

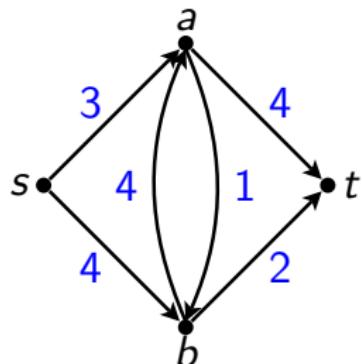
Combinatorial Optimization and Graph Theory

ORCO

Execution of the Push-Relabel Algorithm

Zoltán Szigeti

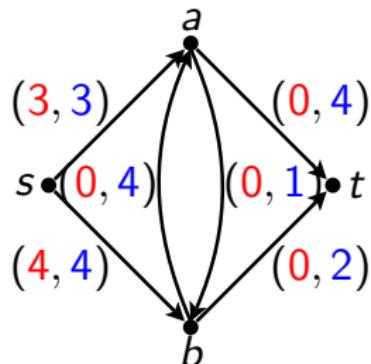
Execution of the Push-Relabel algorithm



Problem

Find in the network (D, g) a g -feasible (s, t) -flow of **maximum** value and an (s, t) -cut of **minimum** capacity!

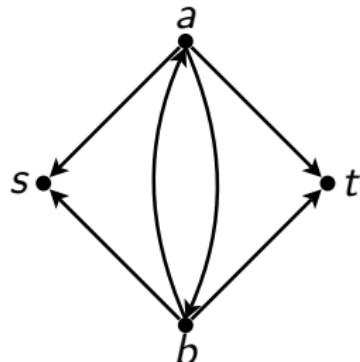
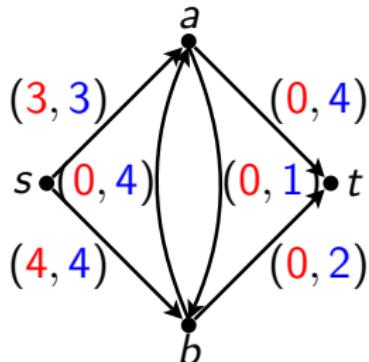
Execution of the Push-Relabel algorithm



INITIALIZATION

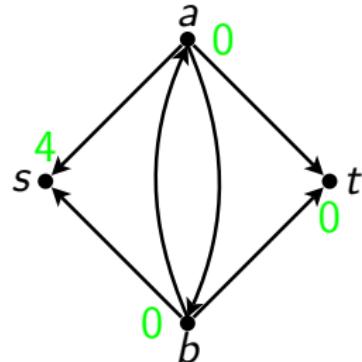
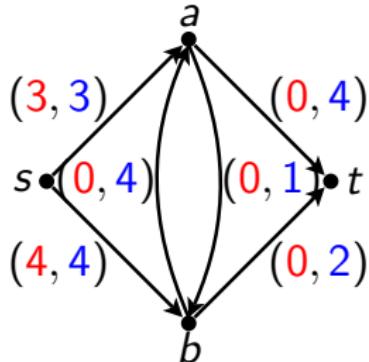
Preflow x

Execution of the Push-Relabel algorithm



Construction
Auxiliary graph D_x

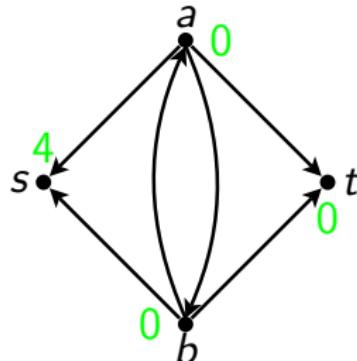
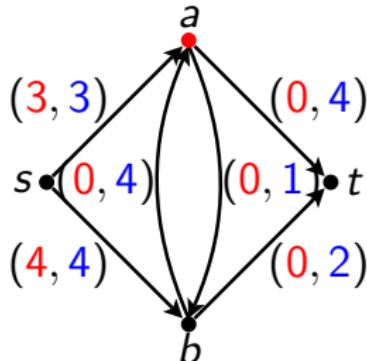
Execution of the Push-Relabel algorithm



INITIALIZATION

Labelling ℓ

Execution of the Push-Relabel algorithm

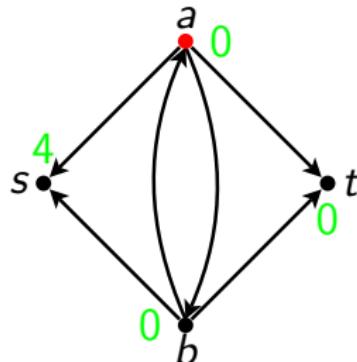
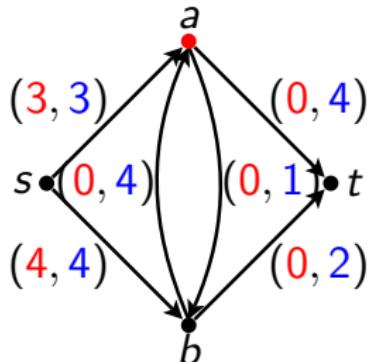


Looking for active vertex

$$\begin{aligned}f_x(a) &= d_x^-(a) - d_x^+(a) \\&= (3 + 0) - (0 + 0) = 3 > 0\end{aligned}$$

$\implies a$ is x -active

Execution of the Push-Relabel algorithm

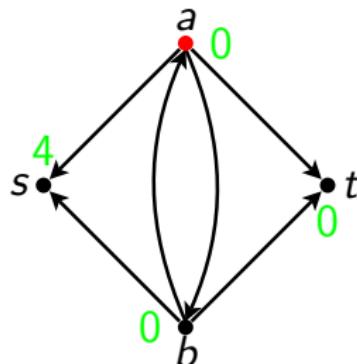
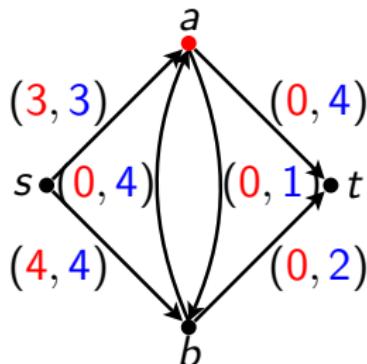


Looking for tight arc leaving a

$$\ell(a) = 0 \neq 1 = \ell(b) + 1 = \ell(t) + 1 \\ \neq 5 = \ell(s) + 1$$

\Rightarrow no ℓ -tight arc leaving a exists

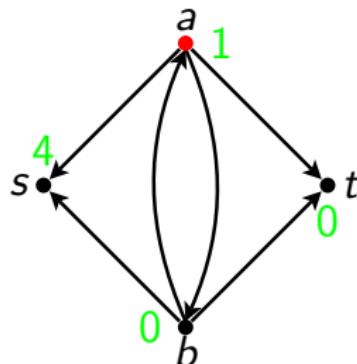
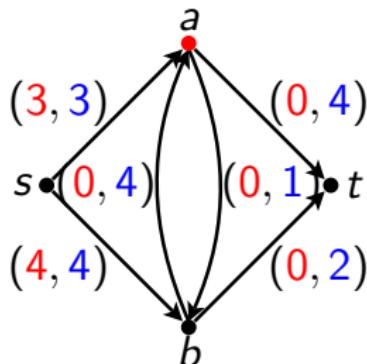
Execution of the Push-Relabel algorithm



RELABEL at a

$$\begin{aligned}\ell'(a) &= \min\{\ell(v) + 1 : av \in A_x\} \\ &= \min\{0 + 1, 0 + 1, 4 + 1\} = 1\end{aligned}$$

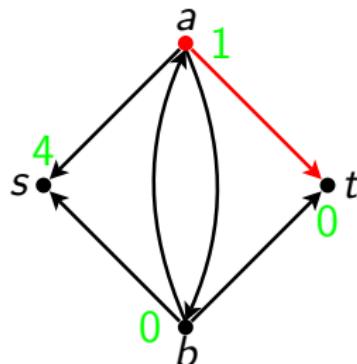
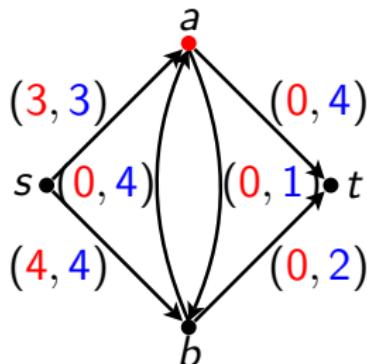
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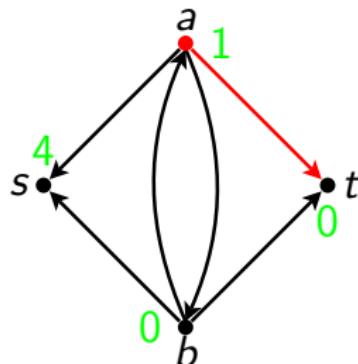
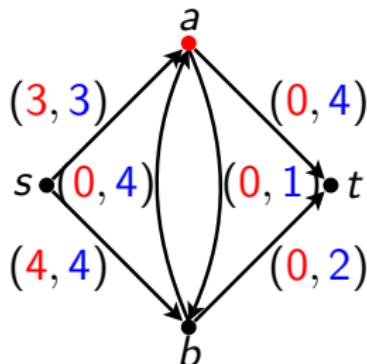


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arc at becomes ℓ' -tight

Execution of the Push-Relabel algorithm



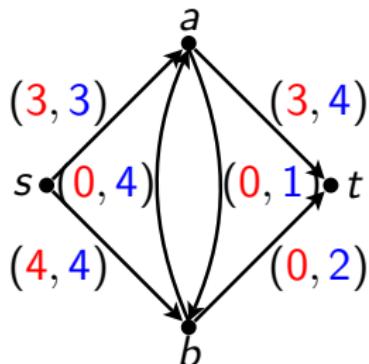
PUSH on at

$$\begin{aligned}\varepsilon &= \min\{\bar{g}(at), f_x(a)\} \\ &= \min\{4 - 0 + 0, 3\} = 3\end{aligned}$$

$$x'(at) = x(at) + \varepsilon = 0 + 3 = 3$$

$$x'(e) = x(e) \quad \forall e \in A \setminus \{at\}$$

Execution of the Push-Relabel algorithm

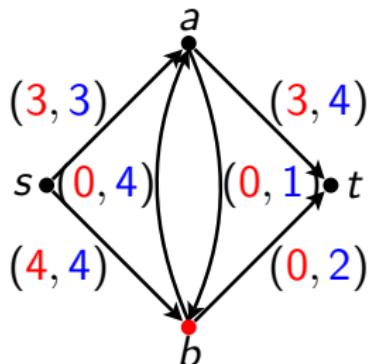


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a is not x' -active anymore

Execution of the Push-Relabel algorithm

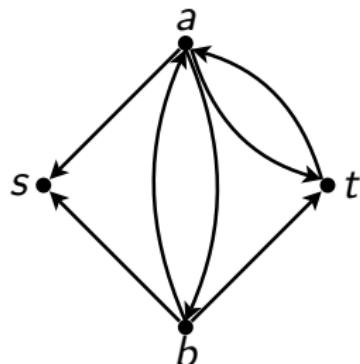
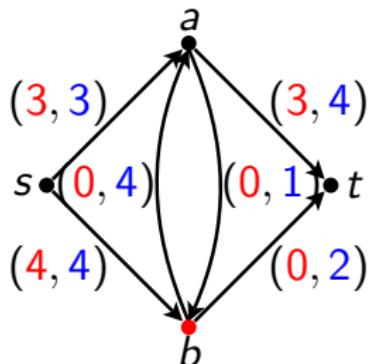


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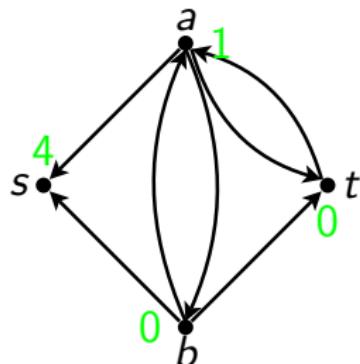
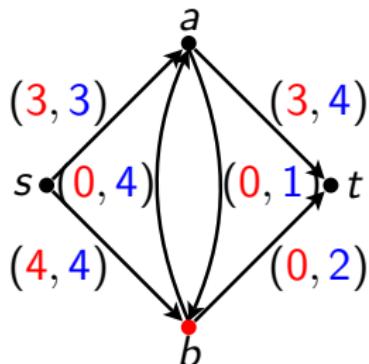
$\Rightarrow b$ is x -active

Execution of the Push-Relabel algorithm



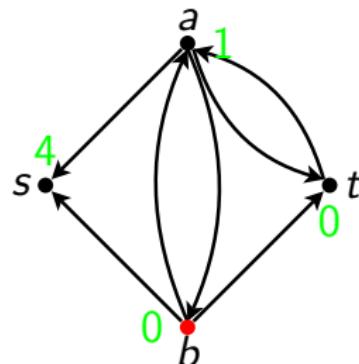
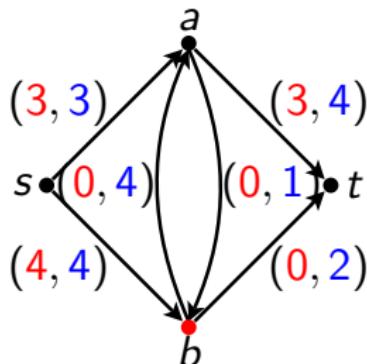
Construction
Auxiliary graph D_x

Execution of the Push-Relabel algorithm



Previous labelling

Execution of the Push-Relabel algorithm

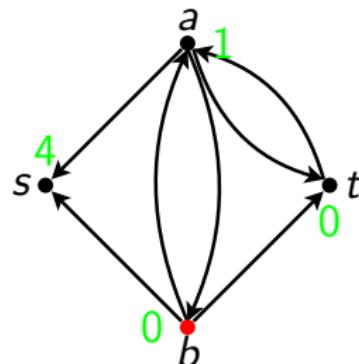
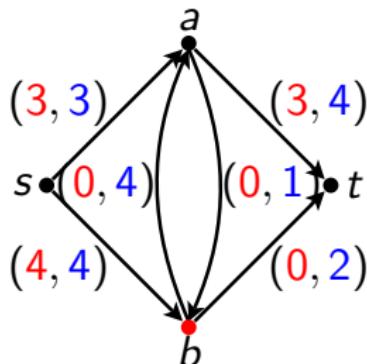


Looking for tight arc leaving b

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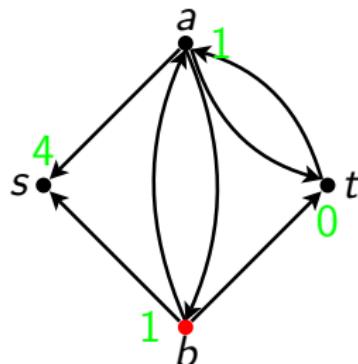
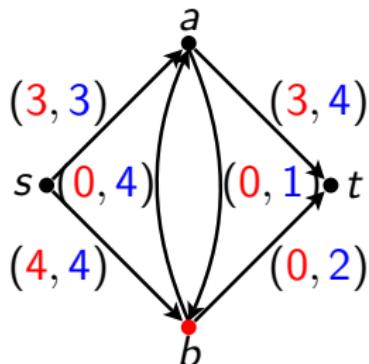
Execution of the Push-Relabel algorithm



RELABEL at b

$$\begin{aligned}\ell'(b) &= \min\{\ell(v) + 1 : bv \in A_x\} \\ &= \min\{1 + 1, 0 + 1, 4 + 1\} = 1\end{aligned}$$

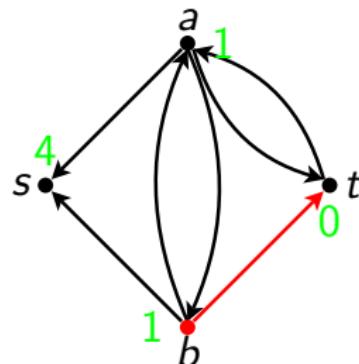
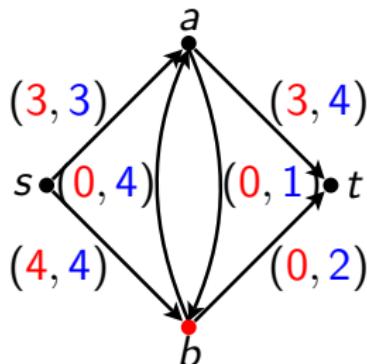
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Execution of the Push-Relabel algorithm

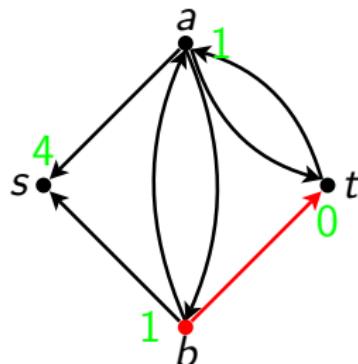
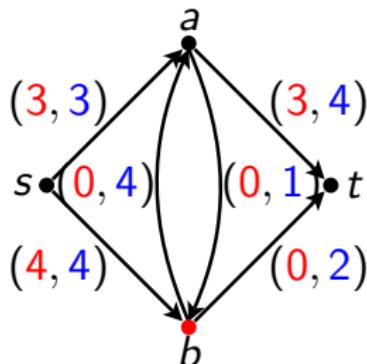


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arc bt becomes ℓ' -tight

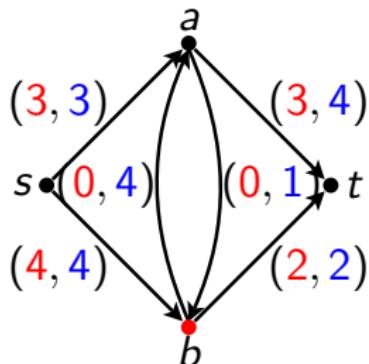
Execution of the Push-Relabel algorithm



PUSH on bt

$$\begin{aligned}\varepsilon &= \min\{\bar{g}(bt), f_x(b)\} \\ &= \min\{2 - 0 + 0, 4\} = 2 \\ x'(bt) &= x(bt) + \varepsilon = 0 + 2 = 2\end{aligned}$$

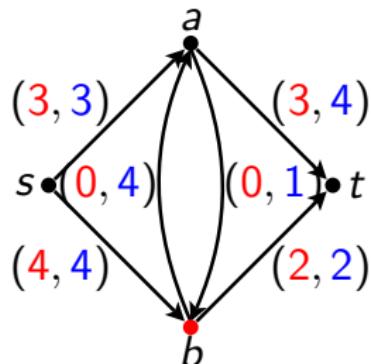
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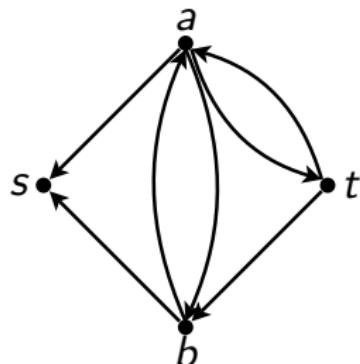
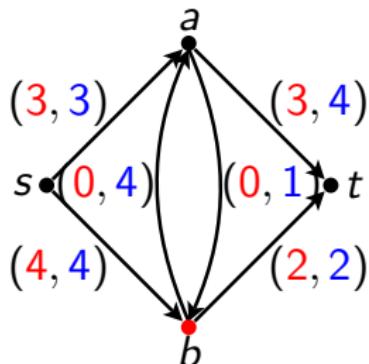


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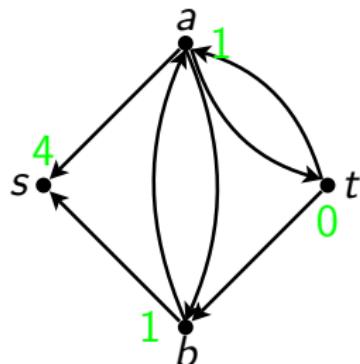
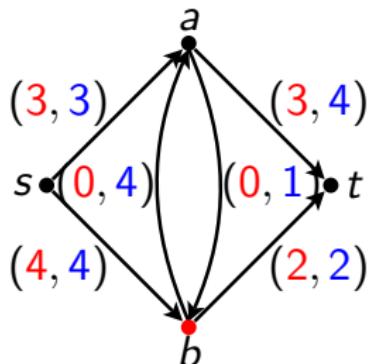
b is still x' -active

Execution of the Push-Relabel algorithm



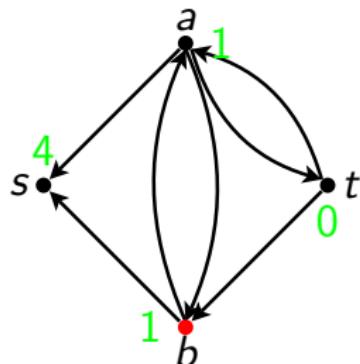
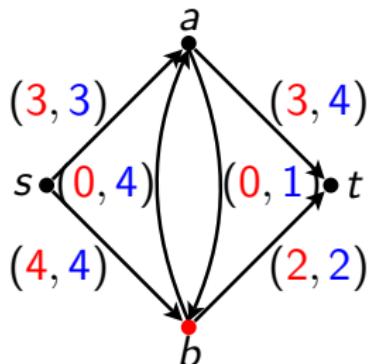
Construction
Auxiliary graph D_x

Execution of the Push-Relabel algorithm



Previous labelling

Execution of the Push-Relabel algorithm

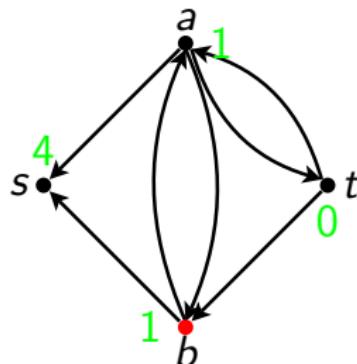
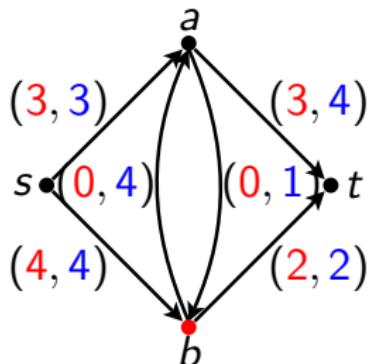


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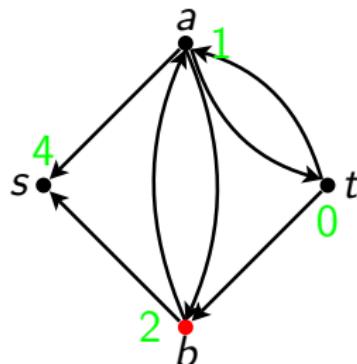
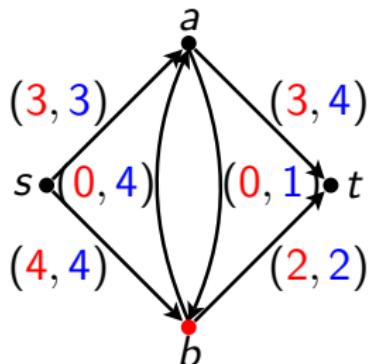
Execution of the Push-Relabel algorithm



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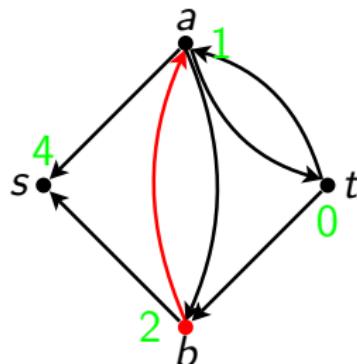
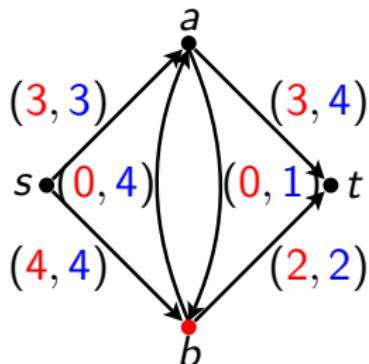
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Execution of the Push-Relabel algorithm

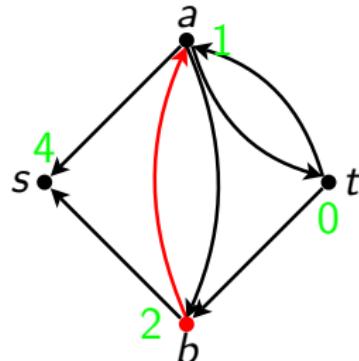
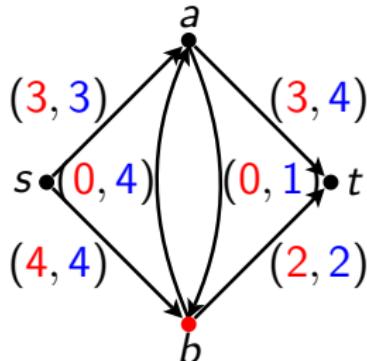


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arc ba becomes ℓ' -tight

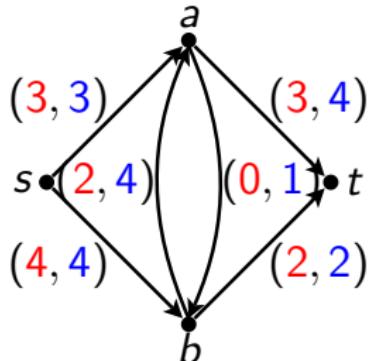
Execution of the Push-Relabel algorithm



PUSH on ba

$$\begin{aligned}\varepsilon &= \min\{\bar{g}(ba), f_x(b)\} \\ &= \min\{4 - 0 + 0, 2\} = 2 \\ \varepsilon' &= \min\{x(ab), \varepsilon\} = \min\{0, 2\} = 0 \\ x'(ba) &= x(ba) + \varepsilon - \varepsilon' = 0 + 2 - 0 = 2 \\ x'(ab) &= x(ab) - \varepsilon' = 0 - 0 = 0\end{aligned}$$

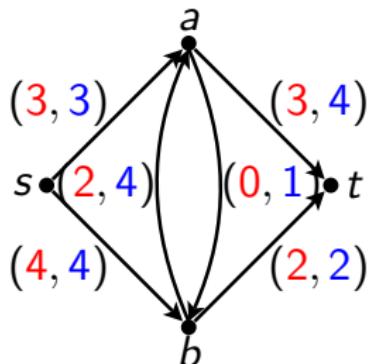
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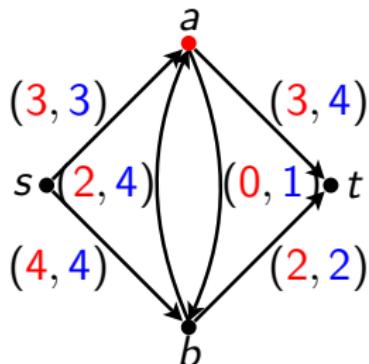


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b is not x' -active anymore

Execution of the Push-Relabel algorithm

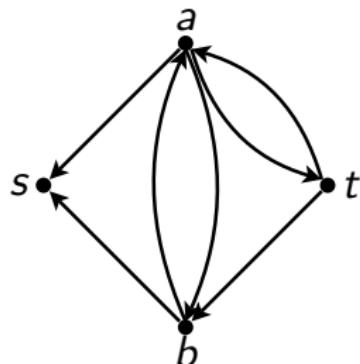
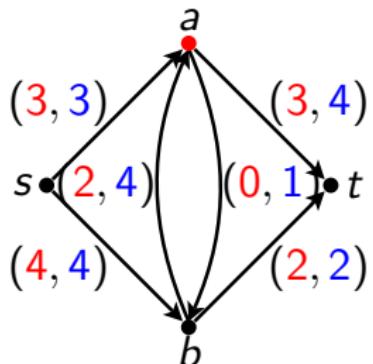


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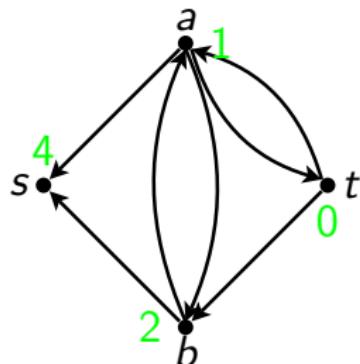
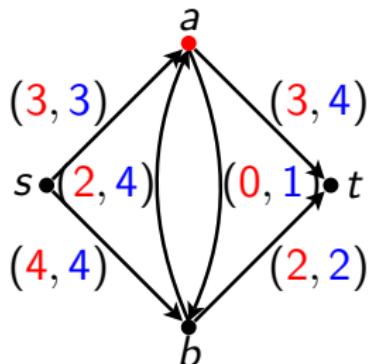
$\Rightarrow a$ is x' -active

Execution of the Push-Relabel algorithm



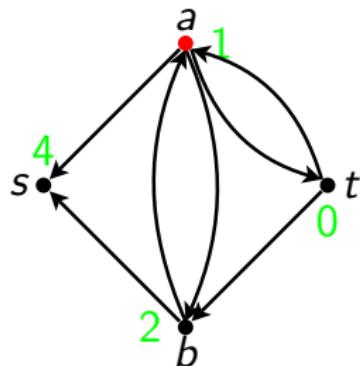
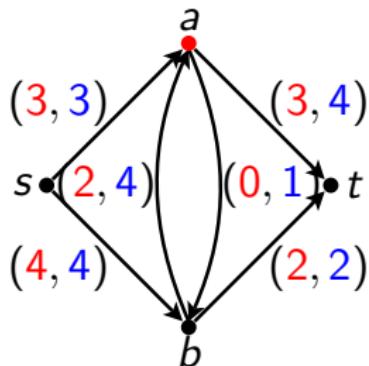
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Execution of the Push-Relabel algorithm



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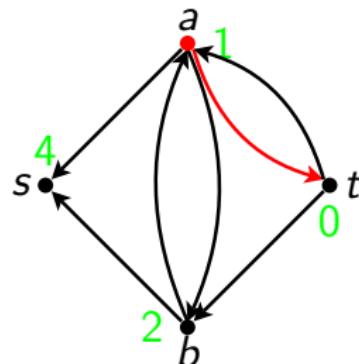
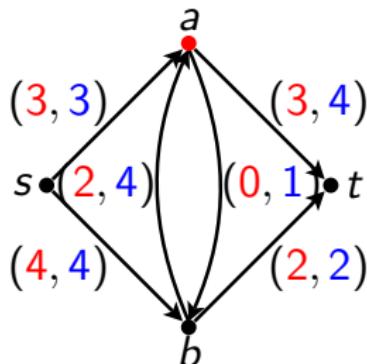
Execution of the Push-Relabel algorithm



Looking for tight arc leaving a

$$\ell(a) = 1 = \ell(t) + 1$$

Execution of the Push-Relabel algorithm

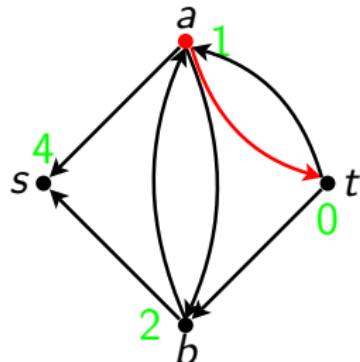
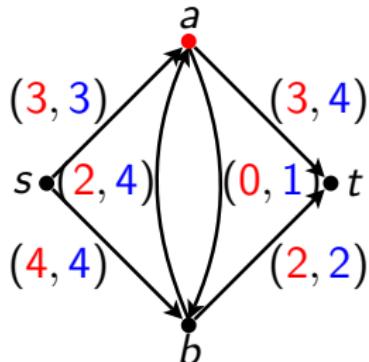


Looking for tight arc leaving a

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\Rightarrow arc at is ℓ -tight

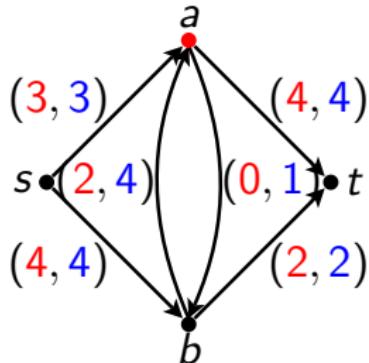
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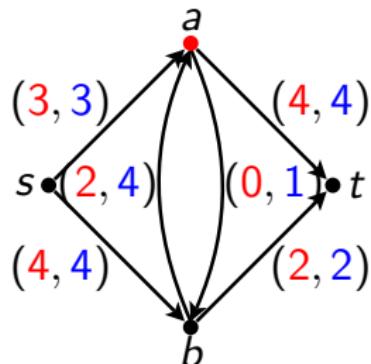
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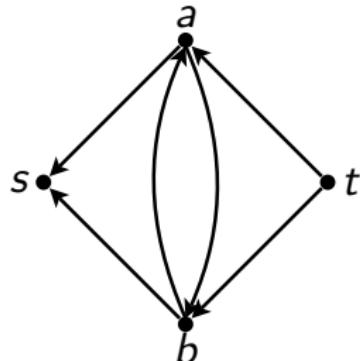
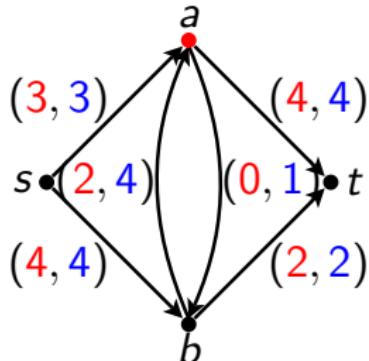


PUSH on at

$$\begin{aligned}\varepsilon &= \min\{\bar{g}(at), f_x(a)\} \\ &= \min\{4 - 3 + 0, 2\} = 1 \\ x'(at) &= x(at) + \varepsilon = 3 + 1 = 4\end{aligned}$$

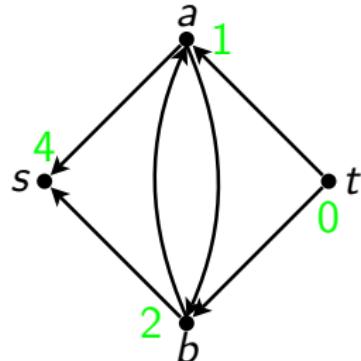
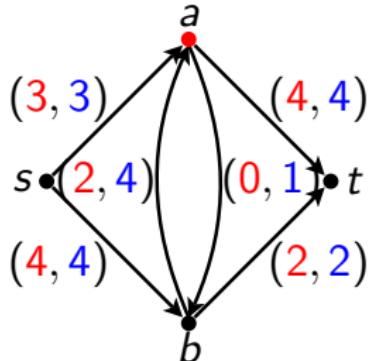
a is still x' -active

Execution of the Push-Relabel algorithm



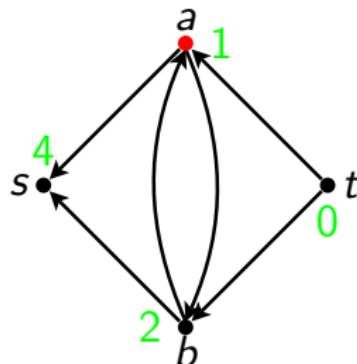
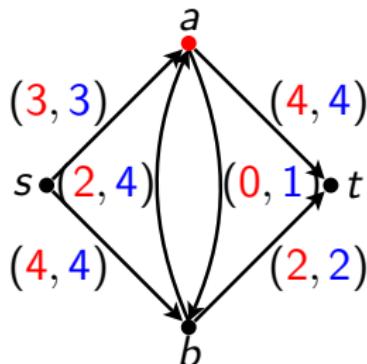
Construction
Auxiliary graph D_x

Execution of the Push-Relabel algorithm



Previous labelling

Execution of the Push-Relabel algorithm

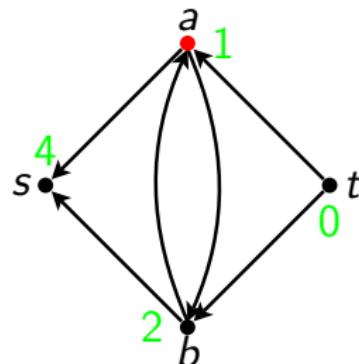
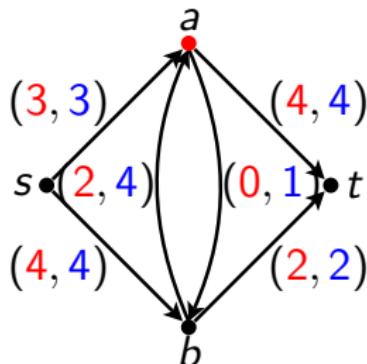


Looking for tight arc leaving a

$$\ell(a) = 1 \neq 3 = \ell(b) + 1 \\ \neq 5 = \ell(s) + 1$$

\Rightarrow no ℓ -tight arc leaving a exists

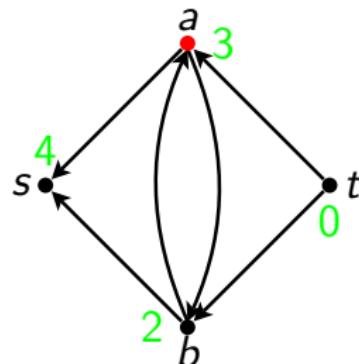
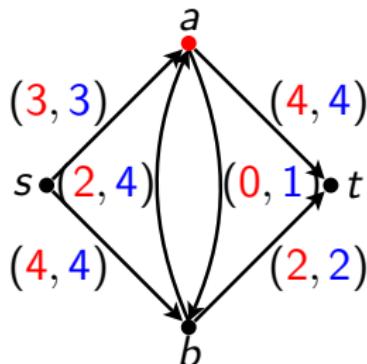
Execution of the Push-Relabel algorithm



RELABEL at a

$$\begin{aligned}\ell'(a) &= \min\{\ell(v) + 1 : av \in A_x\} \\ &= \min\{2 + 1, 4 + 1\} = 3\end{aligned}$$

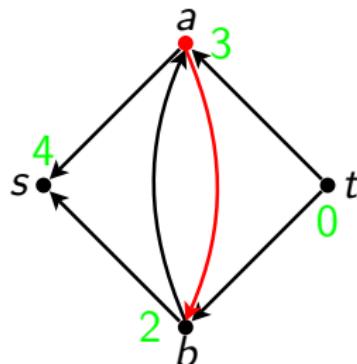
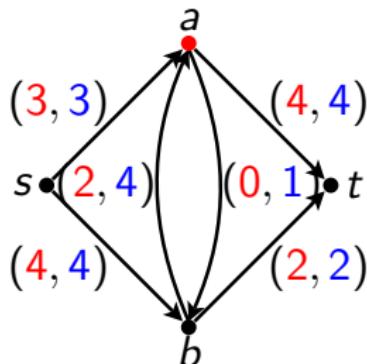
Execution of the Push-Relabel algorithm



RELABEL at a

$$\begin{aligned}\ell'(a) &= \min\{\ell(v) + 1 : av \in A_x\} \\ &= \min\{2 + 1, 4 + 1\} = 3\end{aligned}$$

Execution of the Push-Relabel algorithm

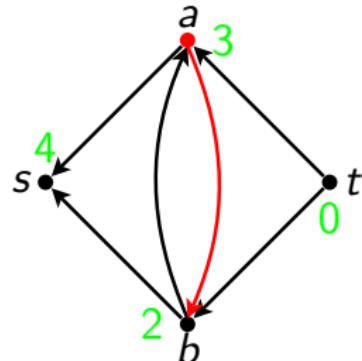
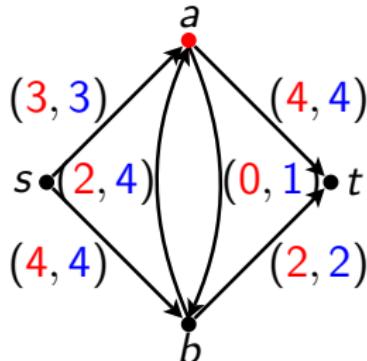


RELABEL at a

$$\begin{aligned}\ell'(a) &= \min\{\ell(v) + 1 : av \in A_x\} \\ &= \min\{2 + 1, 4 + 1\} = 3\end{aligned}$$

arc ab becomes ℓ' -tight

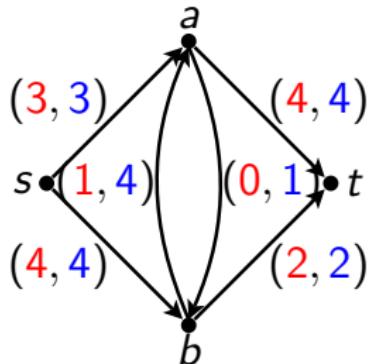
Execution of the Push-Relabel algorithm



PUSH on ab

$$\begin{aligned}\varepsilon &= \min\{\bar{g}(ab), f_x(a)\} \\ &= \min\{1 - 0 + 2, 1\} = 1 \\ \varepsilon' &= \min\{x(ba), \varepsilon\} = \min\{2, 1\} = 1 \\ x'(ab) &= x(ab) + \varepsilon - \varepsilon' = 0 + 1 - 1 = 0 \\ x'(ba) &= x(ba) - \varepsilon' = 2 - 1 = 1\end{aligned}$$

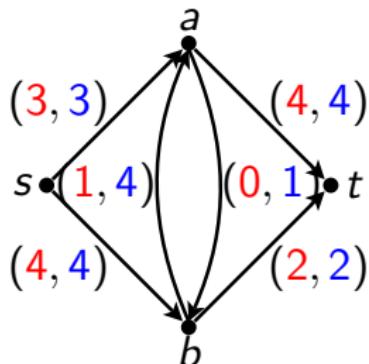
Execution of the Push-Relabel algorithm



PUSH on ab

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Execution of the Push-Relabel algorithm

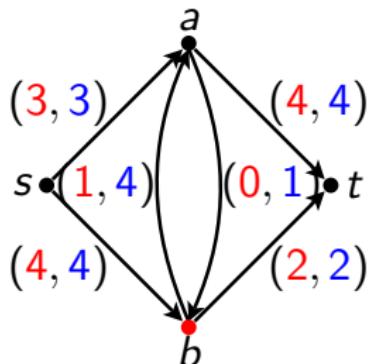


PUSH on ab

$$\begin{aligned}\varepsilon &= \min\{\bar{g}(ab), f_x(a)\} \\ &= \min\{1 - 0 + 2, 1\} = 1 \\ \varepsilon' &= \min\{x(ba), \varepsilon\} = \min\{2, 1\} = 1 \\ x'(ab) &= x(ab) + \varepsilon - \varepsilon' = 0 + 1 - 1 = 0 \\ x'(ba) &= x(ba) - \varepsilon' = 2 - 1 = 1\end{aligned}$$

a is not x' -active anymore

Execution of the Push-Relabel algorithm

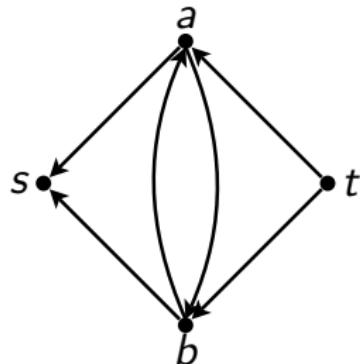
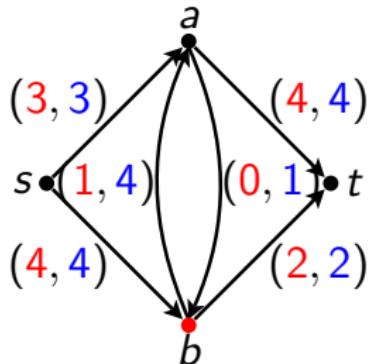


Looking for active vertex

$$\begin{aligned}f_{x'}(b) &= d_{x'}^-(b) - d_{x'}^+(b) \\&= (4 + 0) - (1 + 2) = 1 > 0\end{aligned}$$

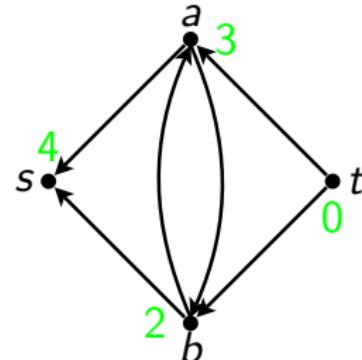
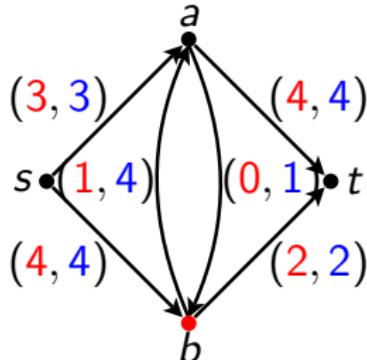
$\Rightarrow b$ is x' -active

Execution of the Push-Relabel algorithm



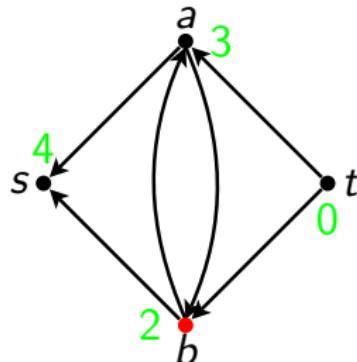
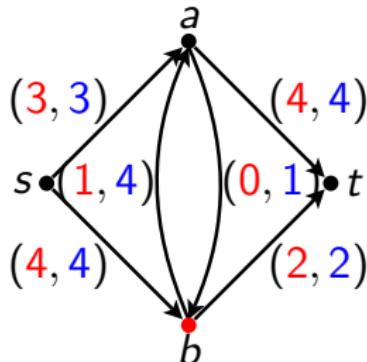
Construction
Auxiliary graph D_x

Execution of the Push-Relabel algorithm



Previous labelling

Execution of the Push-Relabel algorithm

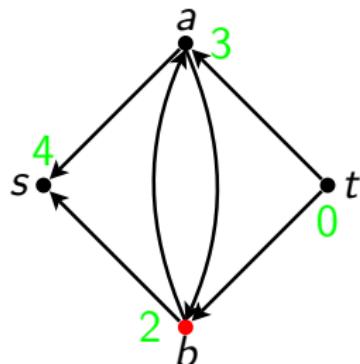
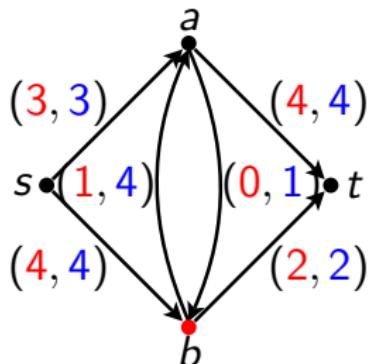


Looking for tight arc leaving b

$$\begin{aligned}\ell(b) &= 2 \neq 4 = \ell(a) + 1 \\ &\neq 5 = \ell(s) + 1\end{aligned}$$

\Rightarrow no ℓ -tight arc leaving b exists

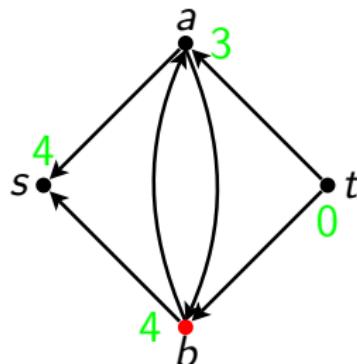
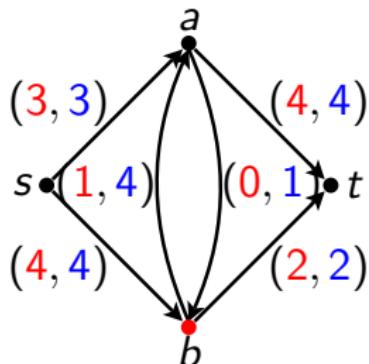
Execution of the Push-Relabel algorithm



RELABEL at b

$$\begin{aligned}\ell'(b) &= \min\{\ell(v) + 1 : bv \in A_x\} \\ &= \min\{3 + 1, 4 + 1\} = 4\end{aligned}$$

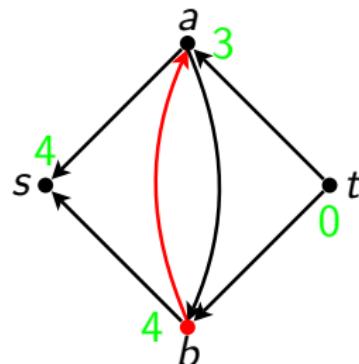
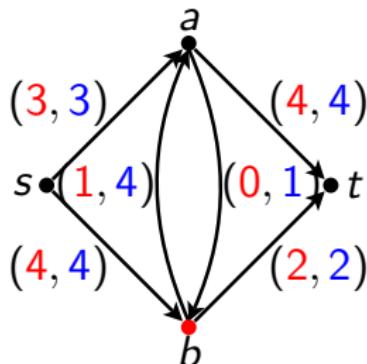
Execution of the Push-Relabel algorithm



RELABEL at b

$$\begin{aligned}\ell'(b) &= \min\{\ell(v) + 1 : bv \in A_x\} \\ &= \min\{3 + 1, 4 + 1\} = 4\end{aligned}$$

Execution of the Push-Relabel algorithm

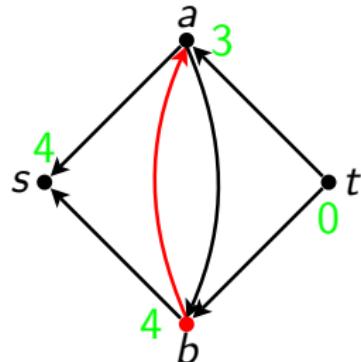
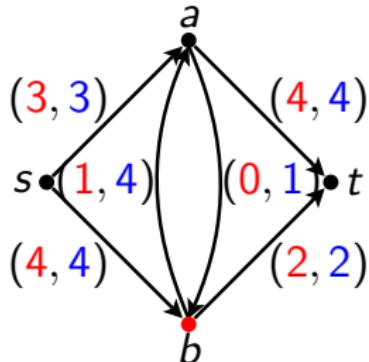


RELABEL at b

$$\begin{aligned}\ell'(b) &= \min\{\ell(v) + 1 : bv \in A_x\} \\ &= \min\{3 + 1, 4 + 1\} = 4\end{aligned}$$

arc ba becomes ℓ' -tight

Execution of the Push-Relabel algorithm



PUSH on ba

$$\varepsilon = \min\{\bar{g}(ba), f_x(b)\}$$

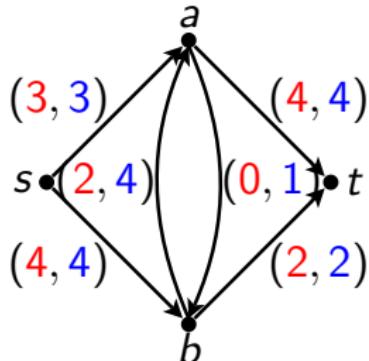
$$= \min\{4 - 1 + 0, 1\} = 1$$

$$\varepsilon' = \min\{x(ab), \varepsilon\} = \min\{0, 1\} = 0$$

$$x'(ba) = x(ba) + \varepsilon - \varepsilon' = 1 + 1 - 0 = 2$$

$$x'(ab) = x(ab) - \varepsilon' = 0 - 0 = 0$$

Execution of the Push-Relabel algorithm



PUSH on ba

$$\varepsilon = \min\{\bar{g}(ba), f_x(b)\}$$

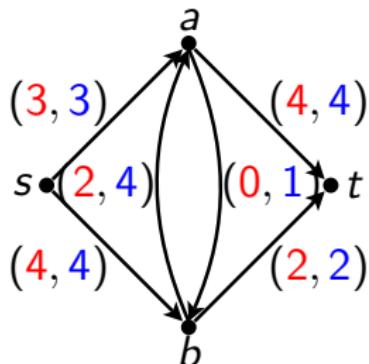
$$= \min\{4 - 1 + 0, 1\} = 1$$

$$\varepsilon' = \min\{x(ab), \varepsilon\} = \min\{0, 1\} = 0$$

$$x'(ba) = x(ba) + \varepsilon - \varepsilon' = 1 + 1 - 0 = 2$$

$$x'(ab) = x(ab) - \varepsilon' = 0 - 0 = 0$$

Execution of the Push-Relabel algorithm

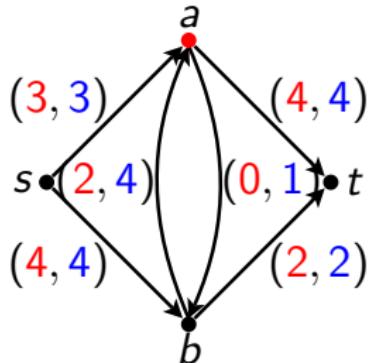


PUSH on ba

$$\begin{aligned}\varepsilon &= \min\{\bar{g}(ba), f_x(b)\} \\ &= \min\{4 - 1 + 0, 1\} = 1 \\ \varepsilon' &= \min\{x(ab), \varepsilon\} = \min\{0, 1\} = 0 \\ x'(ba) &= x(ba) + \varepsilon - \varepsilon' = 1 + 1 - 0 = 2 \\ x'(ab) &= x(ab) - \varepsilon' = 0 - 0 = 0\end{aligned}$$

b is not x' -active anymore

Execution of the Push-Relabel algorithm

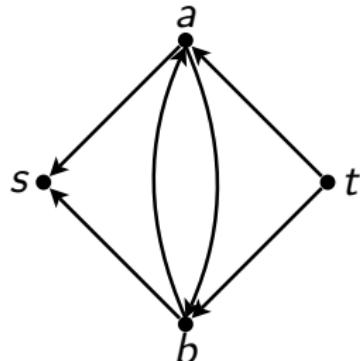
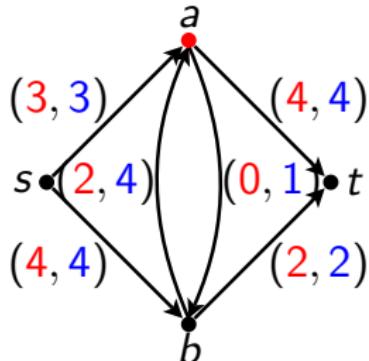


Looking for active vertex

$$\begin{aligned}f_{x'}(a) &= d_{x'}^-(a) - d_{x'}^+(a) \\&= (3 + 2) - (4 + 0) = 1 > 0\end{aligned}$$

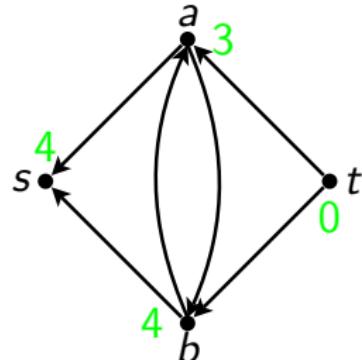
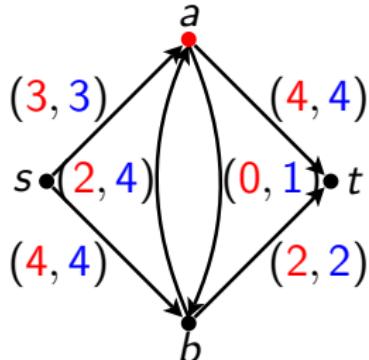
$\Rightarrow a$ is x' -active

Execution of the Push-Relabel algorithm



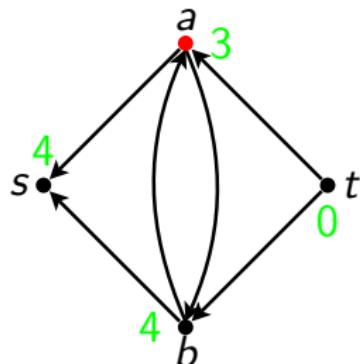
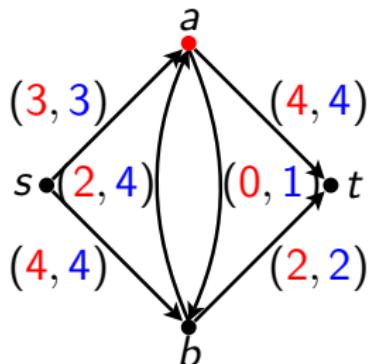
Construction
Auxiliary graph D_x

Execution of the Push-Relabel algorithm



Previous labelling

Execution of the Push-Relabel algorithm

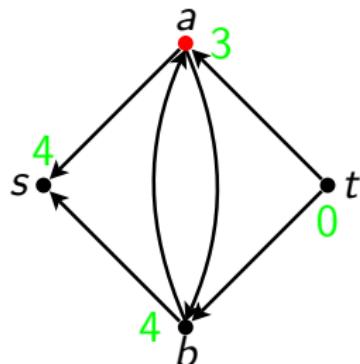
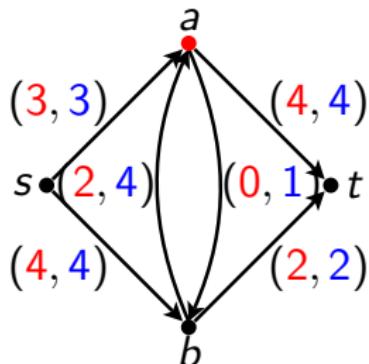


Looking for tight arc leaving a

$$\begin{aligned}\ell(a) &= 3 \neq 5 = \ell(b) + 1 \\ &\neq 5 = \ell(s) + 1\end{aligned}$$

\Rightarrow no ℓ -tight arc leaving a exists

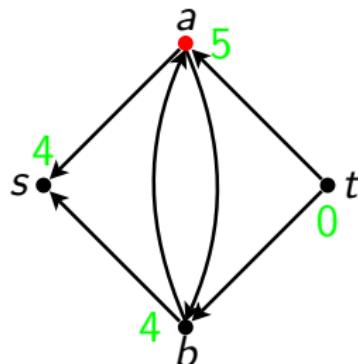
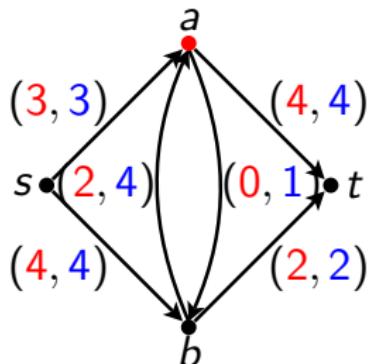
Execution of the Push-Relabel algorithm



RELABEL at a

$$\begin{aligned}\ell'(a) &= \min\{\ell(v) + 1 : av \in A_x\} \\ &= \min\{4 + 1, 4 + 1\} = 5\end{aligned}$$

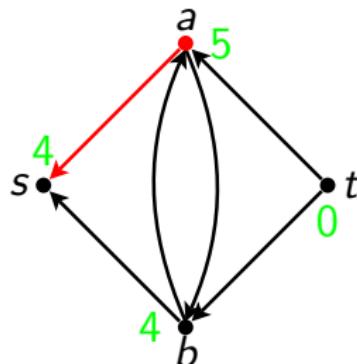
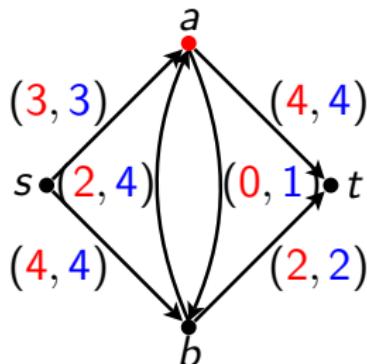
Execution of the Push-Relabel algorithm



RELABEL at a

$$\begin{aligned}\ell'(a) &= \min\{\ell(v) + 1 : av \in A_x\} \\ &= \min\{4 + 1, 4 + 1\} = 5\end{aligned}$$

Execution of the Push-Relabel algorithm

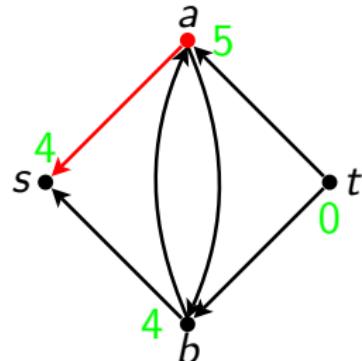
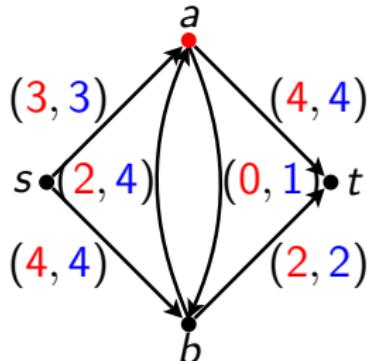


RELABEL at a

$$\begin{aligned}\ell'(a) &= \min\{\ell(v) + 1 : av \in A_x\} \\ &= \min\{4 + 1, 4 + 1\} = 5\end{aligned}$$

arc as becomes ℓ' -tight

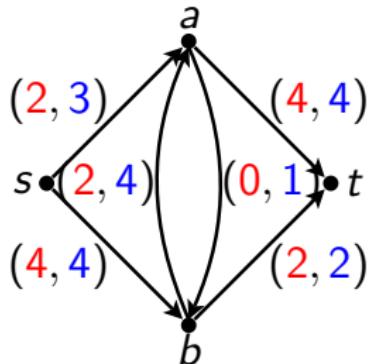
Execution of the Push-Relabel algorithm



PUSH on as

$$\begin{aligned}\varepsilon &= \min\{\bar{g}(as), f_x(a)\} \\ &= \min\{0 - 0 + 3, 1\} = 1 \\ \varepsilon' &= \min\{x(sa), \varepsilon\} = \min\{3, 1\} = 1 \\ x'(sa) &= x(sa) - \varepsilon' = 3 - 1 = 2\end{aligned}$$

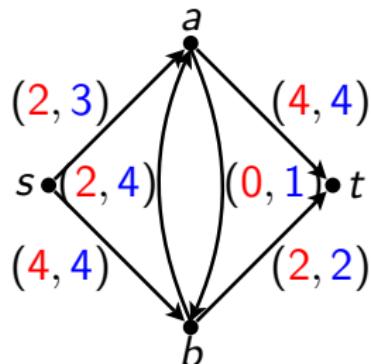
Execution of the Push-Relabel algorithm



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$$\begin{aligned}\varepsilon &= \min\{\bar{g}(as), f_x(a)\} \\ &= \min\{0 - 0 + 3, 1\} = 1 \\ \varepsilon' &= \min\{x(sa), \varepsilon\} = \min\{3, 1\} = 1 \\ x'(sa) &= x(sa) - \varepsilon' = 3 - 1 = 2\end{aligned}$$

Execution of the Push-Relabel algorithm



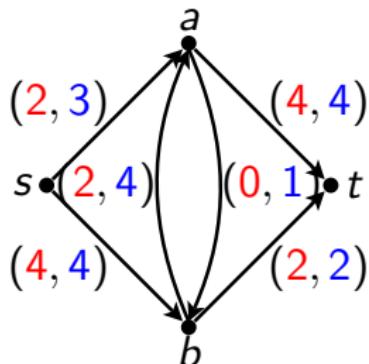
PUSH on as

$$\begin{aligned}\varepsilon &= \min\{\bar{g}(as), f_x(a)\} \\ &= \min\{0 - 0 + 3, 1\} = 1 \\ \varepsilon' &= \min\{x(sa), \varepsilon\} = \min\{3, 1\} = 1 \\ x'(sa) &= x(sa) - \varepsilon' = 3 - 1 = 2\end{aligned}$$

Looking for active vertex

no x' -active vertex exists

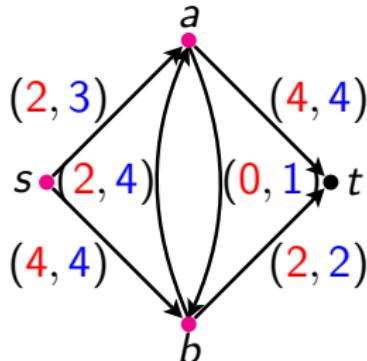
Execution of the Push-Relabel algorithm



Optimal flow

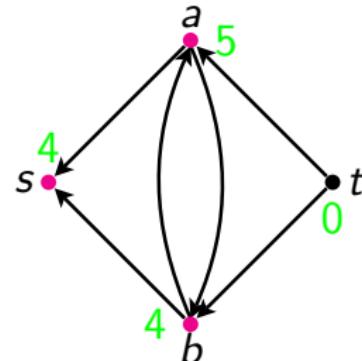
$\implies x$ is a g -feasible (s, t) -flow of maximum value

Execution of the Push-Relabel algorithm



Optimal flow

$\Rightarrow x$ is a g -feasible (s, t) -flow of maximum value



Optimal cut

ℓ does not take 1
 $Z = \{v : \ell(v) > 1\}$ is an (s, t) -cut of minimum capacity