A Polynomial-Time Cutting-Plane Algorithm for Matchings

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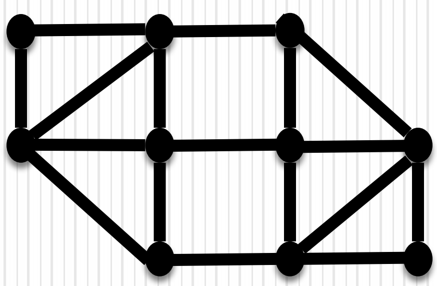
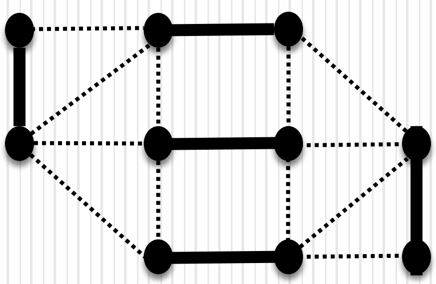
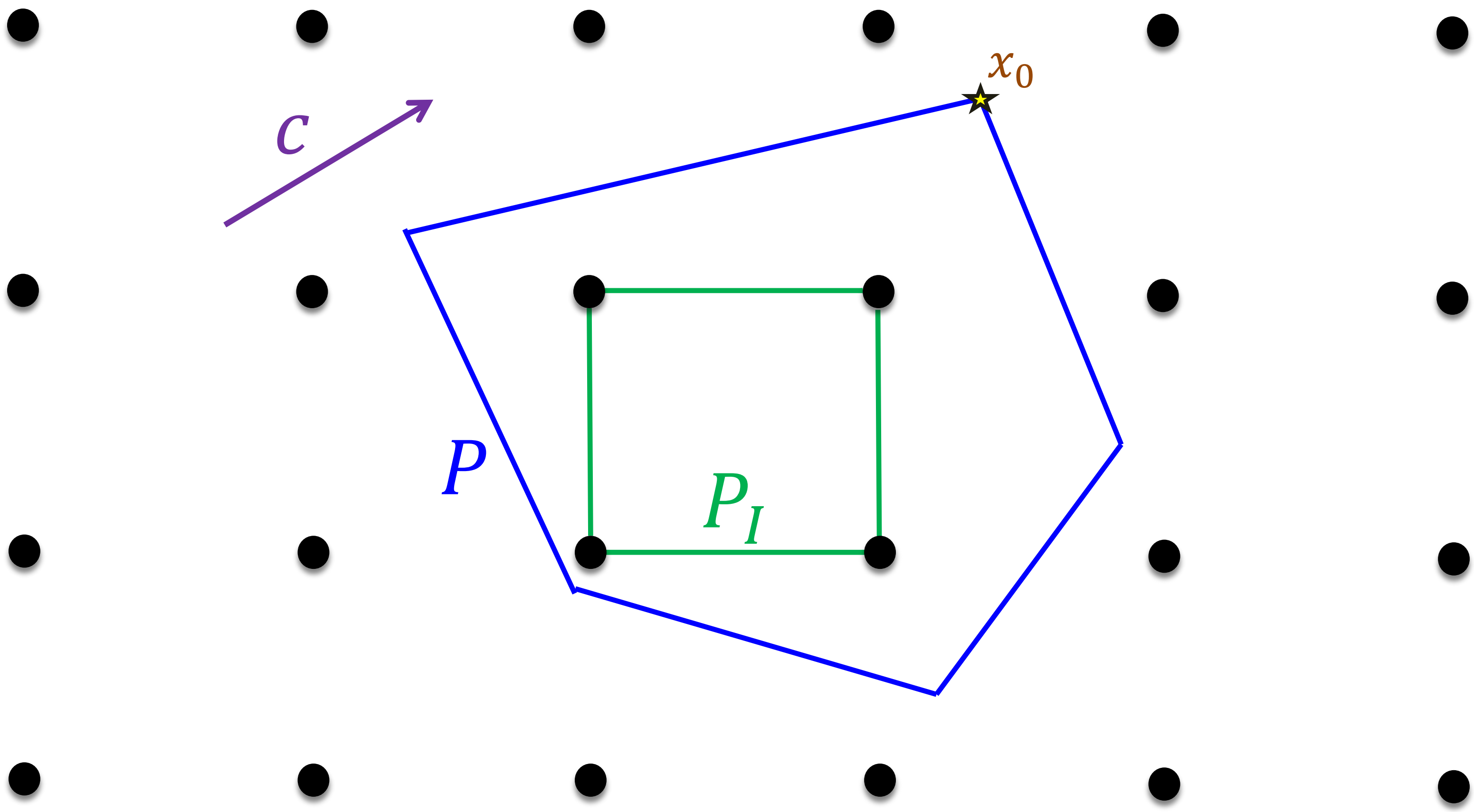




Joint work with Karthik Chandrasekaran & Santosh Vempala.
Slides thanks to Karthik.

Cutting Plane Method



$P = \{x \in R^n : Ax \leq b\}$

$P_I = \text{conv-hull}(P \cap Z^n)$

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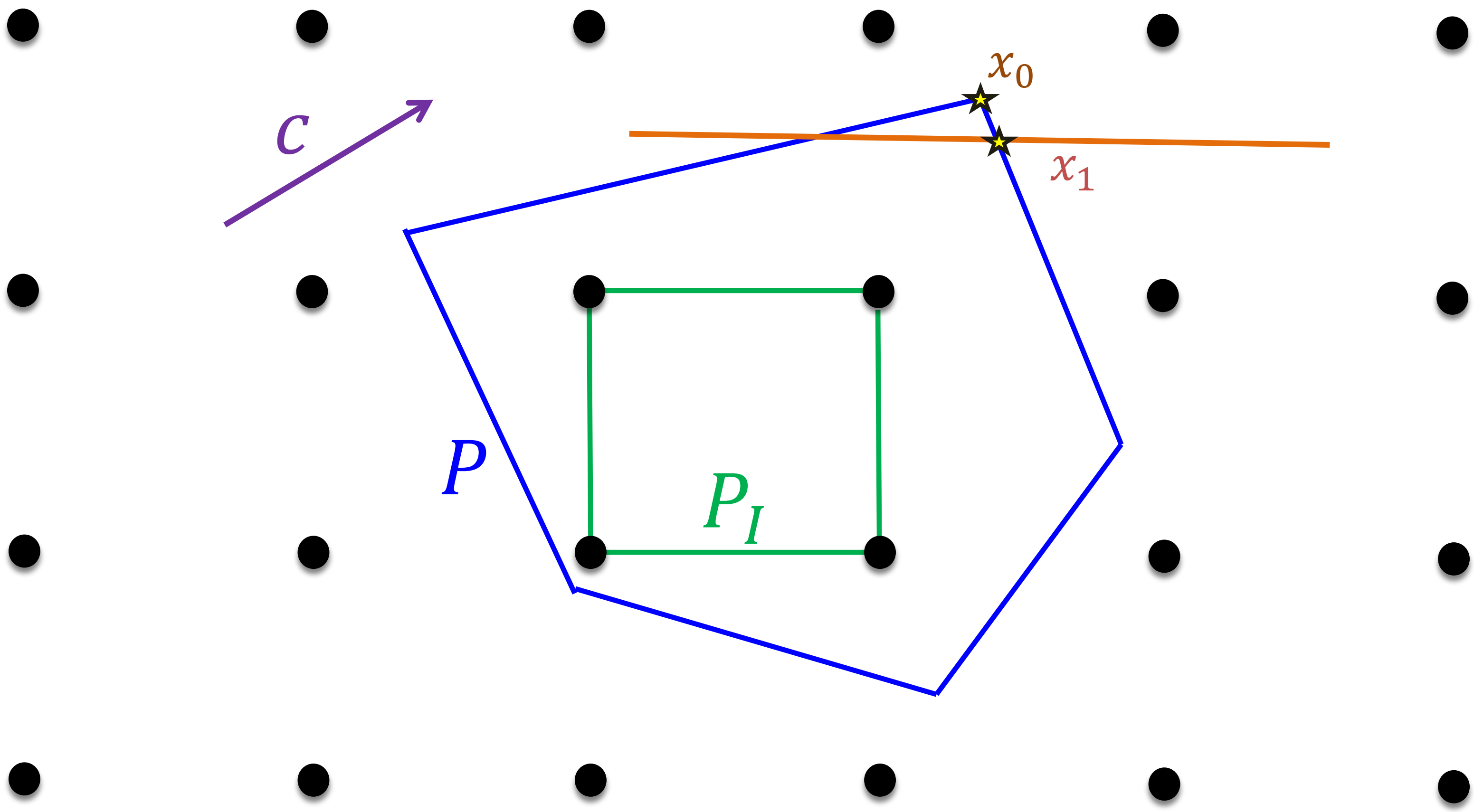

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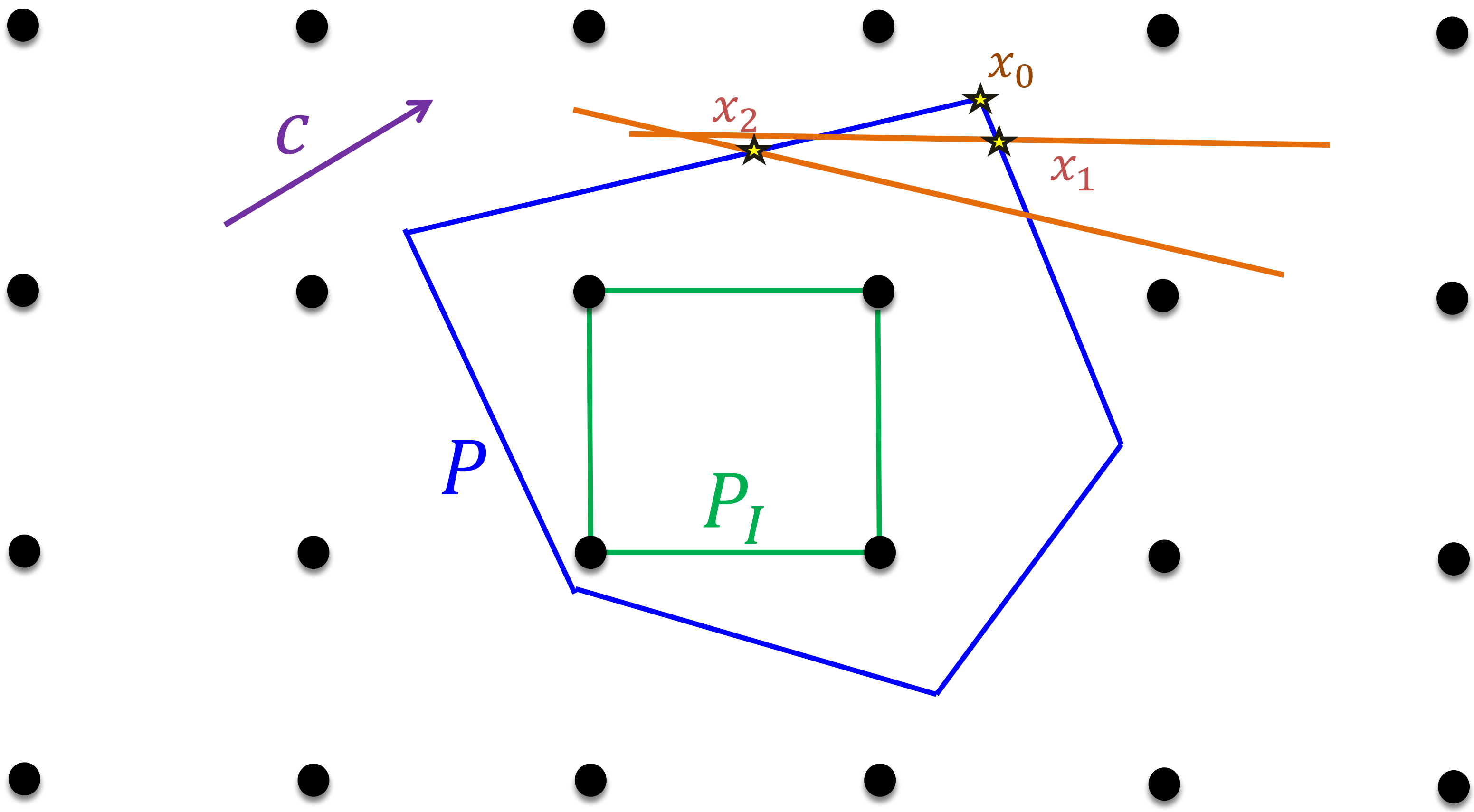

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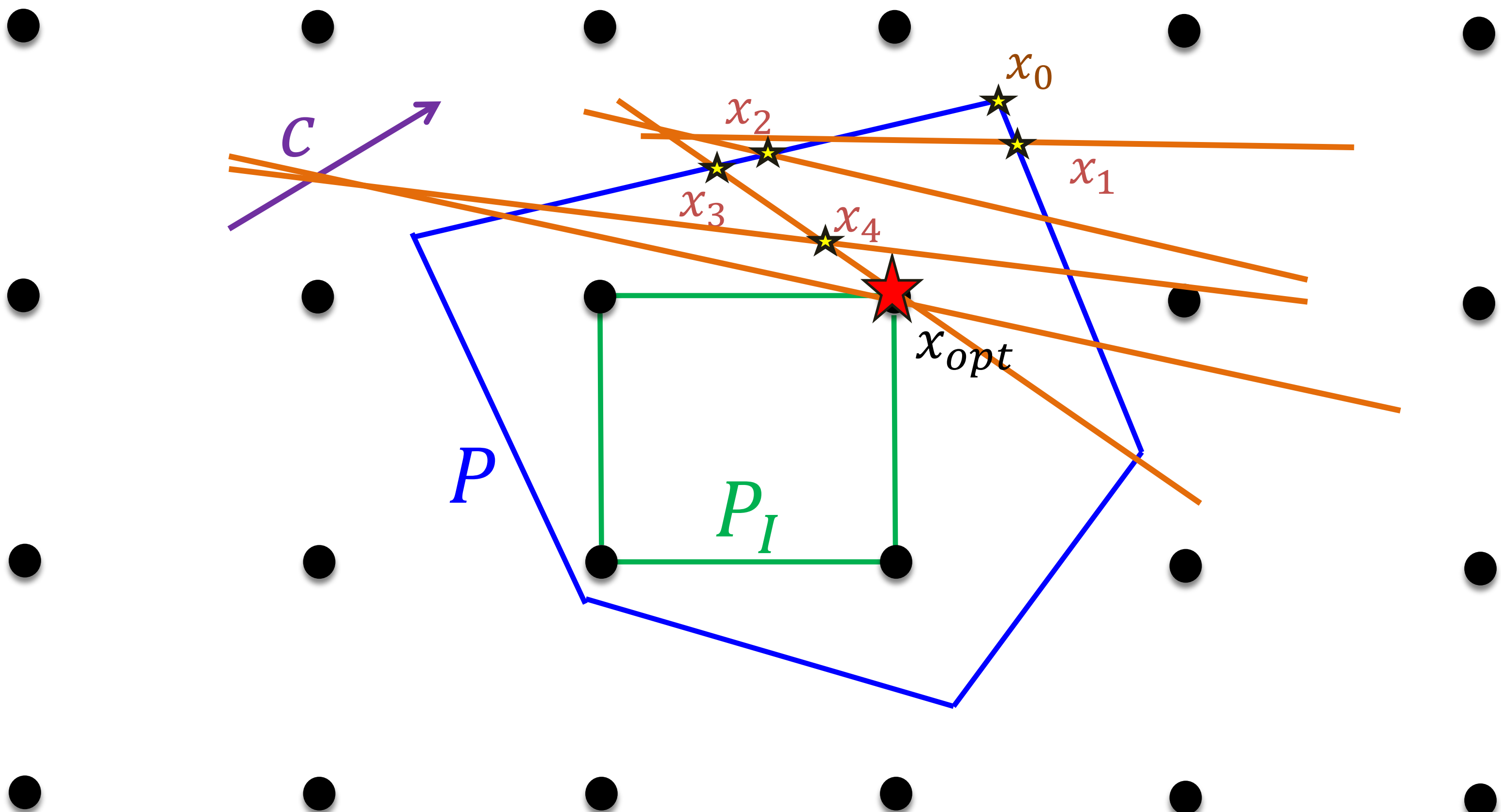

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Cutting Plane Method

1. **Starting LP.** Start with the LP relaxation of the given IP to obtain basic optimal solution x
2. Repeat until x is integral:
    a. **Add Cuts.** Find a linear inequality that is valid for the convex hull of integer solutions but violated by x and add it to the LP
    b. **Re-solve LP.** Obtain basic optimal solution x

Cutting Plane Method

- Proposed by Dantzig-Fulkerson-Johnson (1954)
- Several cut-generation procedures
  - Gomory cuts [Gomory (1958)]
  - Intersection cuts [Balas (1971)]
  - Disjunctive cuts [Balas (1979)]
  - Split cuts [Cook-Kannan-Schrijver (1990)]
  - MIR Inequalities [Nemhauser-Wolsey (1990)]
  - Lift-and-project methods
    [Sherali-Adams (1990), Lovász-Schrijver (1991), Balas-Ceria-Cornuéjols (1993)]
- Closure properties of polytopes [Chvátal (1973)]
  - Chvatal-Gomory rank
- Cutting plane proof system
  [Chvátal-Cook-Hartmann (1989)]
- Implemented in commercial IP solvers

Requirements – Efficient and Correct

- Efficient cut-generation procedure [Gomory(1958)]
- Convergence to integral solution [Gomory(1958)]
- Fast convergence (number of cuts to reach integral solution)
  - At most 2^n cuts for $0/1$-IP [Gomory(1958)]
  - No hope for faster theoretical convergence using Gomory cuts
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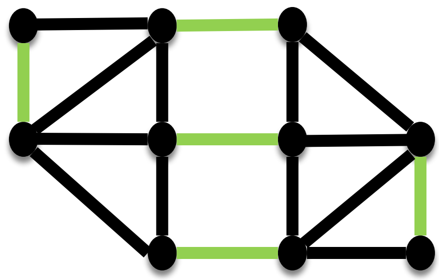
<u>Question:</u> Can we explain the efficiency in practical implementations?

Minimum-Cost Perfect Matching

- Problem: Given an edge cost function in a graph, find a minimum cost perfect matching



Minimum-Cost Perfect Matching

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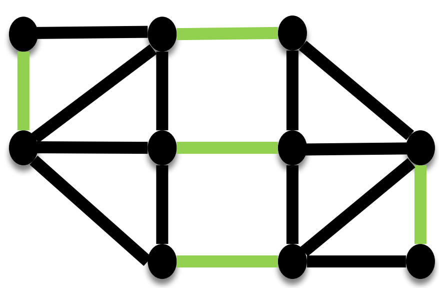
Edmonds (1965)	$O(n^2 m)$
Gabow (1973), Lawler (1976)	$O(n^3)$
Galil-Micali-Gabow (1986)	$O(nm \log n)$
Gabow (1990)	$O\big(n(m + n \log n)\big)$
Gabow-Tarjan (1991)	$O\big(m \log(n\|c\|_\infty) \sqrt{n \log n}\big)$



Minimum-Cost Perfect Matching

- Problem: Given an edge cost function in a graph, find a minimum cost perfect matching

- IP formulation:

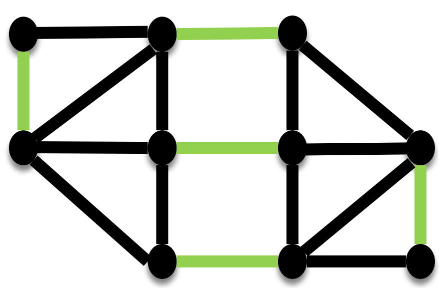
$$\min \sum_{uv \in E} c(uv)x(uv)$$

$$x(\delta(u)) = 1 \ \forall u \in V \qquad \text{(degree constraints)}$$

$$x \geq 0 \qquad \text{(non-negativity constraints)}$$

$$x \in Z^E$$

$$x(\delta(u)) := \sum_{e:e=uv} x(e)$$



LP Relaxation

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LP Relaxation (Bipartite Relaxation)

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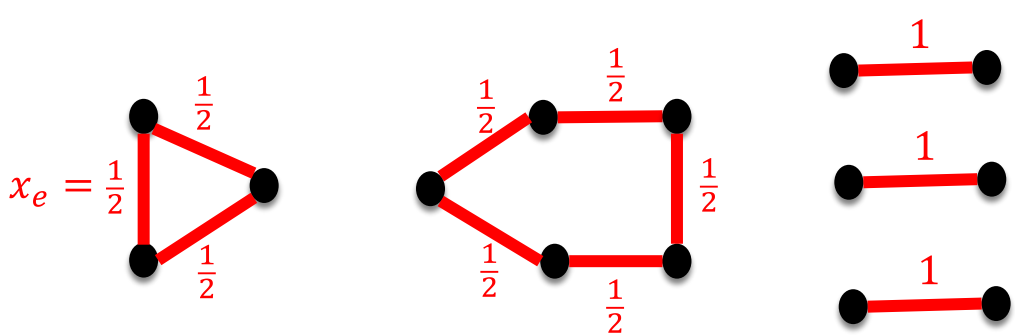
- In bipartite graphs, the solution is the indicator vector of a perfect matching in the graph

- In non-bipartite graphs, the solution is not necessarily integral

    - Solution is half-integral and supported by a disjoint union of edges and odd cycles



Polyhedral Characterization [Edmonds(1965)]

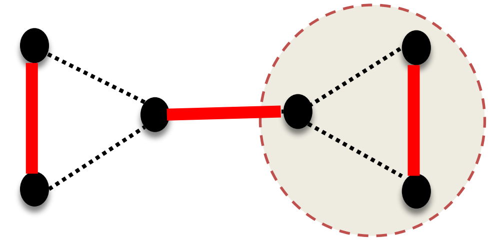
$\min \sum_{uv \in E} c(uv)x(uv)$

$x(\delta(u)) = 1 \ \forall u \in V$ (degree constraints)

$x \geq 0$ (non-negativity constraints)

$x(\delta(S)) \geq 1 \ \forall S \subseteq V, |S| odd$ (odd-set inequalities) – *Exponentially many!*

$$x(\delta(S)) := \sum_{e=uv: u \in S,\, v \in V \setminus S} x(e)$$



- Odd-set inequalities
  - Gomory cuts derived from bipartite relaxation

    [Chvátal (1973)]

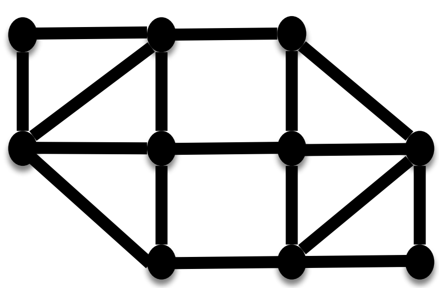
- Efficient separation oracle exists
  - For any given point x, we can efficiently verify if x satisfies all constraints and if not output a violated constraint

    [Padberg-Rao (1982)]

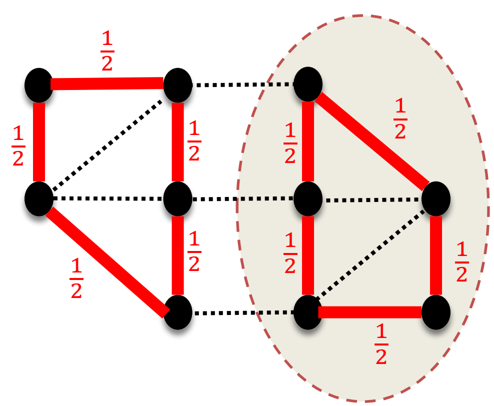
Cutting Plane Algorithm [Padberg – Rao(1982)]

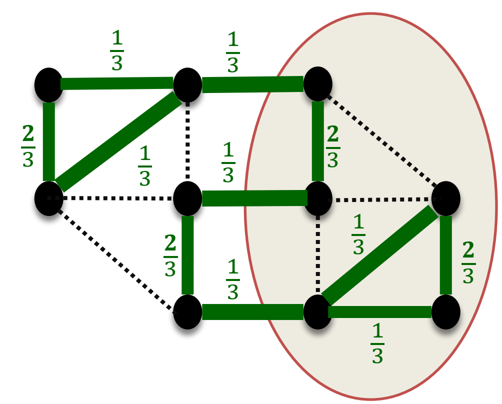
1. **Starting LP.** Start with the <u>bipartite relaxation</u> to obtain basic optimal solution x
2. Repeat until x is integral:
   a. **Add Cuts.** Find an <u>odd-set inequality</u> that is violated by x using the Padberg-Rao procedure and add it to the LP
   b. **Re-solve LP.** Obtain basic optimal solution x

- Easy to find cuts in the first step
  - Starting LP optimum is half-integral and supported by a disjoint union of odd cycles and edges
- Half-integral structure is not preserved in later steps - need Padberg-Rao for cut generation



$c_e = 1 \; \forall e$





Cutting Plane Algorithm [Padberg – Rao(1982)]

- Discussed by Lovász-Plummer (1986)
- Implementations by Grötschel-Holland (1985), Trick (1987), Fischetti-Lodi (2010)
  - Computationally efficient

Result: A cutting plane algorithm for minimum-cost perfect matching
- Convergence guarantee: $O(n \log n)$ rounds of cuts for an n-vertex graph
- Black-box LP solver after each round of cut addition
- Intermediate LP optima are half-integral

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- ➢ Black-box LP solver after each round of cut addition
- ➢ Intermediate LP optima are half-integral

➢ Bunch, PhD thesis in Chemical Engineering (1997): similar (weaker) result, using specifically implemented simplex rule.

Outline

- Cutting Plane Method for Integer Programs
- Perfect Matching and Linear Constraints
- Results
- New Cutting Plane Algorithm for Matching
  - Analysis
    - Half-integral Structure
    - Progress
      - Coupling with a new Combinatorial Primal-Dual Algorithm for Matching
- Future Directions

New Cutting Plane Algorithm

Primal - P_F

Min $\sum_{uv \in E} c_{uv} x_{uv}$

$$x(\delta(u)) = 1 \ \forall u \in V$$

$$x(\delta(S)) \geq 1 \ \forall S \in F$$

$$x \geq 0$$

1. Perturb the integral cost function by adding $\frac{1}{2^i}$ to edge i
2. **Starting LP.** Bipartite relaxation $(F = \emptyset)$
3. Repeat until x is integral
   - **(a) Retain old cuts.**
   - **(b) Choose new cuts.**
   - **(c) Re-solve LP.** Find an optimal solution x to P_F

To ensure uniqueness of intermediate optima

Final integral optimum to perturbed cost function is also optimum to original cost function

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Dual - D_F

$$\text{Max} \sum_{S \in V \cup F} \Pi(S)$$
$$\sum_{S \in V \cup F : uv \in \delta(S)} \Pi(S) \leq c(uv) \ \forall uv \in E$$
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     $$H' = \{S \in F : \Pi(S) > 0\}$$
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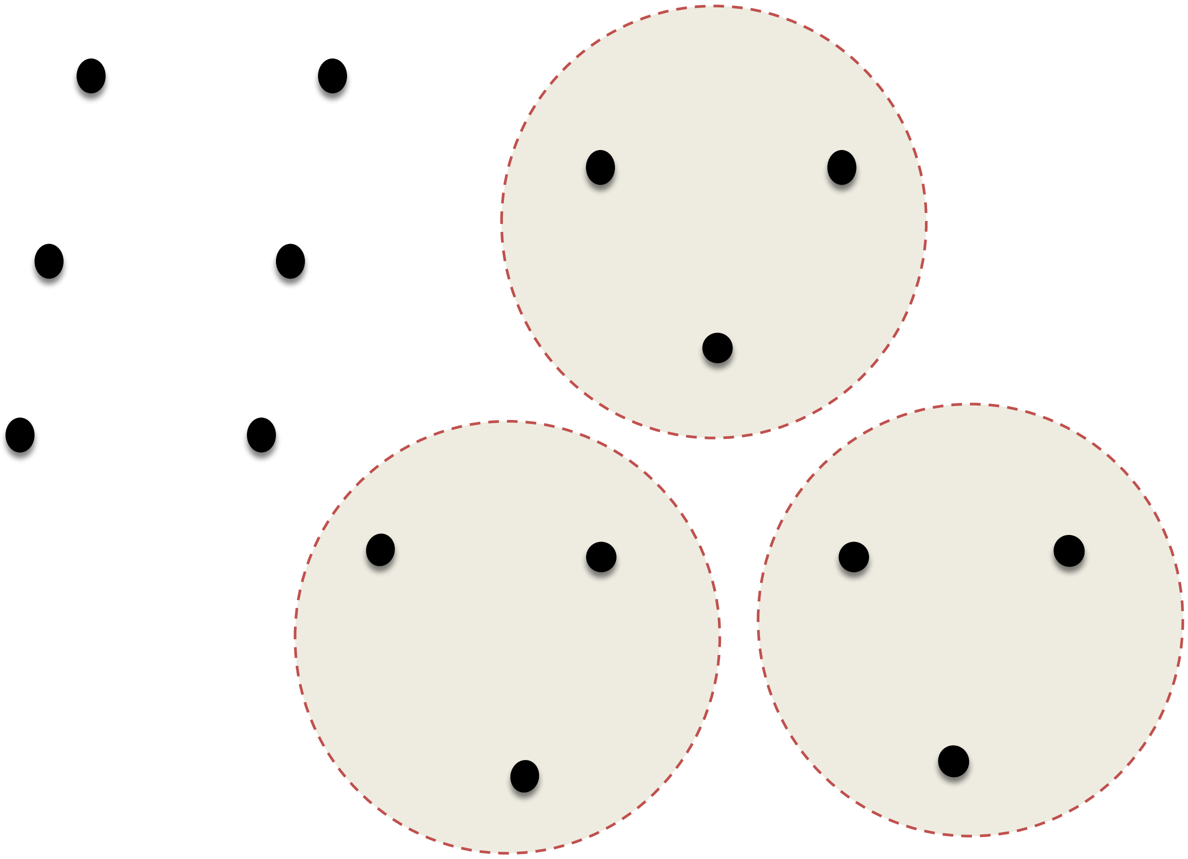
$$\text{Min} \sum_{S \in F} \frac{|\Pi(S) - \Pi_{prev}(S)|}{|S|}$$

An Example

$$F = \{S_1, S_2, S_3\}$$



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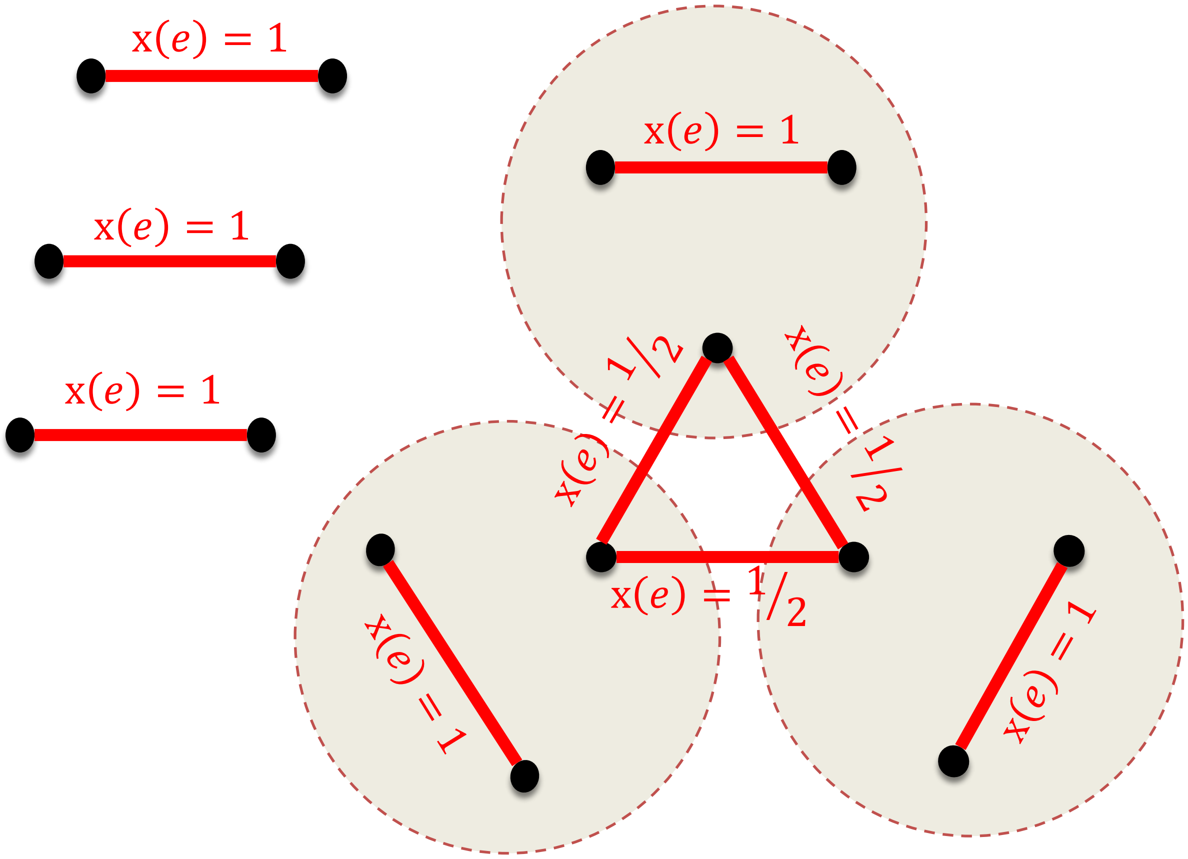

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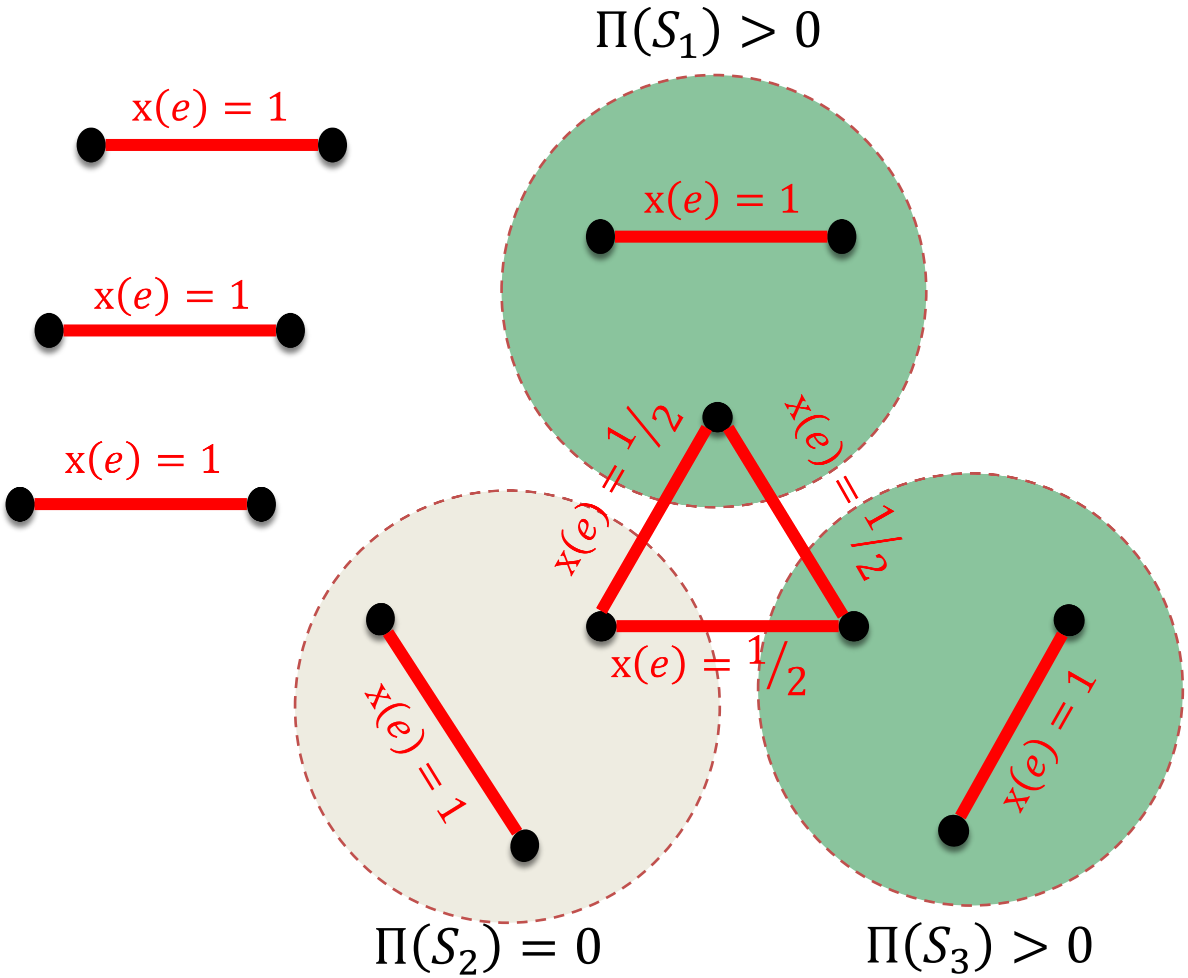

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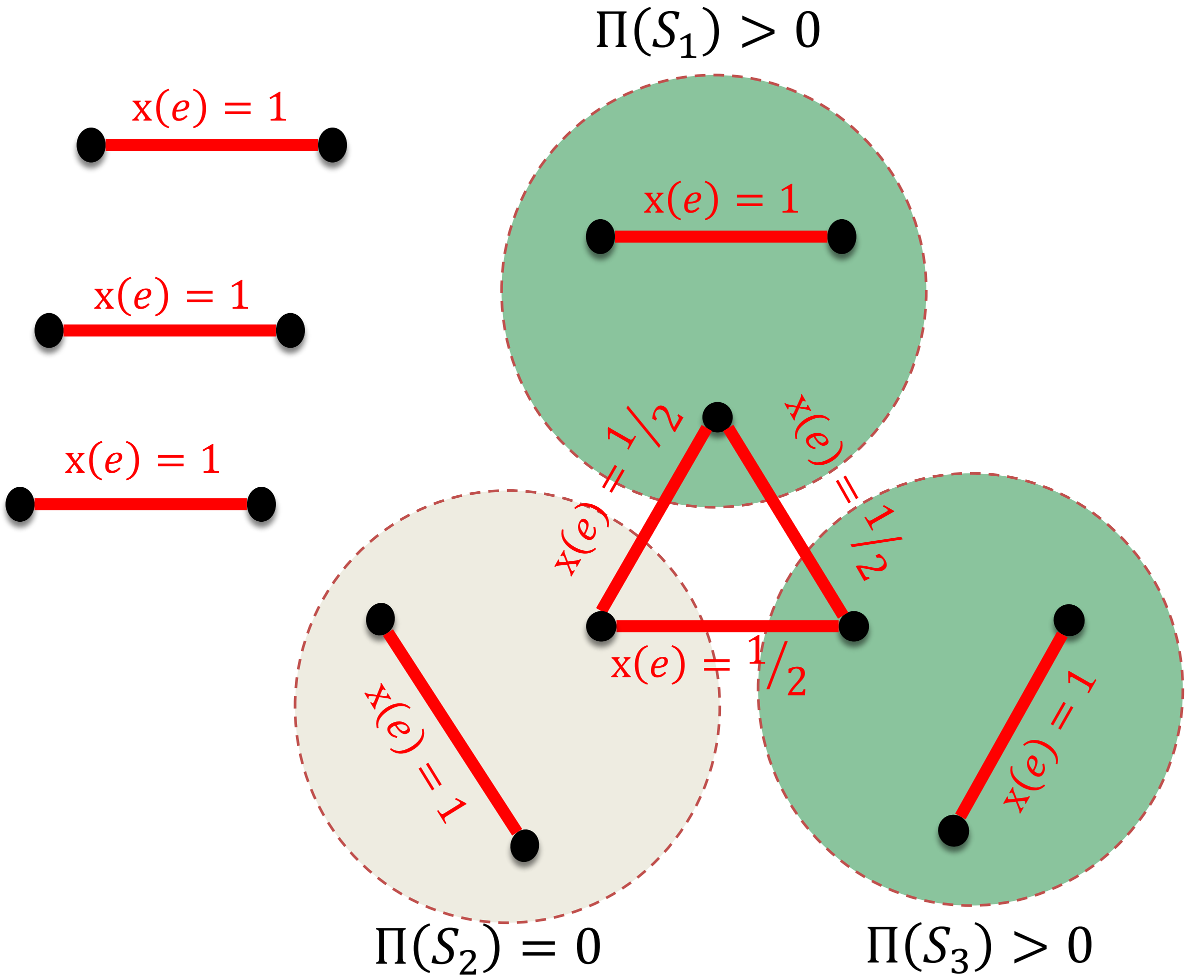

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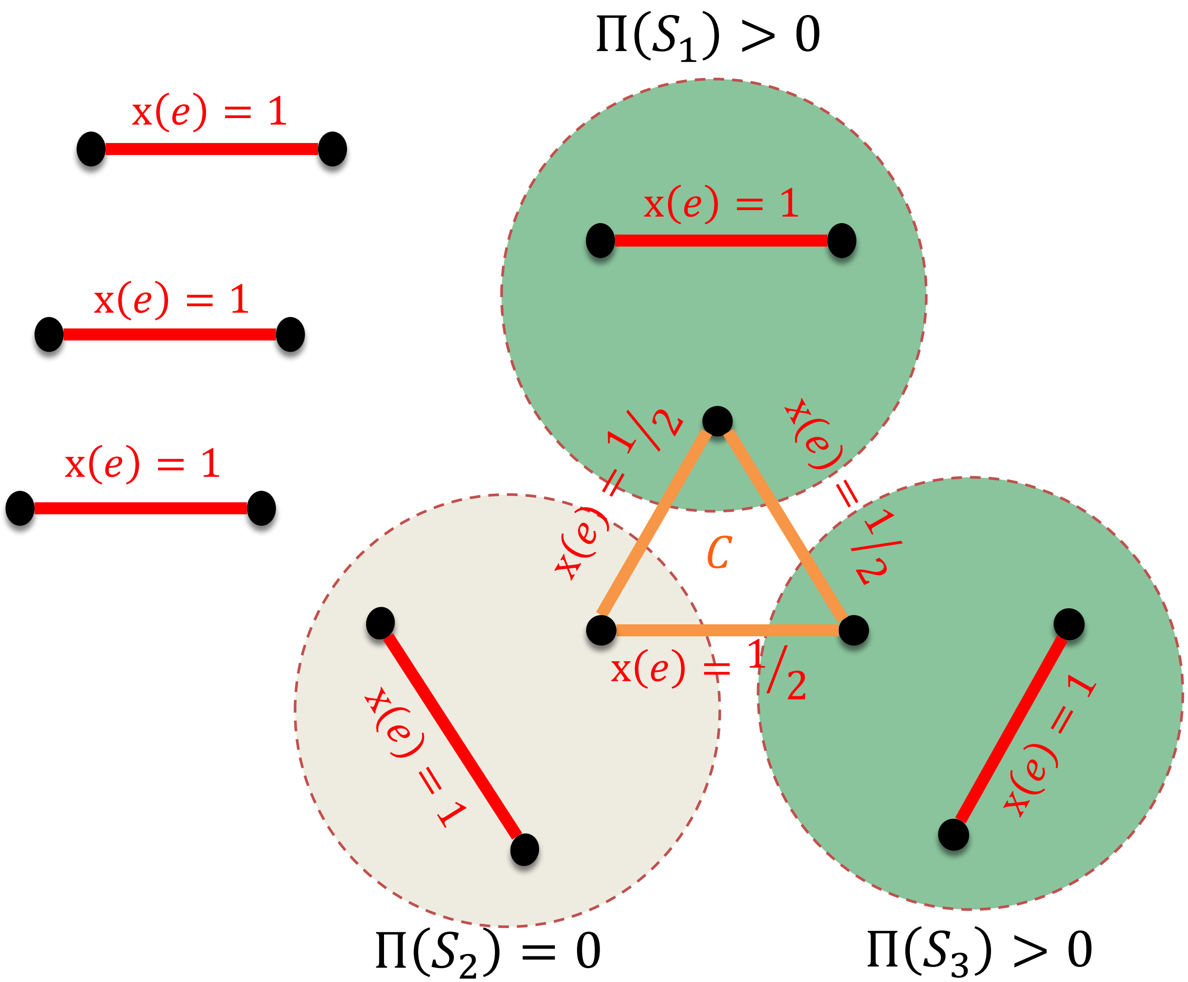

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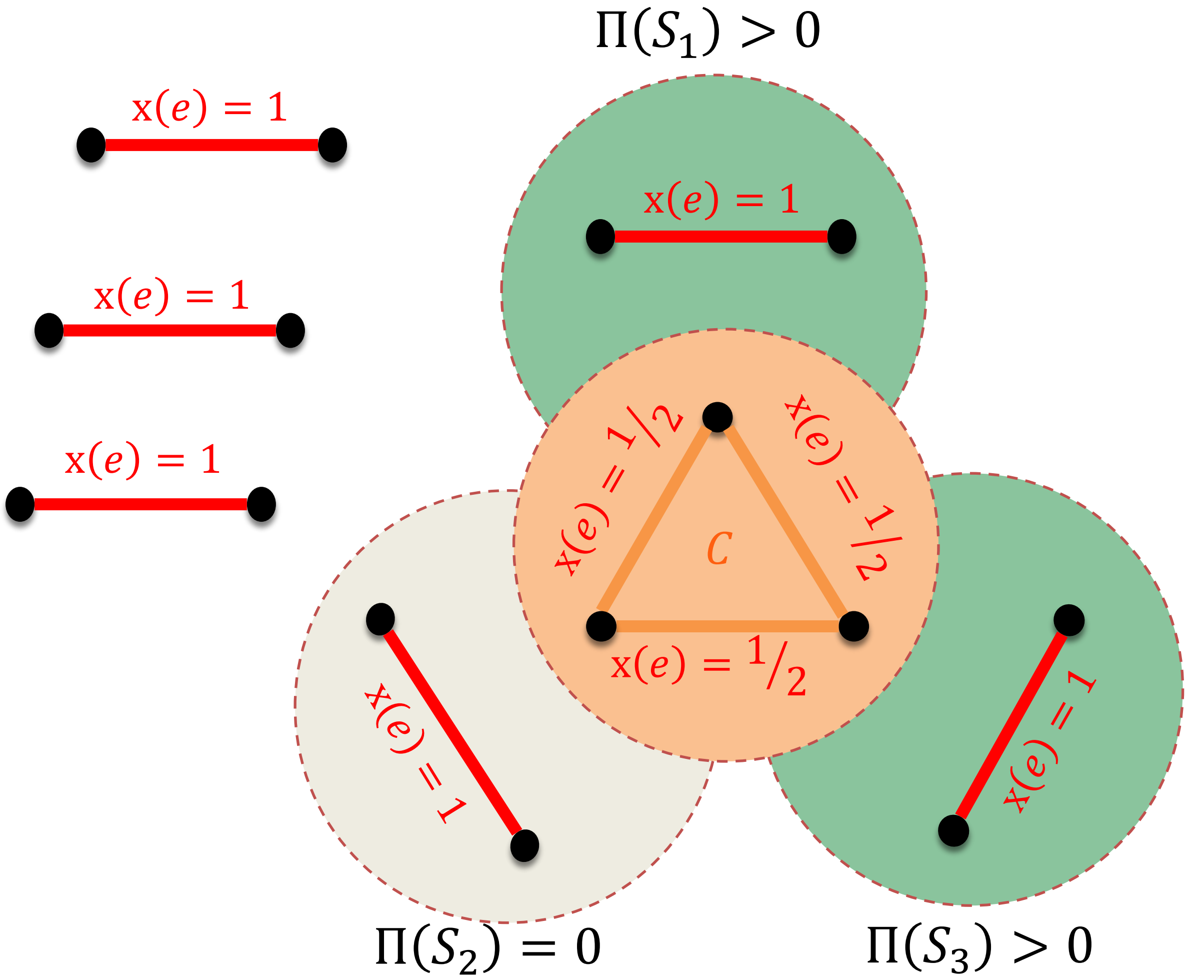

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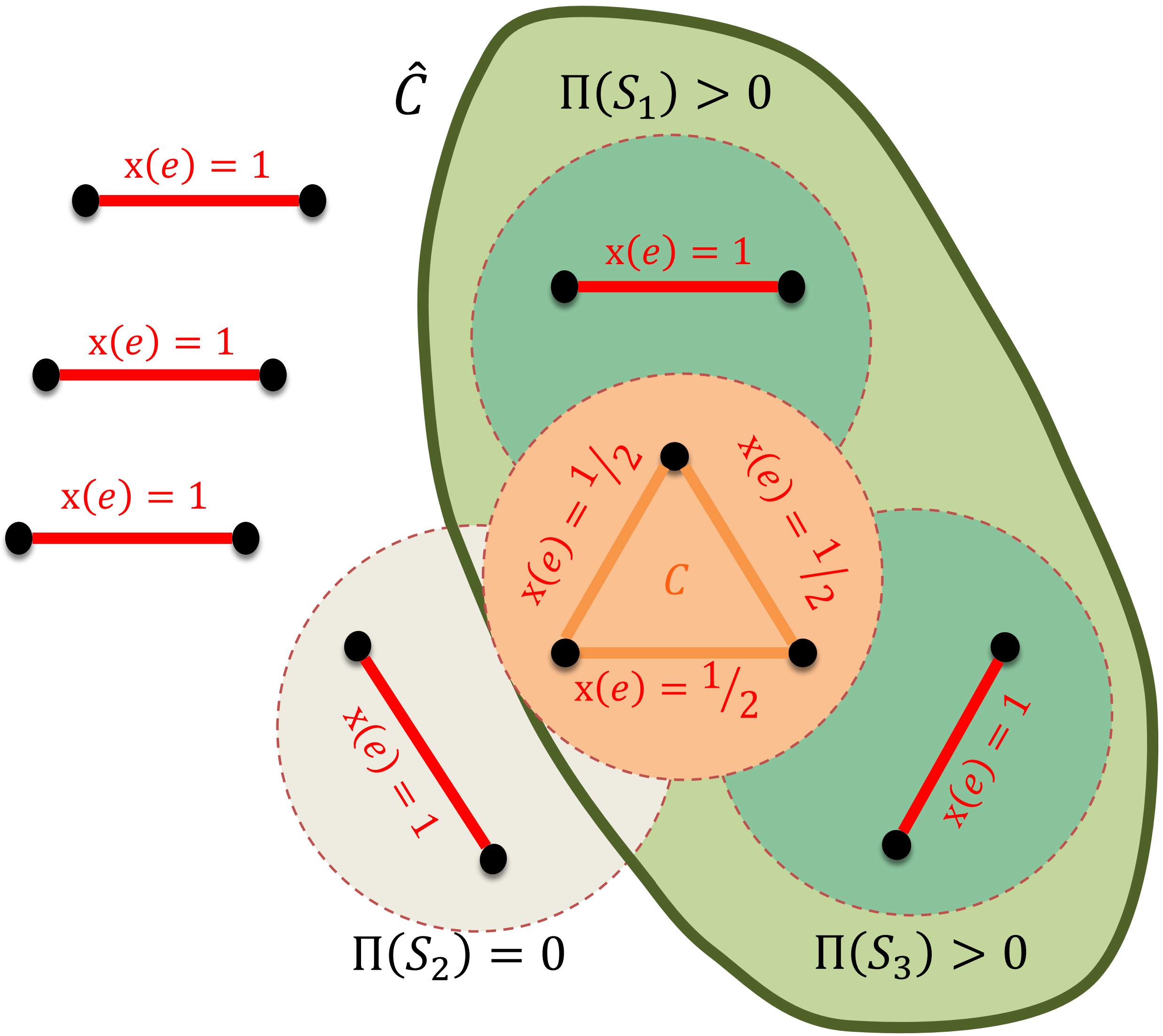

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$H'' = \{\hat{C}\}$

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   - **(b) Choose new cuts.** For each cycle $C \in \text{supp}(x)$, define $\hat{C}$ as the union of $V(C)$ and the inclusionwise maximal sets of H' intersecting $V(C)$

     $$H'' = \{\hat{C} : C \in \text{supp}(x)\}$$

   - (c) Set the new $F = H' \cup H''$
   - **(d) Re-solve LP.** Find an optimal solution x to P_F

Analysis Overview

- **Laminarity:** Intermediate LPs are defined by a laminar family F of odd sets

  [⇒At most $\frac{n}{2}$ odd-set inequalities in intermediate LPs]

(i) Structural Guarantee: Intermediate LP optima are half-integral and supported by a disjoint union of odd cycles and edges

[⇒Cut-generation in $O(n)$ time]

(ii) Progress: The number of odd cycles $\text{odd}(x)$ in the support of the intermediate LP optima x

- Non-increasing
- Decreases by one in at most $\frac{n}{2odd(x)}$ rounds of cut addition

[⇒Number of rounds of cut addition is $O(n \log n)$]

Half-integral Structure

- Conjecture 0: All intermediate solutions are Half-integral

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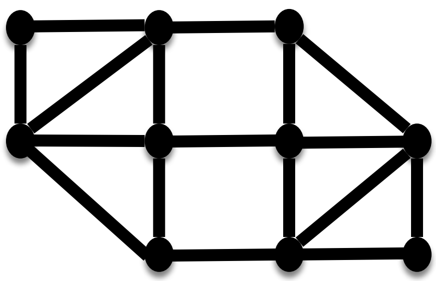

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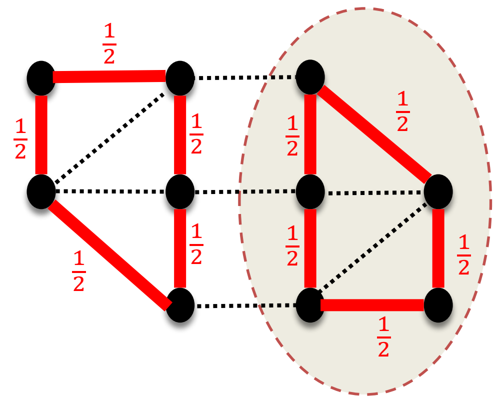

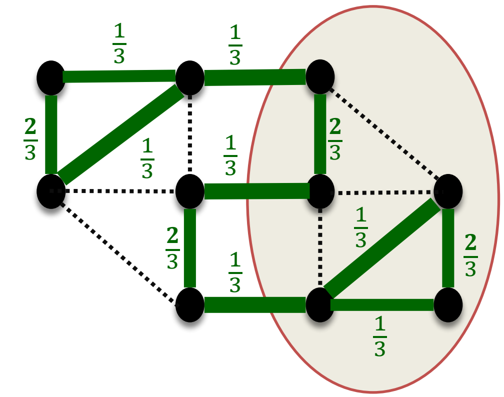

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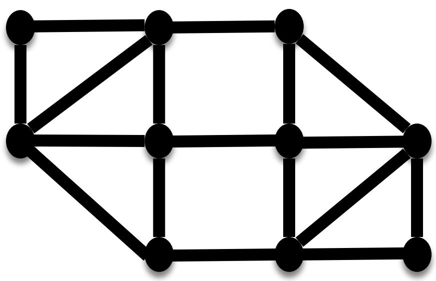

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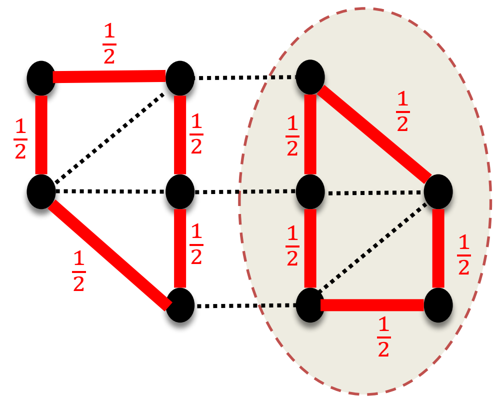

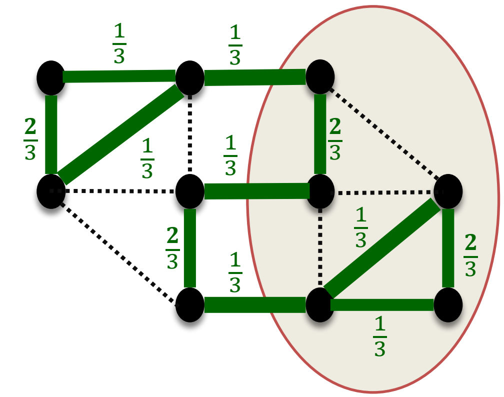

Hey look – the optima are not unique!

Half-integral Structure

- Conjecture 2: Half-integral if the optimum is unique and odd-set inequalities correspond to a laminar family

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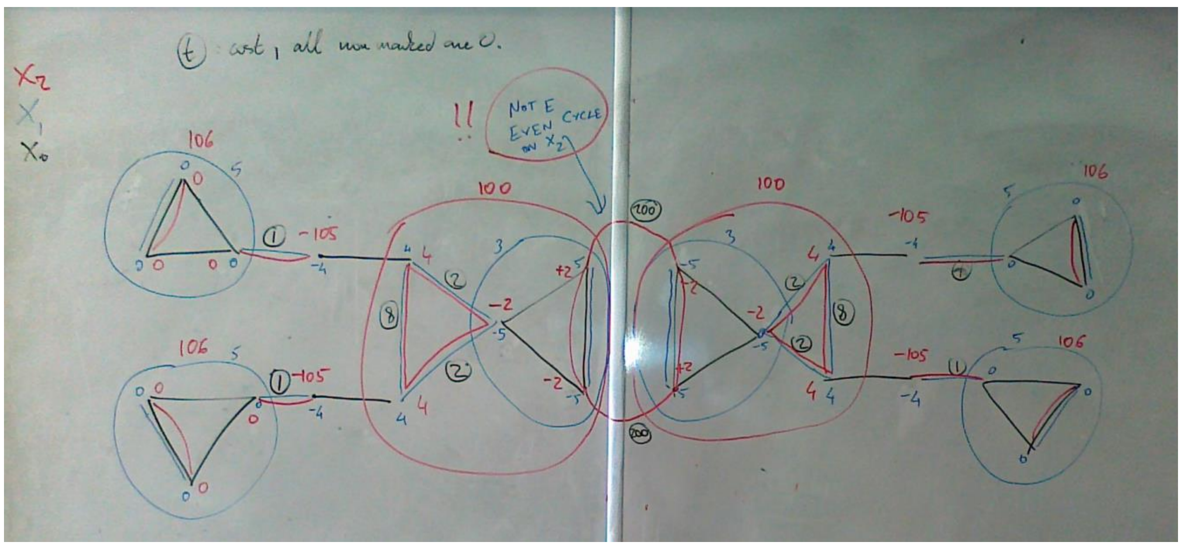

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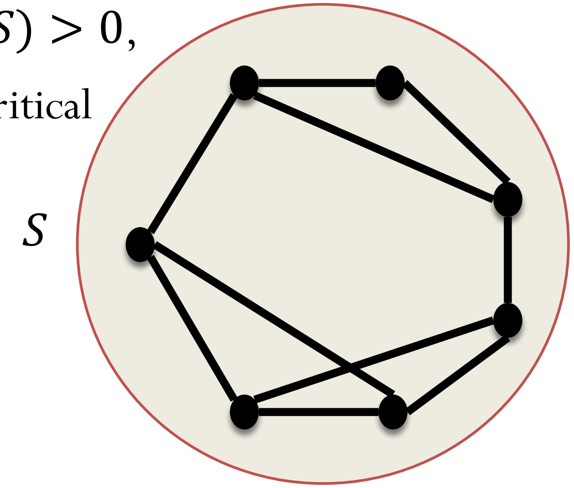
- **Lemma:** If x is unique, F is laminar, and D_F has an F-critical dual optimal solution, then x is half-integral
- For simplicity, say the sets in F are disjoint
- **Definition.** Π is a F-critical dual solution to D_F if $\forall S \in F: \Pi(S) > 0$, the induced graph over S using the tight edges wrt Π is factor-critical
  - For every $u \in S$, there exists a matching M_u
    - M_u covers all vertices in $S \setminus u$



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  - Same notion appears in Edmonds' blossom algorithm.

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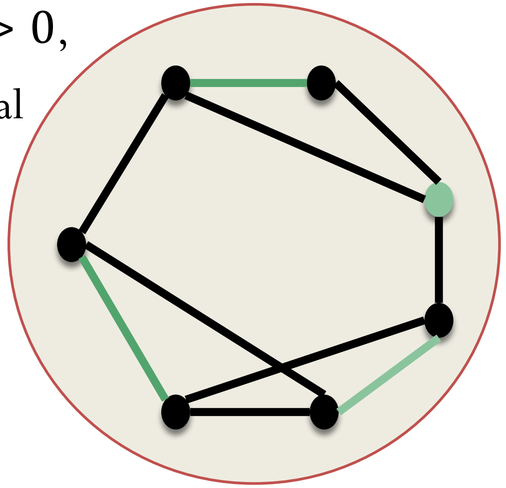

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$$\text{Min} \sum_{uv \in E} c_{uv} x_{uv}$$
$$x(\delta(u)) = 1 \ \forall u \in V$$
$$x(\delta(S)) \geq 1 \ \forall S \in F$$
$$x \geq 0$$

Dual - D_F

$$\text{Max} \sum_{S \in V \cup F} \Pi(S)$$
$$\sum_{S \in V \cup F : uv \in \delta(S)} \Pi(S) \leq c(uv) \ \forall uv \in E$$
$$\Pi(S) \geq 0 \ \forall \ S \in F$$

1. Perturb the integral cost function by adding $\frac{1}{2^i}$ to edge i
2. **Starting LP.** Bipartite relaxation $(F = \emptyset)$
3. Repeat until x is integral
   - **(a) Retain old cuts.** Find a specific dual optimal solution Π to D_F.
     $$H' = \{S \in F : \Pi(S) > 0\}$$
   - **(b) Choose new cuts.** For each cycle $C \in \text{supp}(x)$, define $\hat{C}$ as the union of $V(C)$ and the inclusionwise maximal sets of H' intersecting $V(C)$
     $$H'' = \{\hat{C} : C \in \text{supp}(x)\}$$
   - (c) Set the new $F = H' \cup H''$
   - **(d) Re-solve LP.** Find an optimal solution x to P_F

Analysis Overview

(i) Half-integral Structure: Intermediate LP optima are half-integral and supported by a disjoint union of odd cycles and edges

(ii) Cut Retention: If $\mathrm{odd}(x)$ remains the same in iterations $i, i+1, \ldots, j$ then all cuts added in iterations $i, i+1, \ldots, j$ are retained up to the j'th iteration

Proof of Cut Retention: Coupling with the intermediate solutions of a new Half-Integral Primal-Dual Algorithm for matching

- The choice of specific dual optimal solution to retain cuts comes from this coupling

Half-Integral Primal-Dual Algorithm

Edmonds' Algorithm

- Intermediate Primal – integral
- Unique way to augment primal
- Build an alternating tree using tight edges and repeatedly attempt to augment primal and change dual values until there are no more exposed nodes
- Deshrink if dual value on a set decreases to zero
- Shrink if a set forms a blossom

Half-Integral Algorithm

- Intermediate Primal – half-integral
- 3 ways to augment primal
- Build an alternating tree using tight edges and repeatedly attempt to augment primal and change dual values until there are no more exposed nodes
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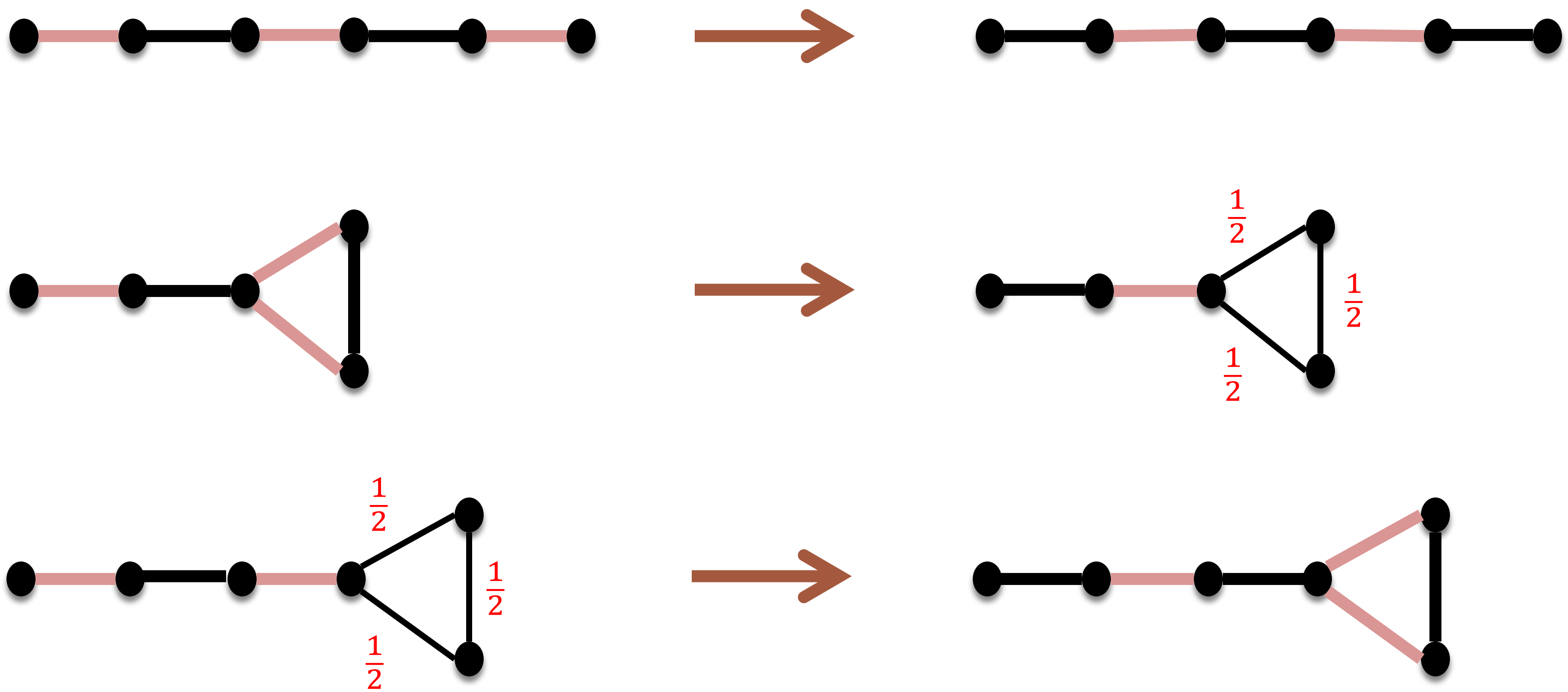
Half-Integral Primal-Dual Algorithm

Primal Augmentations



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Half-Integral Algorithm

- Intermediate Primal – half-integral
- 3 ways to augment primal
- Build an alternating tree using tight edges and repeatedly attempt to augment primal and change dual values until there are no more exposed nodes
- Deshrink if dual value on a set decreases to zero
- Augment primal if a set forms a blossom
- After covering all exposed nodes, shrink all odd cycles to exposed nodes and proceed again until no more odd cycles

Summary

- Main ingredients
  - Drop cuts with zero dual values
    - Common in implementations of cutting plane method to ensure LPs do not blow up in size
  - Add cuts to maintain laminarity

- Tools from the analysis
  - New polyhedral results about the matching polytope
    - Solution to the LP with some odd-set inequalities is half-integral provided certain conditions are satisfied
  - New combinatorial algorithm for matching
    - An alternate primal-dual algorithm for matching where the intermediate solutions are half-integral

Future Directions

- Implications of the Dual-based cut-retention procedure for other poly-time solvable combinatorial problems
  - Combinatorial polytopes with Chvátal rank one (Edmonds-Johnson matrices)

- Efficient cutting plane algorithms for optimization over
  - intersection of two matroid polytopes
  - subtour elimination polytope

Happy Birthday, András!