Approximation algorithms for the traveling salesman: recent advances and future directions

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April 25, 2014

Asymmetric TSP

Given a finite set V of cities and distances $c : V \times V \to \mathbb{R}_{\geq 0}$, find a tour (a list $v_0, \ldots, v_k$ containing each vertex at least once, with $v_0 = v_k$) of minimum total length $\sum_{i=1}^{k} c(v_{i-1}, v_i)$.

- equivalent to visiting each city exactly once if triangle inequality holds
- *NP*-hard (even if c is 1-2, or 1-∞, and symmetric) (Karp [1972])
- $O(\log n)$-approximation algorithm, where $n = |V|$ (Frieze, Galbiati, Maffioli [1982])
- $O(\log n / \log \log n)$-approximation algorithm (Asadpour, Goemans, Mądry, Oveis Gharan, Saberi [2010])
- no $\frac{75}{74}$-approximation algorithm exists unless $P = NP$ (Karpinski, Lampis, Schmied [2013])

Symmetric TSP

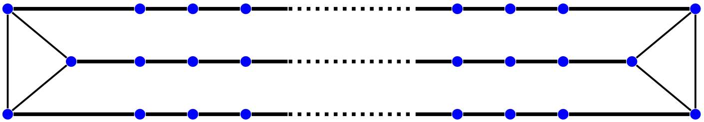
Assume $c(v, w) = c(w, v)$ for all $v, w \in V$.

- best known approximation ratio $\frac{3}{2}$ (Christofides [1976])
- no $\frac{123}{122}$-approximation algorithm exists unless $P = NP$ (Karpinski, Lampis, Schmied [2013])
- integrality ratio of subtour relaxation between $\frac{4}{3}$ and $\frac{3}{2}$ (Wolsey [1980])

Subtour relaxation (assuming triangle inequality):

$$\min\left\{ c^\top x \;:\; x(\delta(v)) = 2\,(v \in V),\;\; x(\delta(U)) \geq 2\,(\emptyset \neq U \subset V),\;\; x \geq 0 \right\}$$

(Dantzig, Fulkerson, Johnson [1954], Held, Karp [1970], Cornuéjols, Fonlupt, Naddef [1985], Cunningham, Monma, Munson, Pulleyblank [1990])



Graph-TSP

Given a graph $G = (V, E)$, let $c(v, w) = \begin{cases} 1 & \text{if } \{v, w\} \in E \\ \infty & \text{if } \{v, w\} \notin E \end{cases}$ for $v, w \in V$

Equivalently, look for a smallest Eulerian multi-subgraph of G.

Improved approximation ratios for subcubic graphs:

- $\frac{4}{3}$ (Mömke, Svensson [2011]) (before, for cubic graphs: (Boyd, Sitters, van der Ster, Stougie [2011]))
- $\frac{685}{684}$ impossible unless $P = NP$ (Karpinski, Schmied [2013])

Improved approximation ratios for general graphs:

- $1.5 - \epsilon$ (Oveis Gharan, Saberi, Singh [2011])
- 1.461 (Mömke, Svensson [2011])
- 1.445 (Mucha [2012])
- 1.4 (Sebő, V. [2012])

s-*t*-path TSP

Given a TSP instance and two cities *s* and *t*, find a shortest tour that begins in *s* and ends in *t*.

General symmetric weights:

- 1.667 (Christofides [1976], Hoogeveen [1991])
- 1.619 (An, Kleinberg, Shmoys [2012])
- 1.6 (Sebő [2013])

In graphs:

- 1.586 (Mömke, Svensson [2011])
- 1.584 (Mucha [2012])
- 1.578 (An, Kleinberg, Shmoys [2012])
- 1.5 (Sebő, V. [2012])
- 1.5 (Gao [2013])

2ECSS

Given a graph G, compute a smallest 2-edge-connected spanning subgraph of G.

- *APX*-hard (Fernandes [1998])

Approximation ratios:

- $\frac{3}{2}$ (Khuller and Vishkin [1994])
- $\frac{17}{12}$ (Cheriyan, Sebő, Szigeti [2001])
- $\frac{4}{3}$ (Sebő, V. [2012])
- $\frac{4}{3}$ (Hunkenschröder, Vempala, Vetta [2014])

Approximation ratios for generalizations:

- 2 for general weights (Khuller and Vishkin [1994])
- $\frac{3}{2}$ for metric weights (and in general if doubling edges is allowed) (Christofides [1976], Wolsey [1980], Alexander, Boyd, Elliott-Magwood [2006])

Integrality ratios

2ECSS, general weights:

- between $\frac{6}{5}$ and $\frac{3}{2}$ (Alexander, Boyd, Elliott-Magwood [2006])

2ECSS, unit weights:

- between $\frac{8}{7}$ (Boyd, Fu, Sun [2014]) and $\frac{4}{3}$ (Sebő, V. [2012])

TSP, general weights:

- between $\frac{4}{3}$ and $\frac{3}{2}$ (Wolsey [1980])

TSP, unit weights:

- between $\frac{4}{3}$ and $\frac{7}{5}$ (Sebő, V. [2012])

s-t-path TSP, general weights:

- between $\frac{3}{2}$ and $\frac{8}{5}$ (Sebő [2013])

s-t-path TSP, unit weights:

- $\frac{3}{2}$ (Sebő, V. [2012])

ATSP, general weights:

- between 2 (Boyd, Elliott-Magwood [2005], Charikar, Goemans, Karloff [2006]) and $8 \log n / \log \log n$ (Asadpour, Goemans, Mądry, Oveis Gharan, Saberi [2010])

ATSP, unit weights:

- between $\frac{3}{2}$ (Gottschalk [2013]) and $8 \log n / \log \log n$

How important are integrality ratios?

- We cannot solve the LPs combinatorially in polynomial time.
- Integrality ratios do not imply lower bounds on approximability.

Example: Euclidean TSP

- approximation scheme (Arora [1998])
- subtour LP has integrality ratio $\frac{4}{3}$ (Hougardy [2014])

- Integrality ratios imply bounds on what we can achieve
  if we use this LP as lower bound.

Sketch of the $\frac{7}{5}$-approximation algorithm

(Sebő, V. [2012])

In the critical case,

- *G* is factor-critical (i.e., has an odd ear-decomposition).
  Otherwise minimize the number of even ears (Frank [1993])
  and get a stronger lower bound (Cheriyan, Sebő, Szigeti [2001]).

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  Otherwise use Rado's theorem to get a stronger lower bound.
- there are $\frac{n}{10}$ pendant ears.
  If more, take them and continue like Christofides.
  If fewer, delete trivial ears, mark one edge per pendant ear and two edges per non-pendant ear as removable, and apply the Mömke-Svensson lemma.

Removable pairing by s-t-numbering

Delete 1-ears. Then m edges left, assume 2-vertex-connected.
Let $s = v_1, v_2 \ldots, v_{n-1}, v_n = t$ be an s-t-numbering.

(Every vertex except s has a left neighbour, and every vertex except t has a right neighbour.)

Then for every vertex, declare all edges going left except one as removable (Svensson [2012]). Let L be the removable edges.
Similarly let R be a removable set (right instead of left).

$$|L| + |R| \;\geq\; 2m - n - n_2$$

where n_2 is the number of degree 2 vertices.

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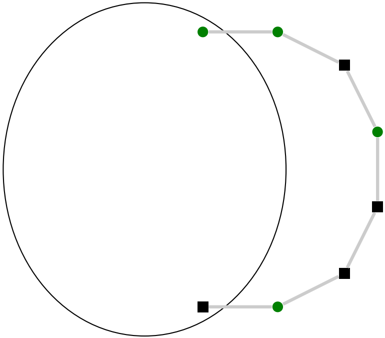
If the $k = m - n + 1$ ears form a forest or at least have distinct representative endpoints (cf. Gottschalk [2013]), $n_2 \leq n - k$. Then the bigger of the two removable sets is $\geq m - n + \frac{k}{2}$. With the Mömke-Svensson lemma we get a tour of length $<$

$$\frac{4}{3}n + \frac{k}{3},$$

which is less than $\frac{7}{5}n$ if $k < \frac{n}{5}$.

Enhanced ear induction



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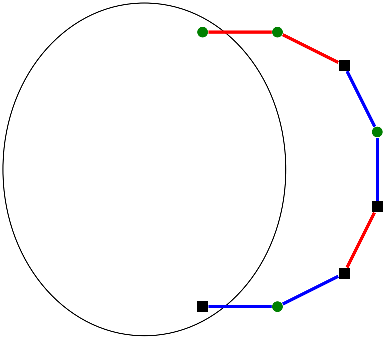

- Split ear at the vertices that have wrong parity so far.

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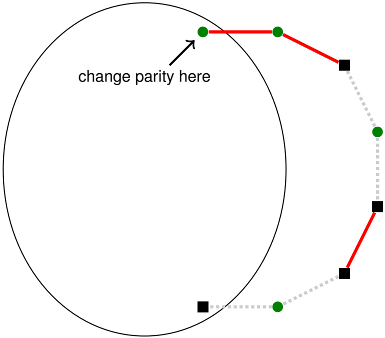

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- Take smaller part for obtaining a T-join.

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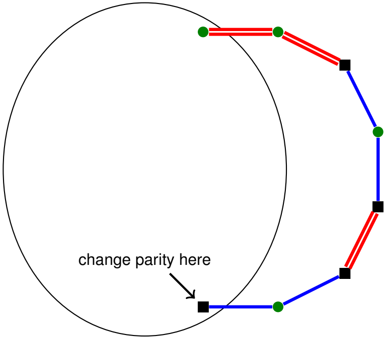

- Split ear at the vertices that have wrong parity so far.
- Take smaller part for obtaining a T-join.
- Double smaller part for obtaining a T-tour.

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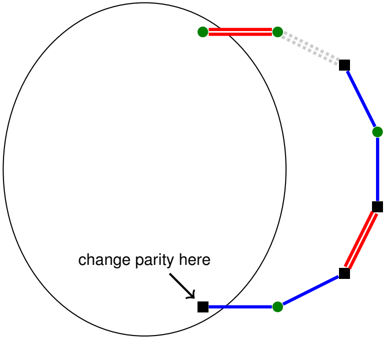

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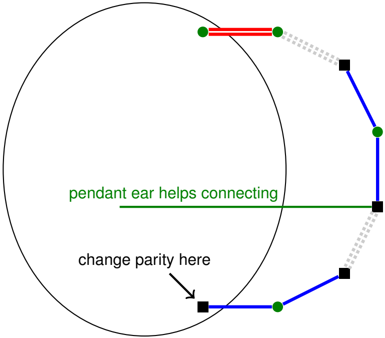

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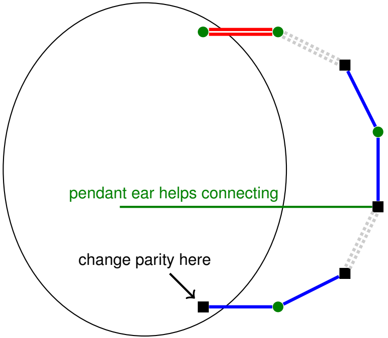

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Need at most $\frac{3}{2}|\mathrm{in}(P)| + \frac{1}{2}\varphi(P) - \max\{1, k\}$ edges if length > 3 and k pendant ears help. Can assume that all pendant ears are helpful. Worst case: $k = 1$ for all $\frac{n}{10}$ non-pendant ears.

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Cheap parity correction by weighted ear induction?

Let

$$\psi(G, c) \ := \ \max\bigl\{\min\{c(J) : J \text{ is a } T\text{-join in } G\} : |T| \text{ even}\bigr\}.$$

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Then for any (generalized) ear-decomposition

$$\psi(G, c) \;\leq\; \sum_{P \text{ ear}} \psi(\bar{P}, c),$$

where $\bar{P}$ arises from P by contracting the endpoints.

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If $c \equiv 1$, the right-hand side is $\sum_P \frac{|\text{in}(P)|+\varphi(P)}{2} = \frac{|V(G)|-1+k_{\text{even}}}{2}$, and there is an ear-decomposition with equality ($\varphi(G)$ even ears), and can be computed in polynomial time (Frank [1993]).

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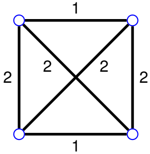
In general, the minimum r.h.s. can be bigger than $\psi(G,c)$ (by at most a factor $\frac{3}{2}$?), but not bigger than $\frac{1}{2}$OPT.



(Gottschalk, V.)

Thank you!

BONN 2014

17th Conference on Integer Programming and Combinatorial Optimization

www.or.uni-bonn.de/ipco

Location: Bonn, Germany
Date: June 23–25, 2014
PC chair: Jon Lee
Local org.: S. Held, J. Vygen

Summer school: June 20–22, 2014

Gérard Cornuéjols, András Frank
Thomas Rothvoß, David Shmoys

Extras:

- welcome reception, Arithmeum
- poster session
- Rhine river cruise with dinner