Combinatorial Optimization and Graph Theory ORCO Matchings in general graphs

Zoltán Szigeti

Matchings in general graphs

Planning

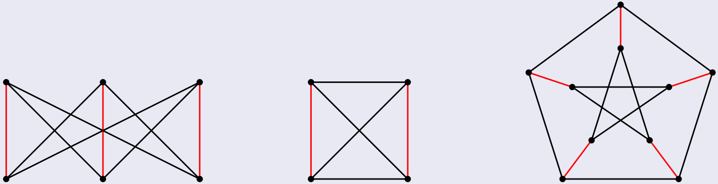
1. Theorems of existence and min-max,
2. Algorithms to find a perfect matching / maximum cardinality matching,
3. Structure theorem.

Application : Assignment of pilots

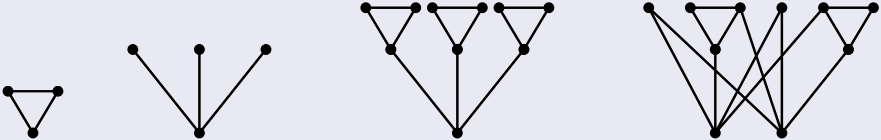
- The manager of an airline wants to fly as many planes as possible at the same time.
- Two pilots must be assigned to each plane.
- Unfortunately, some pilots can not fly together.
- The manager knows all the pilotes, he knows hence wether two pilots are compatible or not.
- How would you model this problem?

Examples

Graphs with perfect matching



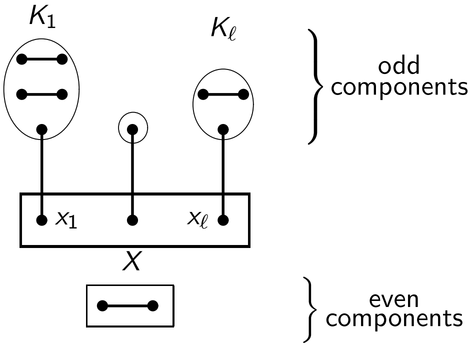
Graphs without perfect matching



Definitions

Definitions

1. Graph odd/even : $|V(G)|$ is odd/even.
2. $c_o(G-X)$: number of odd connected components of $G-X$.
3. barrier : $X \subseteq V(G)$ such that $c_o(G-X) = |X|$.



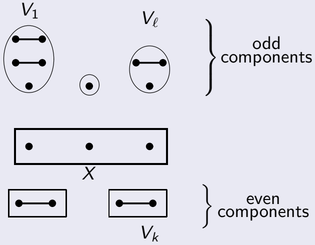
Lemma: $G=(V,E)$ graph, $X \subseteq V$.

1. $|V| \equiv c_o(G-X)+|X| \quad (2)$.
2. If $|V|$ is even, then $c_o(G-X) \equiv |X| \quad (2)$.

Proof

$\{X, V_1, V_2, \dots, V_k$ the connected components of $G-X\}$ = partition of V.

$$\begin{aligned}
|V| &= |X| + \sum_1^k |V_i| \\
&= |X| + \sum_{|V_i|\ odd} |V_i| + \sum_{|V_i|\ even} |V_i| \\
&\equiv |X| + \sum_{|V_i|\ odd} 1 + \sum_{|V_i|\ even} 0 \\
&= |X| + c_o(G-X) \quad (2).
\end{aligned}$$



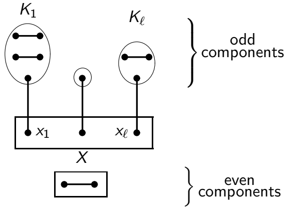
Necessary condition to have a perfect matching

Lemma

If G has a perfect matching M, then $\forall X \subseteq V(G),\ c_o(G-X) \leq |X|$.

Proof

1. Let $K_1, ..., K_\ell$ be the odd connected components of $G - X$.
2. M is a perfect matching and $|V(K_i)|$ is odd, there exists thus an edge of M that connects K_i to a vertex x_i in X.
3. Since M is a matching, $x_i \neq x_j$ if $i \neq j$.
4. $c_o(G-X) = \ell = |\{x_1, ..., x_\ell\}| \leq |X|$.



Characterization of the existence of a perfect matching

Theorem of Tutte

G has a perfect matching $\Longleftrightarrow c_o(G-X) \leq |X| \ \ \forall X \subseteq V(G) \ \ (T)$.

Proof

Let G be a connected graph that satisfies (T) and X a maximal barrier.

1. $|V(G)|$ is even.
2. $X \neq \emptyset$.
3. $G-X$ has no even connected component.
4. For each vertex v of each odd connected component K of $G-X$, $K-v$ satisfies (T).
5. Let $B_X = (X, Y; F)$ be the bipartite graph obtained from G by contracting each odd connected component of $G-X$ and by deleting all the edges in $G[X]$. B_X has a perfect matching.
6. These imply that G has a perfect matching.

Proof of Theorem of Tutte

Proof of 1.

1. By (T) for the set $\emptyset$, we have $0 \leq c_o(G - \emptyset) \leq |\emptyset| = 0$.
2. Hence G has no odd connected component.
3. Since G is connected, G has one connected component that is even.

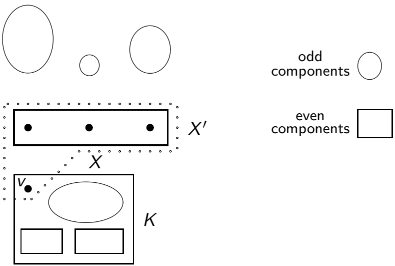
Proof of 2.

1. Let v be an arbitrary vertex of G.
2. By (T) for $\{v\}$, we have, by parity, $1 \leq c_o(G - v) \leq |\{v\}| = 1$.
3. v is hence a barrier.
4. Since X is a maximal barrier, $X \neq \emptyset$.

Proof of Theorem of Tutte

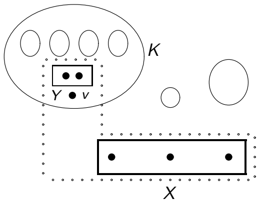
Proof of 3.

1. Suppose by contradiction that K is an even component of $G - X$.
2. By parity, $c_o(K - v) \geq 1 \; \forall v \in V(K)$.
3. By (T) for $X' := X \cup \{v\}$, we have
   $|X'| \geq c_o(G - X') = c_o(G - X) + c_o(K - v) \geq |X| + 1 = |X'|$,
4. X' is hence a barrier.
5. This is a contradiction because X was a maximal barrier.



Proof of 4.

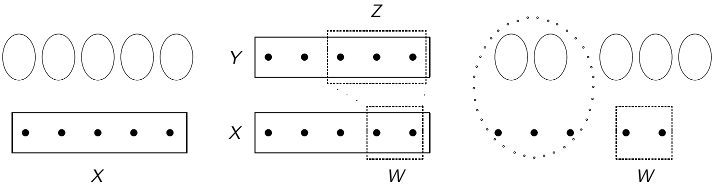
1. Suppose by contradiction that for an odd component K of $G - X$ and a vertex v of K, $H := K - v$ violates the condition (T), that is there exists $Y \subset V(H)$ with $c_o(H - Y) > |Y|$.
2. Since $|V(H)|$ is even, by parity, $c_o(H - Y) \geq |Y| + 2$.
3. By (T) for $X' := X \cup v \cup Y$, we have $|X'| \geq c_o(G - X') = (c_o(G - X) - 1) + c_o(H - Y) \geq (|X| - 1) + (|Y| + 2) = |X'|$,
4. X' is hence a barrier.
5. This is a contradiction because X was a maximal barrier.



Proof of Theorem of Tutte

Proof of 5.

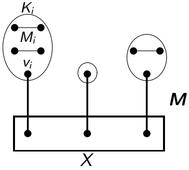
1. Since X is a barrier, we have $|X| = |Y|$.
2. Suppose by contradiction that B_X has no perfect matching. Then, by Hall's Theorem, there exists a set $Z \subseteq Y$ such that for the set $W \subseteq X$ of neighbors of Z, we have $|Z| > |W|$.
3. Each $v \in Z$ corresponds to an odd component of $G - X$,
4. Since $W = \Gamma(Z)$, v corresponds to an odd component of $G - W$.
5. Thus $c_o(G - W) \geq |Z| > |W|$.
6. Then the condition (T) is violated, contradiction.



Proof of Theorem of Tutte

Proof of 6.

1. Let G be a counterexample with $|V(G)|$ minimum.
2. Let X be a maximal barrier of G. By 2, $X \neq \emptyset$.
3. By 5, B_X has a perfect matching M'.
4. For each odd component K_i of $G - X$, M' connects exactly one vertex v_i of K_i to X.
5. By 4, $H_i := K_i - v_i$ satisfies (T) and $|V(H_i)| < |V(G)|$, H_i has hence a perfect matching M_i.
6. By 3, $G - X$ has no even component.
7. $M := M' \cup \bigcup_{K_i} M_i$ is a perfect matching of G,
8. G is not hence a counterexample.



Application

Theorem of Petersen

Every 2-edge-connected 3-regular graph G has a perfect matching.

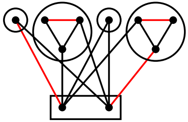
Proof

1. Let X be a subset of vertices of G, $\ell := c_o(G - X)$,
2. $K_1, ..., K_\ell$ the odd components of $G - X$ and
3. E' the set of edges between X and $\bigcup_1^\ell K_i$.
4. Since G is 2-edge-connected and 3-regular, $d(K_i) \geq 2$ and by parity, $d(K_i) \geq 3$, so $3c_o(G - X) = 3\ell \leq \sum_{i=1}^{\ell} d(K_i) = |E'|$.
5. Since G is 3-regular, $|E'| \leq \sum_{v \in X} d(v) = 3|X|$.
6. By consequence, $3c_o(G - X) \leq |E'| \leq 3|X|$.
7. By Tutte's theorem, G has a perfect matching.

Maximum cardinality matching

Formula of Berge – Tutte

1. $\max\{2|M| : M \text{ matching of } G\} = \min\{|V| - c_o(G - X) + |X| : X \subseteq V(G)\}$.
2. $\min\{|M\text{-unsaturated vertices}| : M \text{ matching of } G\} = \max\{c_o(G - X) - |X| : X \subseteq V(G)\}$.



Proof (min $\geq$ max)

Let M be a matching of G, $X \subseteq V(G)$ and $K_1, \dots, K_{c_o(G-X)}$ the odd components of $G - X$.

1. $|V(K_i)|$ odd $\Rightarrow$ each K_i contains at least one vertex v_i such that
   1. either v_i is M-unsaturated,
   2. or v_i is connected to a vertex of X by an edge of M.
2. M matching $\Rightarrow \leq |X|$ of them are connected to X by an edge of M,
3. $\geq c_o(G - X) - |X|$ of them contain an M-unsaturated vertex.

Matchings in general graphs

Theorem of Tutte

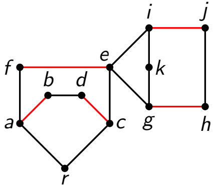
G has a perfect matching $\Longleftrightarrow$ $c_o(G-X) \leq |X| \;\; \forall X \subseteq V(G)$.

Problem

Find a perfect matching.

Theorem of Berge

A matching M of G is of maximum cardinality $\Longleftrightarrow$ there exists no M-augmenting path.



Idea 1

1. Find an augmenting path.
2. As a searching algorithm uses an arborescence to find a path from s to t, we will construct an alternating tree.

Alternating trees

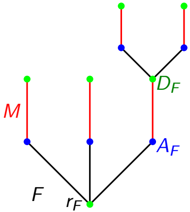
Definition : Given a graph G and a matching M of G,

1. M-alternating tree:
   1. a tree F in G,
   2. a vertex r_F of F is M-unsaturated (r_F is the root of F),
   3. in F, every vertex of odd distance from r_F is of degree 2,
   4. every unique elementary (r_F, v)-path in F is M-alternating.
2. odd/even vertex: $v \in V(F)$ such that $\text{dist}_F(r_F, v)$ is odd/even.
3. A_F and D_F: Set of odd/even vertices of F.

Remark

$|D_F| = |A_F| + 1$:

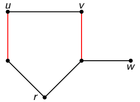
1. $F - r_F$ has a perfect matching connecting always a vertex of A_F to a vertex of D_F,
2. r_F is in D_F.



Blossoms

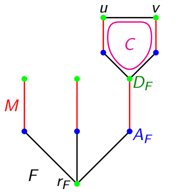
Example

Constructing an alternating tree is not sufficient.



Definition: Given an M-alernating tree F

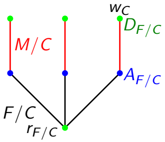
blossom: the unique odd cycle C of $F + uv$ where u and v are two even vertices of F.



Idea 2 of Edmonds

Let's shrink the blossom!

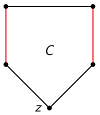
1. We will have pseudo-vertices w_C!
2. M/C is a matching of G/C.
3. F/C is an M/C-alternating tree.
4. Every pseudo-vertex is an even vertex.



Blowing up a blossom

Lemma

For every odd cycle C and for every $z \in V(C)$, $C - z$ has a perfect matching.

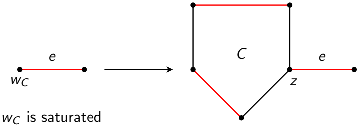


Definition

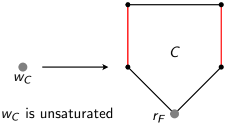
blow up a blossom: w_C is replaced by the odd cycle C.

Lemma

When we blow up a blossom we can extend the matching such that the number of unsaturated vertices does not augment.



when w_C is saturated



when w_C is unsaturated

Perfect matching algorithm of Edmonds

INPUT: $G = (V, E)$ a graph.
OUTPUT: Either a perfect matching M of G
or a set X violating the Tutte condition.

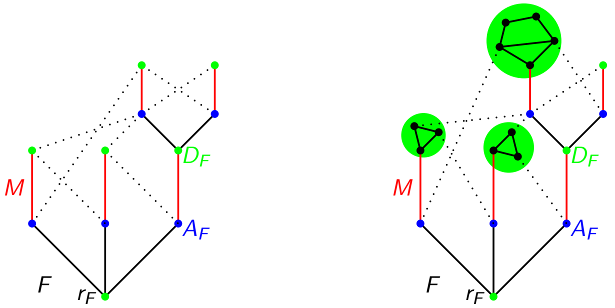
Step 0. *Initialization.*
$M_0 := \emptyset, i := 0,$
$G_0 := G, j := 0.$

Step 1. *Stopping rule 1.*
If all the vertices of G are M_i-saturated then Stop with M_i.

Step 2. *Beginning of the construction of an alternating tree.*
Let r be an M_i-unsaturated vertex.
$F_0 := (r, \emptyset), k := 0.$

Step 3. *Stopping rule 2.*
If every edge of G_j leaving an F_k-even vertex enters
an F_k-odd vertex then stop with $X :=$ set of F_k-odd vertices.

Perfect matching algorithm of Edmonds



Perfect matching algorithm of Edmonds

Step 4. *Choice of the edge $u_k v_k$.*

Let $u_k v_k$ be an edge of G_j such that u_k is an F_k-even vertex and v_k is not an F_k-odd vertex.

Step 5. *Augmenting the alternating tree.*

If v_k is not in F_k and there exists $v_k w_k \in M_i$ then do

$F_{k+1} := F_k + u_k v_k + v_k w_k$, $k := k + 1$,

Go to Step 3.

Step 6. *Shrinking a blossom.*

If v_k is in F_k then do

$C_j :=$ be the unique cycle in $F_k + u_k v_k$,

$M_{i+1} := M_i / C_j$, $i := i + 1$,

$G_{j+1} := G_j / C_j$, $j := j + 1$,

$F_{k+1} := F_k / C_j$, $k := k + 1$,

Go to Step 3.

Perfect matching algorithm of Edmonds

Step 7. *Augmenting the matching in G.*

If v_k is not in F_k and v_k is M_i-unsaturated then do

Constructing an M_i-augmenting path in G_j.

$Q_i :=$ the unique elementary path in F_k from r_F to u_k,

$P_i := Q_i + u_k v_k$.

Augmenting the matching in G_j.

$M_{i+1} := (M_i \setminus E(P_i)) \cup (E(P_i) \setminus M_i), i := i + 1.$

Blowing up the shrunk blossoms in reverse order.

If the shrunk blossoms are $C_0, \dots, C_{j-1}$ then

While $j \neq 0$ do

$e_{j-1} :=$ the edge of M_i incident to $w_{C_{j-1}}$,

$G_{j-1} := G_j \div C_{j-1}$ (blowing up of C_{j-1}),

$z_{j-1} :=$ the vertex of C_{j-1} incident to e_{j-1},

$M_{i+1} := M_i \cup$ the perfect matching of $C_{j-1} - z_{j-1}$.

$i := i + 1, j := j - 1.$

Go to Step 1.

Justification of the perfect matching algorithm

Theorem

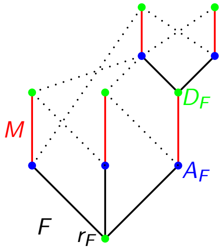
1. If the algorithm stops at Step 3 then
   1. the vertices in D_{F_k} are isolated in $G_j - A_{F_k}$,
   2. every vertex of D_{F_k} corresponds to a set of vertices of G of odd cardinality,
   3. $X = A_{F_k}$ violates Tutte's condition.
2. At Step 7 the size of the matching of G increased.
3. The algorithm stops in polynomial time.

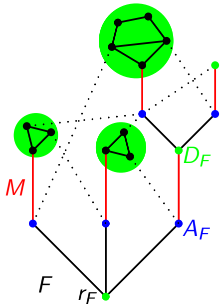
Remark

Edmonds' algorithm implies Tutte's theorem.

INPUT: $G = (V, E)$ a graph.
OUTPUT: Either a perfect matching M of G
or a set X violating the Tutte condition.

Justification of the perfect matching algorithm





Justification of the perfect matching algorithm

Proof

1. If algorithm stops at Step 3: $uv \in E(G_j), u \in D_{F_k} \Longrightarrow v \in A_{F_k}$.
   1. The vertices in D_{F_k} are hence isolated in $G_j - A_{F_k}$.
   2. When we blow up a pseudo-vertex w_C, we replace a vertex by an odd number of vertices since C is a blossom.
   3. By 1 and 2, $c_o(G - X) = c_o(G - A_F) \geq |D_F| = |A_F| + 1 > |X|$.
2. The size of the matching increases:
   1. in the shrunk graph: two M_i-unsaturated vertices become M_{i+1}-saturated and the M_i-saturated vertices remain M_{i+1}-saturated.
   2. Blowing up a blossom does not increase the number of unsaturated vertices (by the blowing up lemma).
3. The algorithm stops in polynomial time:
   1. Every step is polynomial.
   2. Matching augmentation may happen at most $n/2$ times.
   3. Between two matching augmentations:
      1. tree augmentation may happen at most $n/2$ times,
      2. blossom shrinking may happen at most $n/2$ times.

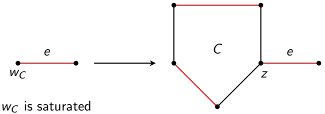
Maximum cardinality matching algorithm of Edmonds

INPUT: $G = (V, E)$ a graph.
OUTPUT: A maximum cardinality matching of G.

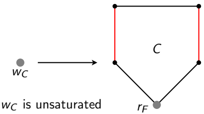
Step 1. *Constructing alternating trees.*
While there exists an unsaturated vertex do
Execute the perfect matching algorithm with the following modification :
before stopping at Step 3 delete the vertices of the alternating tree.

Step 2. *Blowing up the shrunk blossoms in reverse order.*
As in the perfect matching algorithm by adding that
if $w_{C_{j-1}} = r_{F_k}$ then $z_{j-1} := r_{F_{k-1}}$.

Step 3. Stop.



when w_C is saturated

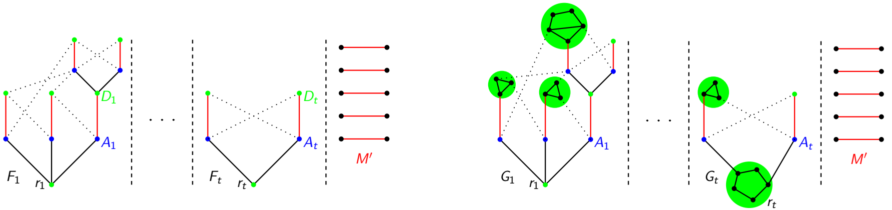


when w_C is unsaturated

Justification of the max. cardinality matching algorithm

Structure

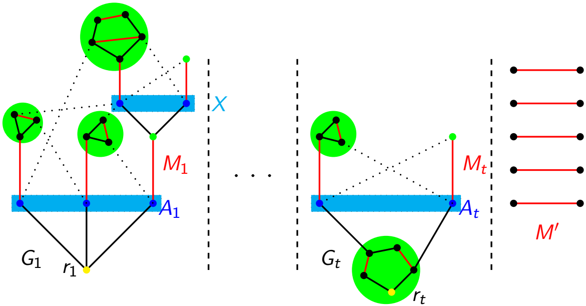
- When the algorithm stops we have t vertex disjoint alternating trees $F_1, \dots, F_t$ and a perfect matching M' of the remaining graph.
- Every vertex of D_i corresponds to an odd connected subgraph of G.
- Every F_i corresponds to a subgraph G_i of G.



Justification of the max. cardinality matching algorithm

Theorem

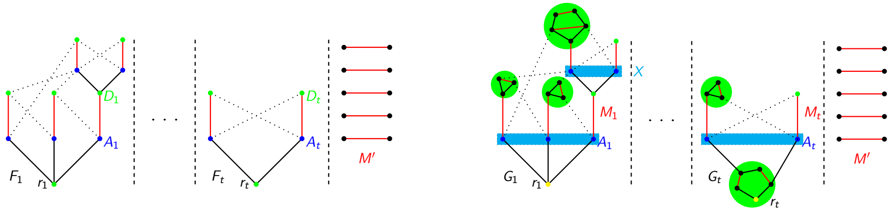
1. G_i has a matching M_i with exactly one M_i-unsaturated vertex r_i in G_i.
2. $c_o(G - X) - |X| = t$ where $X := \bigcup_{i=1}^{t} A_{F_i}$.
3. $M := M' \cup \bigcup_{i=1}^{t} M_i$ is a maximum cardinality matching of G.
4. The algorithm stops in polynomial time.



Justification of the max. cardinality matching algorithm

Proof

1. F_k has one unsaturated vertex (r_{F_k}) and blowing up a blossom does not increase the number of unsaturated vertices (by the blowing up lemma).
2. $c_o(G-X) = c_o(G - \bigcup_{i=1}^t A_{F_i}) = \sum_{i=1}^t |F_i - A_{F_i}| = \sum_{i=1}^t |D_{F_i}| = \sum_{i=1}^t (|A_{F_i}|+1) = |\bigcup_{i=1}^t A_{F_i}| + t = |X| + t$.



Justification of the max. cardinality matching algorithm

Proof

3. By the trivial direction of the formula of Berge –Tutte, 1 and 2, $\max \leq \min \leq |\{M\text{-unsaturated vertices}\}| = t = c_o(G - X) - |X| \leq \max$.
   Thus $\min\{|M\text{-unsaturated vertices}| : M \text{ matching of } G\} = \max\{c_o(G - X) - |X| : X \subseteq V(G)\}$.
4. We execute at most n times the perfect matching algorithm which is polynomial, and hence this algorithm is polynomial.

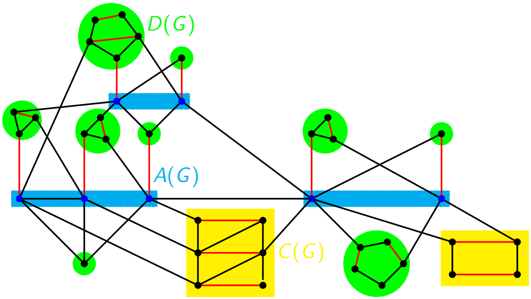
Remark

This algorithm of Edmonds implies the Berge –Tutte formula.

Structure theorem

Definitions

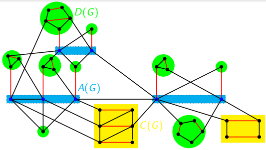
1. near-perfect matching : a matching of G covering all but one vertex,
2. Factor-critical graph : $G - v$ has a perfect matching $\forall v \in V(G)$,
3. $D(G) := \{v \in V : \nu(G - v) = \nu(G)\}$,
4. $A(G) :=$ set of neighbors of $D(G)$ in $V(G) \setminus D(G)$,
5. $C(G) := V(G) \setminus (D(G) \cup A(G))$.



Structure theorem

Theorem (Gallai-Edmonds)

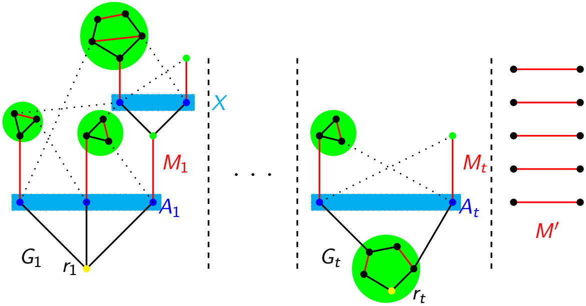
1. The connected components of $G[D(G)]$ are factor-critical.
2. $C(G)$ has a perfect matching.
3. $c_o(G - A(G)) - |A(G)| = \max\{c_o(G - X) - |X| : X \subseteq V\}$.
4. $B_{A(G)}$ has a matching
   1. covering $A(G)$ and
   2. not covering any given vertex $v \notin A(G)$.
5. A matching M is of maximum cardinality if and only if M contains
   1. a near-perfect matching of each connected components of $G[D(G)]$,
   2. a perfect matching of $C(G)$ and
   3. a matching of $B_{A(G)}$ covering $A(G)$.



Structure theorem

Proof

1. Execute Edmonds' maximum cardinality matching algorithm and let $D := \bigcup_{i=1}^{t} D_i$, $A := \bigcup_{i=1}^{t} A_i$, $C := V(G) \setminus (D \cup A)$.
2. D, A, C verify the assertions of the theorem.
3. $D = D(G), A = A(G), C = C(G)$.
4. Thus $D(G), A(G), C(G)$ verify the assertions of the theorem.



Proof of (1) for D

1. By the blowing up lemma, blowing up an odd cycle in a factor-critical graph results a factor-critical graph.
2. Since a connected component of $G[D]$ is obtained from a (pseudo-) vertex (which is factor-critical) by blowing up odd cycles, it is hence factor-critical.

Proof of (2) for C

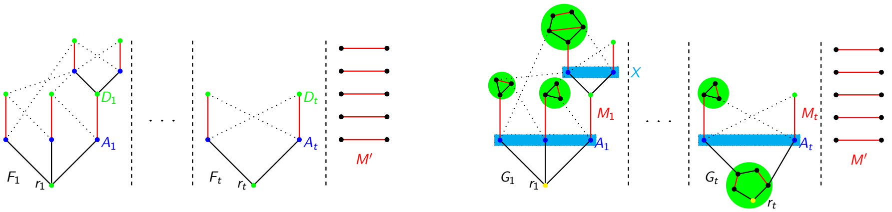
M' is a perfect matching of C.

Proof of (3) for A

$c_o(G - A) - |A| = \max\{c_o(G - X) - |X| : X \subseteq V\}$ was proved in the previous theorem.

Proof of (4) for A

1. B_A is the union of the alternating trees found by the algorithm.
2. For a given $v \in D_i$, let P be the alternating path between r_i and v.
3. By changing the role of matching edges and non-matching edges in P, we get the required matching.



Proof of (5) for D, C, A

It follows directly from (3).

Proof of $D = D(G), A = A(G), C = C(G)$

1. It is enough to prove that $D = D(G)$.
2. Since D, C, A satisfy (5), we have $D(G) \subseteq D$.
3. Since D satisfies (1) and A satisfies (4), we have $D \subseteq D(G)$.