Meeting in Honour of András Sebő: Nash-Williams' Orientation Theorem & Extensions

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Sebő 60: Waterloo Park

Sebő 60: Bill Cunningham 65

Sebő 60: Hilbert

Sebő 60: Bill Cunningham

Sebő 60:

Sebő 60:

Happy Birthday András

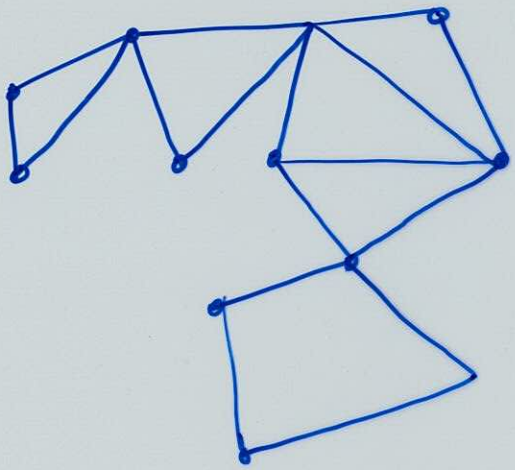
. . . looking forward to the next 60 years

k-edge connected (k-EC) graph

deleting any set of $\leq k-1$ edges leaves a connected graph

E.g. 2EC

(no cut edges)

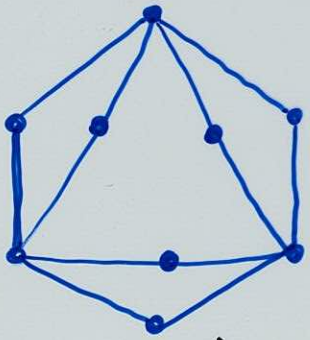


k-node connected (k-NC) graph

- has $\geq k+1$ nodes
- deleting any set of $\leq k-1$ nodes leaves a connected graph

E.g. 2NC

(no cut nodes)



- k-NC $\Longrightarrow$ k-EC

Orientations of Graphs – Nash-Williams' theorem

Nash-Williams' orientation theorem (1960):
The edges of a k-edge connected (undirected) graph can be oriented such that the resulting directed graph is $\lfloor \frac{k}{2} \rfloor$-edge connected.

Orientations of Graphs – Smooth Orientations

Smooth orientation of graph $G = (V, E)$ means:

$$\forall v \in V: \quad |\deg^{out}(v) - \deg^{in}(v)| \quad \leq \quad 1$$

Easy Fact:
Any smooth orientation of a $2k$-edge connected, Eulerian graph $G = (V, E)$ results in a directed graph that is (strongly) k-edge connected.

Proof: Fix any smooth orientation of $E(G)$.
Then, we have

$$\forall v \in V: \quad \deg^{out}(v) \quad = \quad \deg^{in}(v).$$

Hence, for each set of nodes S, we have

$$\text{number of arcs in cut } (S, \bar{S}) \quad = \quad \text{number of arcs in cut } (\bar{S}, S)\ ,$$

that is, $\deg^{out}(S) \quad = \quad \deg^{in}(S)$.
Thus, for each $\emptyset \neq S \subsetneq V$, we have $\deg^{out}(S) \quad = \quad \frac{1}{2}\deg_G(S) \geq k$, hence, the smooth orientation is (strongly) k-edge connected.

□

Orientations of Graphs – Thomassen's conjecture

Thomassen's conjecture (1989):
For all k, there exists a value $f(k)$ such that all $f(k)$-node connected graphs have a k-node connected orientation.

Carsten Thomassen, "Configurations in graphs of large minimum degree, connectivity, or chromatic number," *Ann. NY Acad. Sci.*, 555:402–412, 1989.

Orientations of Graphs – Node Connectivity – Halin Graphs

Halin graph $HG(k, T)$ $\quad$ $k \equiv$ node connectivity,
T given tree s.t. each internal node has degree$\geq k$

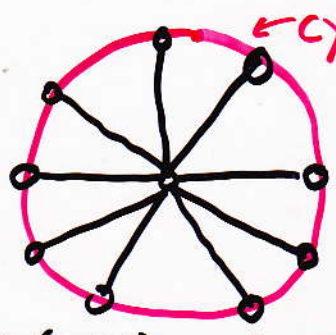
$HG(k, T) \equiv$ (k-2) copies of T, with common leaf nodes but disjoint on set of internal nodes, AND cycle C on leaf nodes

Example: $k = 3$, $\quad$ $T = K_{1,n}$ (star),
then $HG(3, T) \equiv$ wheel graph on $n + 1$ nodes

Halin Graphs $HG(k,T)$

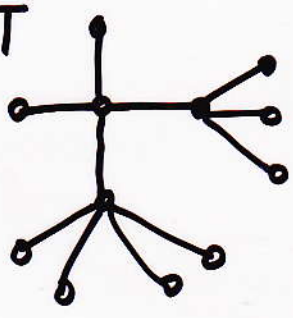
Examples

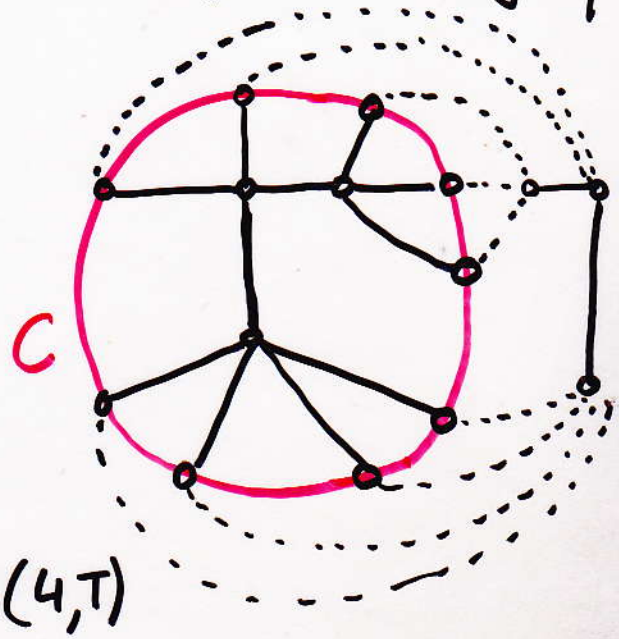
$k=3$ $T=K_{1,n}$ (star)



$HG(3,T)$ = wheel graph

$k=4$ T





$HG(4,T)$

Good orientation of $HG(k,T)$

- consistent orientation of C, e.g., clockwise
- fix any smooth orientation of T
  - $\lfloor \frac{k-2}{2} \rfloor$ copies of T get this orientation
  - remaining copies of T get opposite orientation

Thm (C, Zou '12)

Good orientation of $HG(k,T)$ ...
is a strongly $\lfloor k/2 \rfloor$ node connected digraph

Orientations of Graphs – Node Connectivity – Halin Graphs

Halin graph $HG(k, T)$ $\quad$ $k \equiv$ node connectivity,
T given tree s.t. each internal node has degree$\geq k$

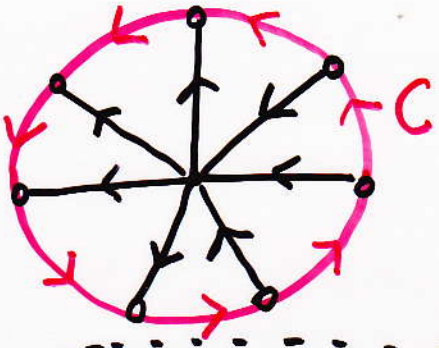
$HG(k, T) \equiv$ (k-2) copies of T, with common leaf nodes but disjoint on set of internal nodes, AND cycle C on leaf nodes

good orientation for Halin graph:
$(*)$ consistent orientation for cycle C on leafs
$(*)$ choose ANY smooth orientation for T, then
$\lfloor \frac{k-2}{2} \rfloor$ copies of T get this orientation
remaining copies of T get "opposite orientation"

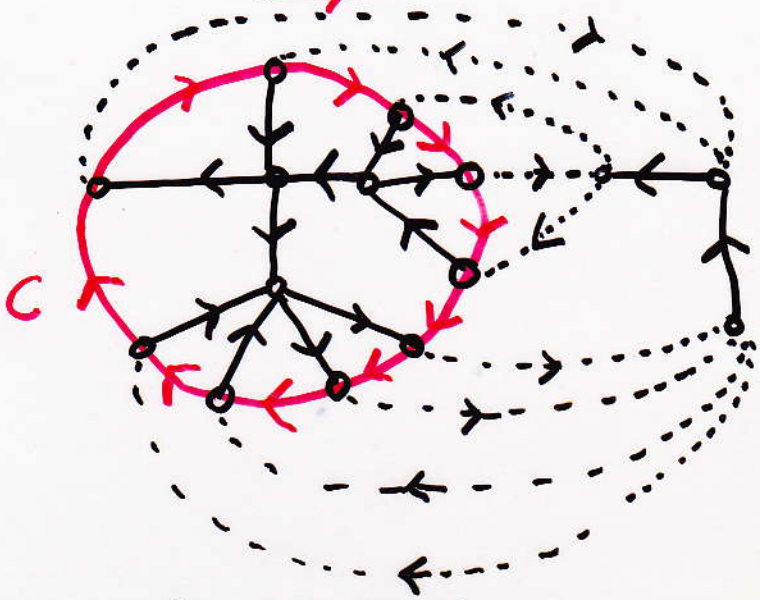
Theorem: (C,Zou 2012)
A good orientation of $HG(k, T)$ results in a digraph that is (strongly) $\lfloor k/2 \rfloor$-node-connected.

Halin Graphs: Good Orientation

$HG(3,T)$



$HG(4,T)$



Good orientation of $HG(k,T)$

- consistent orientation of C
- fix any smooth orientation of T

$\lfloor \frac{k-2}{2} \rfloor$ copies of T get this orientation

remaining copies of T get opposite orientation

Orientations of Graphs – Node Connectivity – S.Even's test

Directed graph $G = (V, E)$
goal: certify that G is (strongly) k-node connected

choose a set S^* of $\geq k$ nodes and check that S^* is k-node connected, that is, for each ordered pair of nodes $v, w \in S^*$, G has k openly disjoint v, w dipaths

then for each node $v \in V - S^*$, check that G has a k-fan from v to S^* (that is, a set of k dipath from v to S^* that are disjoint on $G - v$),

and check that G has a reverse k-fan from S^* to v

Orientations of Graphs – Node Connectivity – Halin Graphs

Halin graph $HG(k, T)$ $\quad$ $k \equiv$ node connectivity,
T given tree s.t. each internal node has degree$\geq k$

$HG(k, T) \equiv$ (k-2) copies of T, with common leaf nodes but disjoint on set of internal nodes, AND cycle C on leaf nodes

Goal: A good orientation of $HG(k, T)$ is (strongly) $\lfloor k/2 \rfloor$-node connected.

Claim: Let L be the set of leaf nodes of T. Then, in the good orientation of $HG(k, T)$, we have: L is $\lfloor k/2 \rfloor$-node connected.

Rest of proof: Let v be any internal node of $HG(k, T)$, thus $v \notin L$.
In the good orientation of $HG(k, T)$:
there exists a $\lfloor k/2 \rfloor$-fan from v to L

Subclaim: Given a smooth orientation $\widehat{T}$ of a tree T and any arc e of $\widehat{T}$, at least one leaf node of T is reachable from e (via a dipath starting with e)

there exists a reverse $\lfloor k/2 \rfloor$-fan from L to v
By S.Even's test, the good orientation of $HG(k, T)$ is $\lfloor k/2 \rfloor$-node connected.

Orientations of Graphs – Node Connectivity – Halin Graphs

Claim: Let L be the set of leaf nodes of T. Then, in the good orientation of $HG(k, T)$, we have: L is $\lfloor k/2 \rfloor$-node connected.

Proof of Claim:
By contradiction.
Let H be the subgraph of $HG(k, T)$ consisting of 2 copies of T and the cycle C on L.
Let $\widehat{H}$ be the induced "good orientation" of H . . . where the two copies of T have opposite orientations.
Let Z be any set of $\leq (\lfloor k/2 \rfloor - 1)$ nodes of L.
(Z is a candidate node cut . . . is $\widehat{H} - Z$ NOT strongly connected?)
Subclaim: for every internal node v of T and its "mate" v', $\widehat{H} - Z$ has a dipath from v to v'.
(Proof: oriented copy of T has $\lfloor k/2 \rfloor$-fan from v to L.)
Subclaim: for every pair of adjacent nodes v, w of T, $\widehat{H} - Z$ has a dipath from v to w.
(Proof: Either $\widehat{H} - Z$ has arc (v, w), or it has v, w diwalk formed by v, v' dipath, arc (v', w'), and w', w dipath.)

Orientations of Graphs – Element Connectivity & Conjecture

Bipartite graph $B = (X, Y, E)$
the set of terminals: Y (one color class, assume $|Y| \geq 2$)
the set of non-terminals, also called Steiner nodes: X (other color class)

Element means a nonterminal or an edge.

Element connectivity means the minimum number of elements whose deletion disconnects a pair of terminals . . .
$\min |S|$: S is a set of terminals, and graph $B - S$ has at least two connected components that each have a terminal.

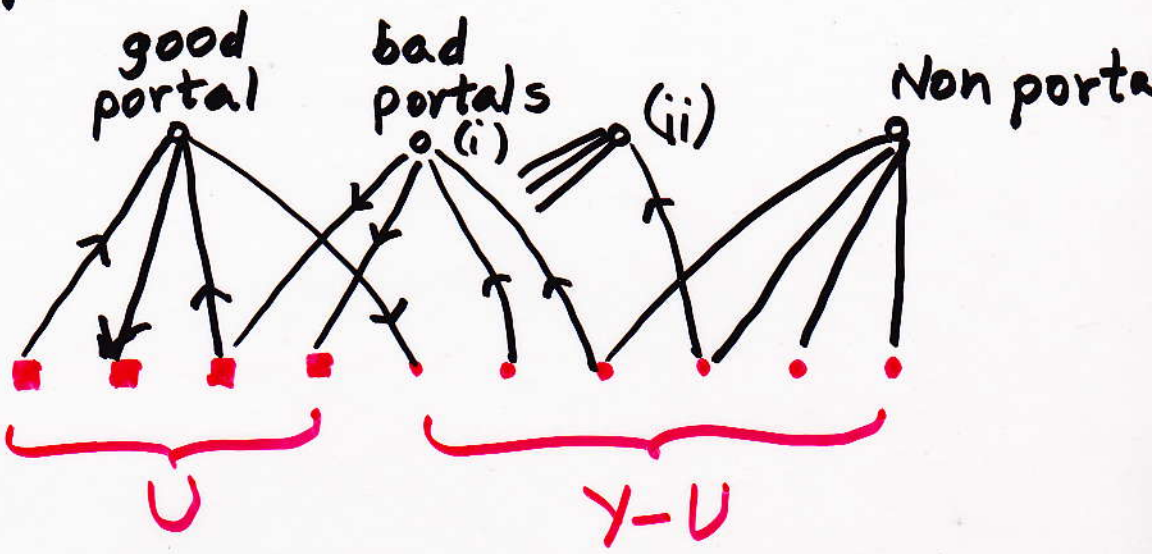
Király and Lau's conjecture (JCTB 2008):
A k-element connected graph has an orientation that is strongly $\lfloor \frac{k}{2} \rfloor$-element connected.

Fact: Conjecture holds when each nonterminal has degree ≤ 3. Apply Nash-Williams' strong orientation theorem.

Element Connectivity

Y : set of terminals
X : set of nonterminals

$B = (X, Y, E)$
bipartite



fix $U \subseteq Y$

portal $\equiv x \in X$ s.t. $\delta(U,x) \neq \emptyset \neq \delta(Y-U,x)$

$\hat{B} \equiv$ smooth orientation of B

good portal $\equiv x \in X$ s.t. $\hat{\delta}(U,x) \neq \emptyset \neq \hat{\delta}(x, Y-U)$

bad portal $\equiv$ NOT good portal

type (i) $\equiv \hat{\delta}(U,x) = \emptyset$

type (ii) $\equiv \hat{\delta}(x, Y-U) = \emptyset$

Orientations of Graphs – Element Connectivity – Portals

Bipartite graph $B = (X, Y, E)$
the set of terminals: Y (one color class, assume $|Y| \geq 2$)
the set of non-terminals, also called Steiner nodes: X (other color class)

Let $\emptyset \neq U \subsetneq Y$ be a subset of terminals (nonempty, proper subset)
Portal means a nonterminal that has a neighbor in U and a neighbor in $Y - U$.

Fix an orientation of the edges of B.
Good portal means a portal x of U such that one of the edges between U and x is directed into x and one of the edges between $Y - U$ and x is directed out of x.
Otherwise, x is called a bad portal.

Observe: a bad portal x either

(i) has all edges between U and x directed out of x, or

(ii) has all edges between $Y - U$ and x directed into x.

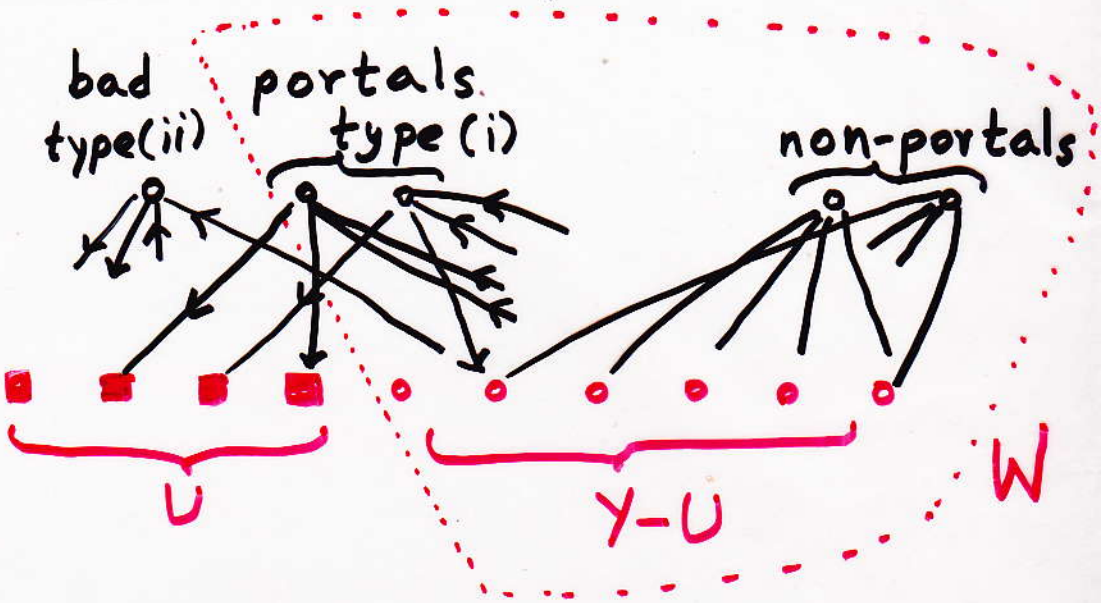
Orientations of Graphs – Element Connectivity – "4" Eulerian

Bipartite graph $B = (X, Y, E)$
the set of terminals: Y (one color class, assume $|Y| \geq 2$)
the set of non-terminals, also called Steiner nodes: X (other color class)

Proposition (C.,Zou, 2012)
Let $B = (X, Y, E)$ be an Eulerian 4-element connected bipartite graph such that every nonterminal node has degree 4.
Let $\widehat{B}$ be any smooth orientation of B.
Then $\widehat{B}$ is strongly 2-element connected.

Element Connectivity

$B = (X, Y, E)$ bipartite



$\hat{B} \equiv$ smooth orientation of B

Suppose $\hat{B}$ NOT (strongly) 2-element conn.

$U \subseteq Y$ witness set for "$\hat{B}$ NOT (strongly) 2-element conn."

$$W = (Y-U) \cup \underbrace{\{x \in Nbr(U) : \hat{\delta}(U,x) = \emptyset\}}_{\text{Bad portals of type (i)}} \cup \underbrace{\{x \in Nbr(U) : \delta(U,x) = \emptyset\}}_{\text{Nonportals}}$$

Contradiction follows from:

No. arcs leaving $W \geq$ No. bad portals in W

≥ 3

No. arcs entering $W \leq 2$ (No. good porals)

≤ 2 ⊠

Orientations of Graphs – Element Connectivity – "4" Eulerian

Proposition (C.,Zou, 2012)
Let $B = (X, Y, E)$ be an Eulerian 4-element connected bipartite graph such that every nonterminal node has degree 4.
Every smooth orientation $\widehat{B}$ of B is strongly 2-element connected.

Proof By contradiction. Suppose the result fails to hold.
Then, it can be seen that there exists a nonempty, proper subset U of Y such that $Nbr^{out}(U)$ has size ≤ 1, that is, U has ≤ 1 good portal and has ≥ 3 bad portals.
Consider the set W of nodes obtained from $Y - U$ by adding every bad portal of U of type (i), and every nonterminal $x \in X$ that is not adjacent to U; thus,

$$W = (Y - U) \bigcup \{x \in Nbr(U) \ : \ \widehat{\delta}(U, x) = \emptyset\} \bigcup \{x \in X \ : \ \delta(U, x) = \emptyset\}.$$

Clearly, the number of arcs leaving W is equal to the number of arcs entering W.
Each bad portal of U contributes at least one arc leaving W, hence, there are ≥ 3 arcs leaving W.
Each of the arcs entering W must have a good portal of U as a tail, hence, there are ≤ 2 arcs entering W.
Contradiction.

Orientations of Graphs – Element Connectivity & Projective Planes

A combinatorial projective plane is a set of points and lines, together with the following incidence relation:

1. Given any two distinct points, there is exactly one line incident with both of them.
2. Given any two distinct lines, there is exactly one point incident with both of them.
3. There exist four points such that no line is incident with more than two of them. (This is used to exclude some degenerate cases.)

Order of projective plane: positive integer $N \geq 2$.
The number of points is $N^2 + N + 1$, the number of lines is $N^2 + N + 1$, each point is incident to $d = N + 1$ lines, and each line is incident to $d = N + 1$ points.
(For each prime power p^j, there exists a projective plane of order p^j.)

Orientations of Graphs – Element Connectivity & Projective Planes

Proposition: (well known)
Let B be the incidence graph of a projective plane of order N. Then B has node connectivity $d = N + 1$.

Hence, the element connectivity of B (with the points as terminals) is equal to the degree d.

Theorem: (C.,Zou, 2012)
Let $B = (X, Y, E)$ be the incidence graph of a projective plane of order $N \geq 2$, let $d = N + 1$ denote the degree, and let each point be a terminal.
If d is even, then let $\widehat{B}$ be any smooth orientation of B.
Otherwise, . . .
The strong element connectivity of $\widehat{B}$ is $\lfloor \frac{d}{2} \rfloor$.

Orientations of Graphs – Element Connectivity & Projective Planes

Proposition: (well known)
Let B be the incidence graph of a projective plane of order N. Then B has node connectivity $d = N + 1$.

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Theorem: (C.,Zou, 2012)
Let $B = (X, Y, E)$ be the incidence graph of a projective plane of order $N \geq 2$, let $d = N + 1$ denote the degree, and let each point be a terminal.
If d is even, then let $\widehat{B}$ be any smooth orientation of B.
Otherwise, let $\widehat{B}$ be the smooth orientation that results by first removing a perfect matching M from B, then fixing a smooth orientation of the remaining graph $B - M = (X, Y, E - M)$, and then fixing arbitrary orientations for the edges of M.
The strong element connectivity of $\widehat{B}$ is $\lfloor \frac{d}{2} \rfloor$.

Orientations of Graphs – Open Questions

Thomassen's conjecture (1989):
For all k, there exists a value $f(k)$ such that all $f(k)$-node connected graphs have a k-node connected orientation.

Strong version of Thomassen's conjecture:
$f(k) = 2k$, that is, every $2k$-node connected graph has a k-node connected orientation.

Frank's conjecture for $k = 2$ (1995):
Let $G = (V, E)$ be an undirected graph with $|V| > 2$. Then G has a 2-node connected orientation if and only if
for all $S \subseteq V$ with $|S| < 2$, $G - S$ is $2(2 - |S|)$-edge connected, that is, G is 4-edge connected and, moreover, $\forall v \in V$, $G - \{v\}$ is 2-edge connected.
(Proved for Eulerian graphs by Berg & Jordán (JGT 2006), and later by Király & Szigeti (JCTB 2006).)

Király and Lau's conjecture (JCTB 2008):

A k-element connected graph has an orientation that is strongly $\lfloor \frac{k}{2} \rfloor$-element connected.