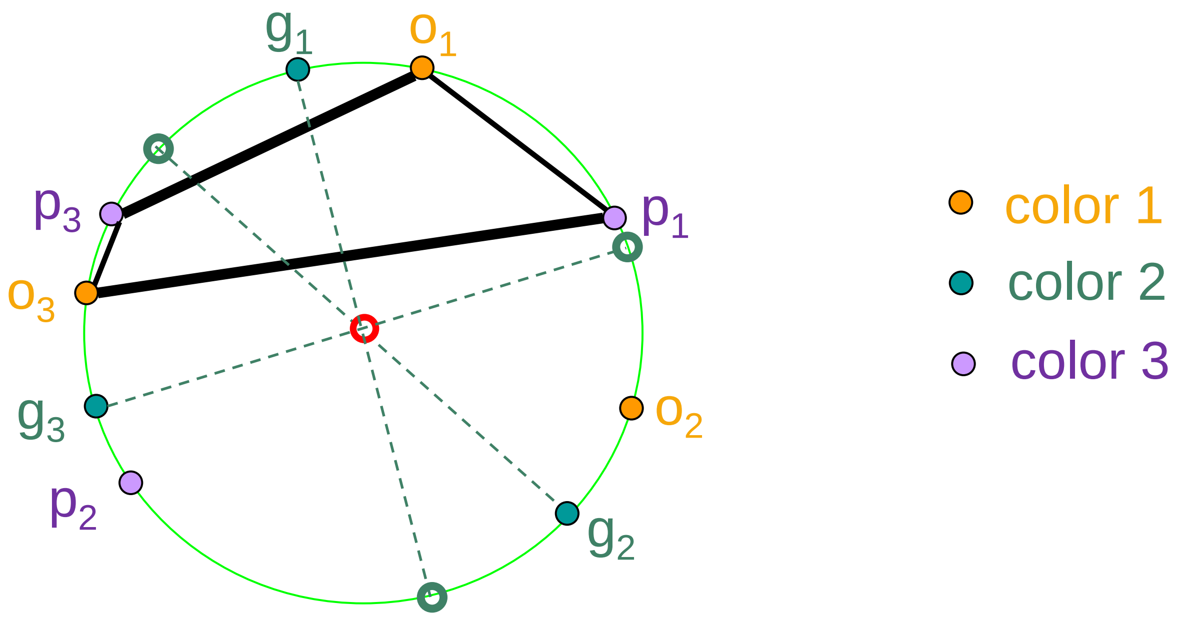
octahedron *not containing* the origin



octahedron $[(o_1,p_3),(o_3,p_1)]$ does not contain *p*

Octahedron Lemma

octahedron *not containing* the origin

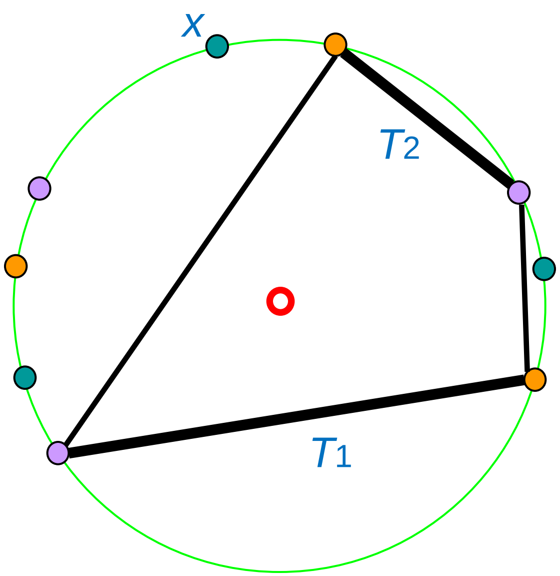


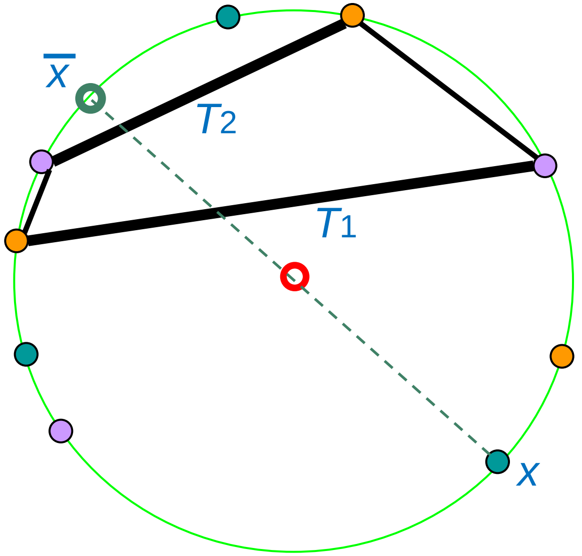
octahedron [(o_1,p_3),(o_3,p_1)] spans any antipode an *even* number of times

Octahedron Lemma

Given 2 disjoint transversals T_1 and T_2, and T_1 spans $\bar{x}$ *(antipode of x),*

- ➢ either octahedron (T_1,T_2) contains p,
- ➢ or there exists a transversal $T \neq T_1$ consisting of points from T_1 and T_2 that spans $\bar{x}$.





Colourful Research Directions

- Generalize the sufficient condition of Bárány for the existence of a colourful simplex
- Determine $\mu(d) = \min_{S,p} depth_S(p)$
- Computational approaches for $\mu(d)$ for small *d*.
- Obtain an efficient algorithm to find a colourful simplex : *Colourful Linear Programming Feasibility* problem

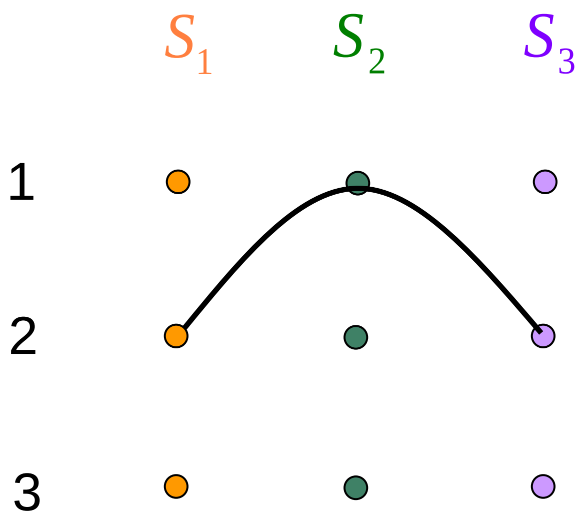
$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

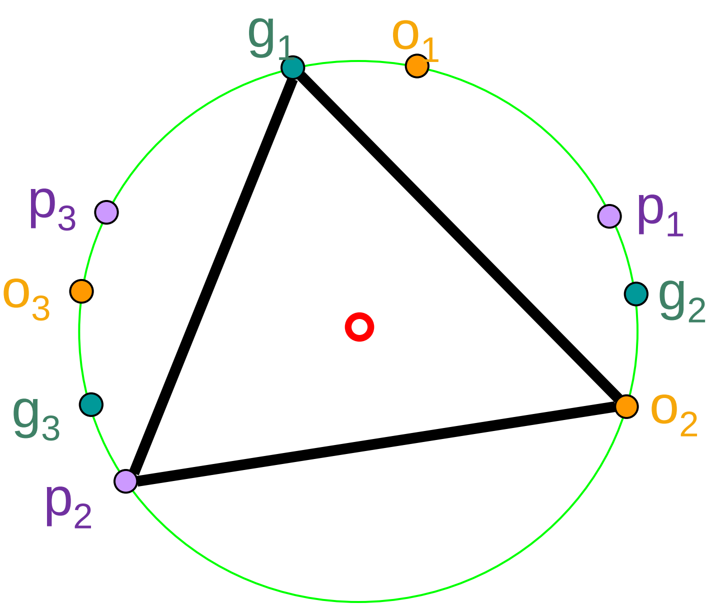
$p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S_2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$

S, *p general position* and $|S_1|, |S_2|, \ldots, |S_{d+1}| \geq d+1$

Computational Approach

(d+1)-uniform (d+1)-partite *hypergraph* representation of *colourful point configurations*



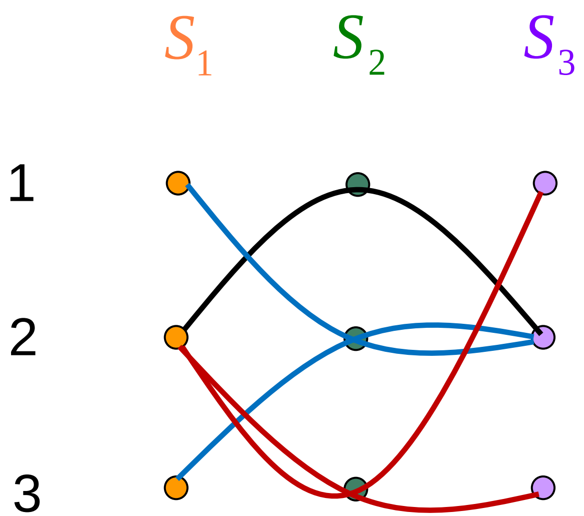


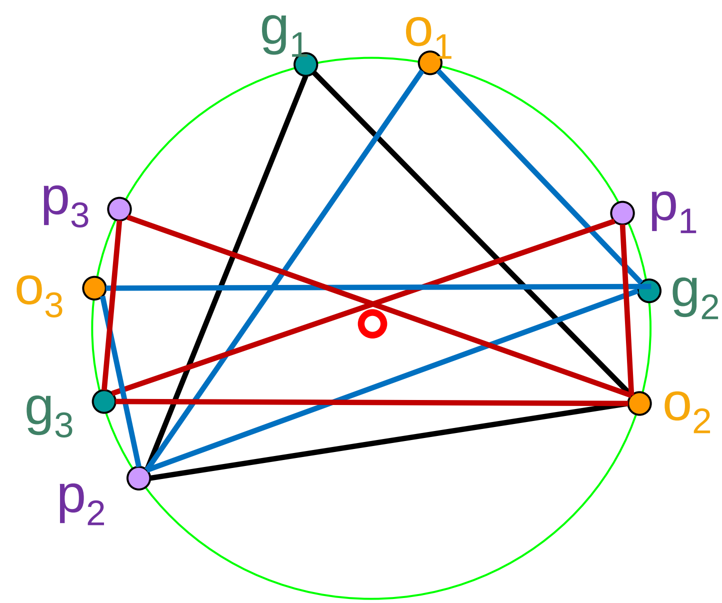
edge: colourful simplex containing *p*

❖ *combinatorial setting suggested by Imre Bárány*

Computational Approach

(d+1)-uniform (d+1)-partite *hypergraph* representation
of *colourful point configurations*





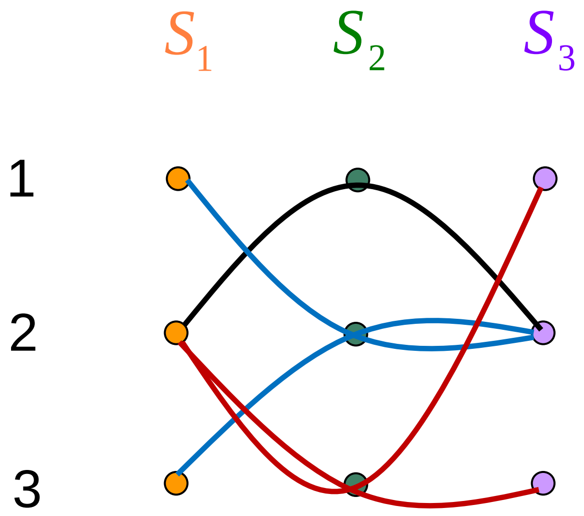
necessary conditions:

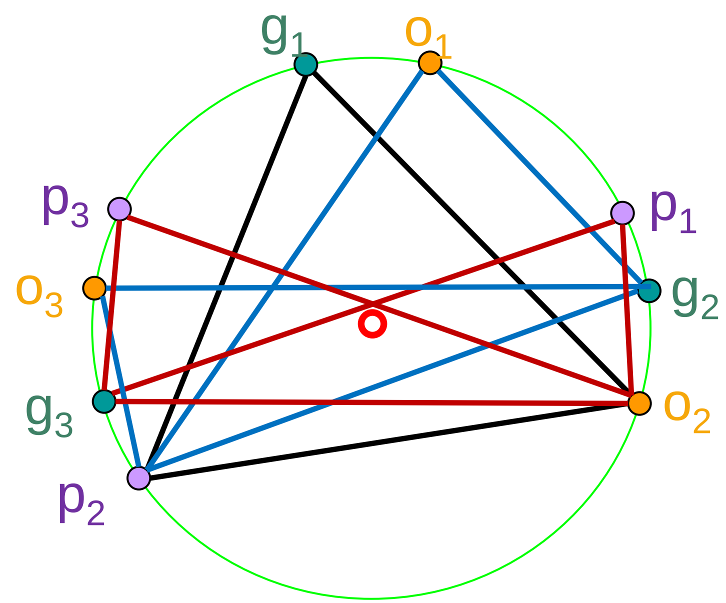
- every vertex belongs to at least 1 edge.
- even number of edges induced by subsets X_i of S_i of size 2

❖ *reformulation of the Octahedron Lemma*

Computational Approach

(d+1)-uniform (d+1)-partite *hypergraph* representation of *colourful point configurations*





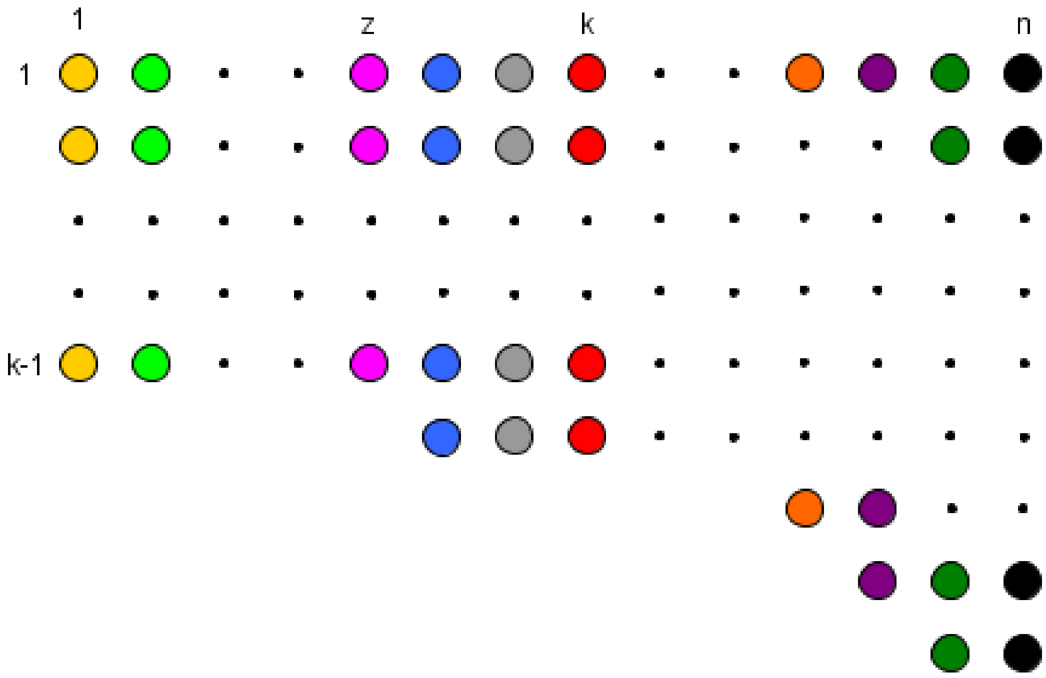
if no hypergraph with t or less hyper-edges satisfies the 2 necessary conditions, then $\mu(d) > t$

$\Rightarrow$ computational proof that $\mu(4) \geq 14$ [D., Stephen, Xie 2013]

❖ *isolated edge argument needed*

Octahedral Systems

n-uniform n-partite hypergraph (S_1,...,S_n ,E) with $|S_i| \geq 2$ such that the number of edges induced by subsets X_i of S_i of size 2 for i=1,...n is even



Octahedral Systems

- octahedral system without isolated vertex, $|S_1| = ... = |S_n| = m$ has at least $m(m+5)/2 - 11$ edges, implying:

$$\mu(d) \geq (d+1)(d+6)/2 - 11$$

[D., Meunier, Sarrabezolles 2014]

- further analysis: $\mu(4) = 17$

[D., Meunier, Sarrabezolles 2014]

Octahedral Systems

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[D., Meunier, Sarrabezolles 2014]

- further analysis: $\mu(d) = d^2 + 1$

[Sarrabezolles 2014]

Colourful Research Directions

- Generalize the sufficient condition of Bárány for the existence of a colourful simplex
- $\mu(d) = \min_{S,p} depth_S(p) = d^2 + 1$ [Sarrabezolles 2014]
- Obtain an efficient algorithm to find a colourful simplex : *Colourful Linear Programming Feasibility* problem

[Bárány, Onn 1997] , [D., Huang, Stephen, Terlaky 2008]
[Meunier, Sarrabezolles 2014]

** bon anniversaire András **

Colourful Research Directions

- Deza, Meunier & Sarrabezolles: A combinatorial approach to colourful simplicial depth. SIAM Journal on Discrete Mathematics (2014) .
- Deza & Meunier : A further generalization of the colourful Carathéodory theorem. Discrete Geometry and Optimization, Fields Institute Communications Series (2013)
- Deza, Stephen, and Xie: A note on lower bounds for colourful simplicial depth. Symmetry (2013)
- Deza, Stephen & Xie: More colourful simplices. Discrete and Computational Geometry (2011)

✓ ***thank you***

Tverberg Theorem

n points can be partitioned into $\left\lfloor \frac{n-1}{d+1} \right\rfloor + 1$ colours, with a point p in convex hull intersection. [Tverberg 1966]

$\binom{\left\lfloor \frac{n-1}{d+1} \right\rfloor + 1}{d+1}$ combinations to choose d+1 *colours*.

If each combination has at least μ colourful simplices. [Bárány 82]

$$\max_p depth_S(p) \geq \mu \binom{\left\lfloor \frac{n-1}{d+1} \right\rfloor + 1}{d+1} = \mu \binom{n}{d+1} + O(n^d)$$

S, *p* *general position*