

# Combinatorial Optimization and Graph Theory

## ORCO

### Push-Relabel Algorithm

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# Max Flow Min Cut

## Problem

- Given
  - a directed graph  $D = (V, A)$ ,
  - a non-negative capacity function  $g$  on the arcs of  $D$  and
  - two vertices  $s, t$  of  $D$ ,
- How to find
  - a  $g$ -feasible  $(s, t)$ -flow of maximum value and
  - an  $(s, t)$ -cut of minimum capacity?

## Reminder

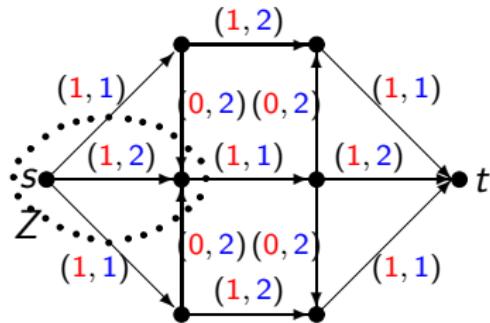
- $(s, t)$ -flow:  $d_x^-(v) = d_x^+(v) \quad \forall v \neq s, t,$
- $g$ -feasible:  $0 \leq x(e) \leq g(e) \quad \forall e \in A,$
- flow value:  $\text{val}(x) = d_x^+(s),$
- $(s, t)$ -cut  $Z$ :  $s \in Z \subseteq V \setminus t,$
- capacity of  $(s, t)$ -cut  $Z$ :  $\text{cap}(Z) = d_g^+(Z).$

## Algorithm of Edmonds-Karp

## ALGORITHM OF EDMONDS-KARP

INPUT : Network  $(G, g)$  such that  $g \geq 0$  et  $s, t \in V : \delta^-(s) = \emptyset = \delta^+(t)$ .

OUTPUT : feasible  $(s, t)$ -flow  $x$  and  $(s, t)$ -cut  $Z$  such that  $\text{val}(x) = \text{cap}(Z)$ .



# Algorithm of Edmonds-Karp

Step 0:  $x_0(e) = 0 \quad \forall e \in A$ ,  $i := 0$ .

Step 1: Construct the auxiliary graph  $G_i := (V, A_i^1 \cup A_i^2)$  where  
 $A_i^1 := \{uv : uv \in A, x_i(uv) < g(uv)\}$  and  
 $A_i^2 := \{vu : uv \in A, x_i(uv) > 0\}$ .

Step 2: Execute algorithm Breadth First Search on  $G_i$  and  $s$  to get  $Z_i \subseteq V$  and an  $s$ -arborescence  $F_i$  of  $G_i[Z_i]$  such that  $\delta_{G_i}^+(Z_i) = \emptyset$ .

Step 3: If  $t \notin Z_i$  then stop with  $x := x_i$  and  $Z := Z_i$ .

Step 4: Otherwise,  $P_i := F_i[s, t]$ , the unique  $(s, t)$ -path in  $F_i$ .

Step 5:  $\varepsilon_i^1 := \min\{g(uv) - x_i(uv) : uv \in A(P_i) \cap A_i^1\}$ ,  
 $\varepsilon_i^2 := \min\{x_i(uv) : vu \in A(P_i) \cap A_i^2\}$ ,  
 $\varepsilon_i := \min\{\varepsilon_i^1, \varepsilon_i^2\}$ .

Step 6:  $x_{i+1}(uv) := \begin{cases} x_i(uv) + \varepsilon_i & \text{if } uv \in A(P_i) \cap A_i^1 \\ x_i(uv) - \varepsilon_i & \text{if } vu \in A(P_i) \cap A_i^2 \\ x_i(uv) & \text{otherwise.} \end{cases}$

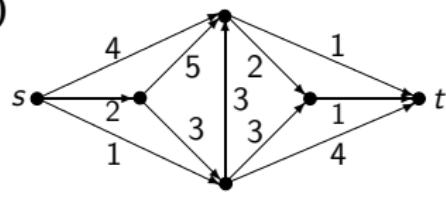
Step 7:  $i := i + 1$  and go to Step 1.

# Execution of Edmonds-Karp algorithm

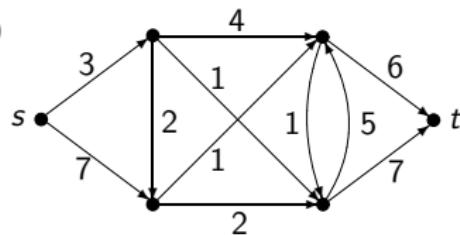
## Example

Execute the algorithm of Edmonds-Karp to find a  $g$ -feasible  $(s, t)$ -flow of maximum value and an  $(s, t)$ -cut of minimum capacity in the following networks:

(a)



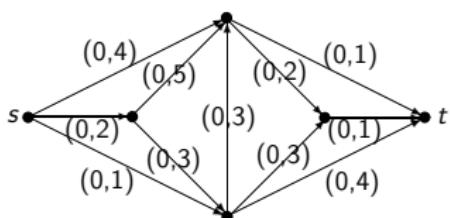
(b)



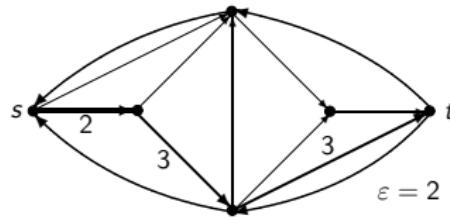
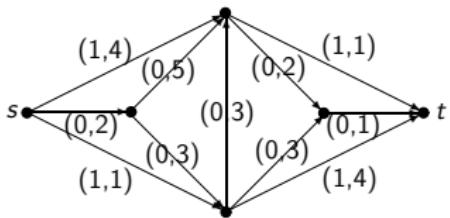
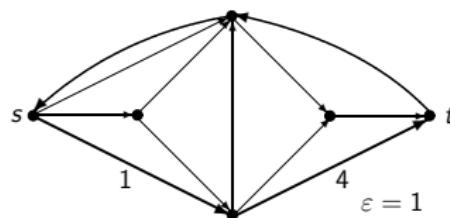
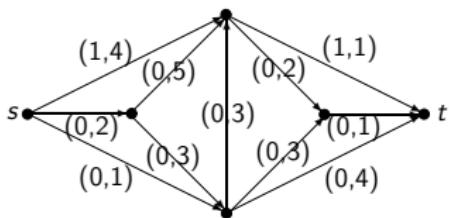
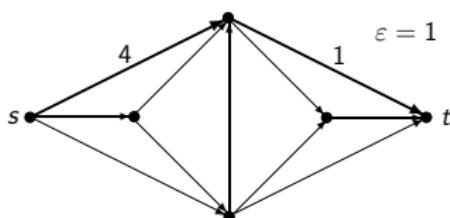
# Execution of Edmonds-Karp algorithm

(a)

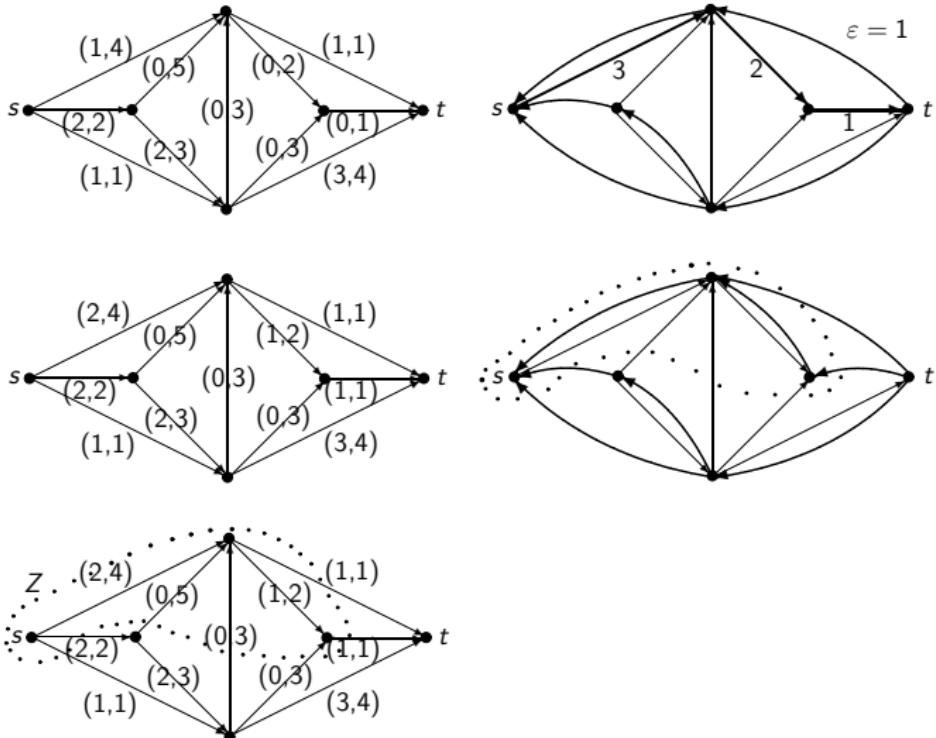
$(x(e), g(e))$



$\overrightarrow{A_1}$   
 $\overrightarrow{A_2}$   
 $\overrightarrow{P}$



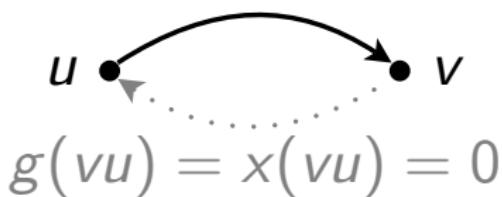
# Execution of Edmonds-Karp algorithm



# Push-Relabel algorithm

## Convention

- Given
  - a directed graph  $D = (V, A)$ ,
  - a non-negative capacity function  $g$  on the arcs of  $D$ ,
  - and two vertices  $s, t$  of  $D$ ,
- we will use the following convention :
  - If  $uv \in A$  and  $vu \notin A$ , then  $g(vu) = 0 = x(vu)$ .  
*(As if the arc  $vu$  existed with capacity 0, but we do not add it.)*



# Auxiliary digraph

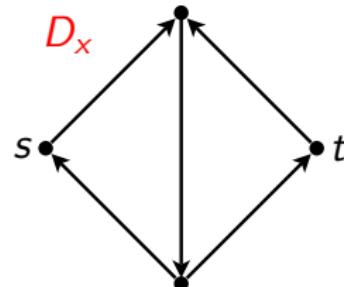
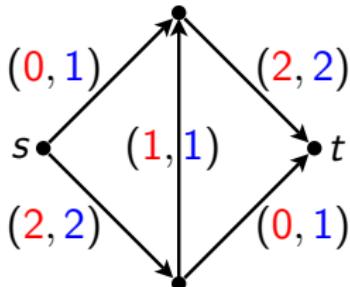
## Definition: auxiliary digraph

- Given

- a directed graph  $D = (V, A)$ ,
- a non-negative capacity function  $g$  on the arcs of  $D$ ,
- and two vertices  $s, t$  of  $D$ ,
- a  $g$ -feasible function  $x$  on  $A$ ,

- Define

- $D_x := (V, A_x)$  where
- $A_x := \{uv : x(uv) < g(uv) \text{ or } x(vu) > 0\}$ .  
(As if  $x$  was a flow, but we do not add parallel arcs.)



# Push

Idea: Given a  $g$ -feasible function  $x$ ,

For  $uv \in A_x$ , we can push

- $g(uv) - x(uv)$  amount on  $uv$  and
- $x(vu)$  amount on  $vu$ ;

without violating  $g$ -feasibility.

(One of these arcs may not exist.)



## Definition

residual capacity:  $\bar{g}(uv) := (g(uv) - x(uv)) + x(vu) \quad \forall uv \in A_x$ .

(This strictly positive value can be pushed from  $u$  to  $v$ .)

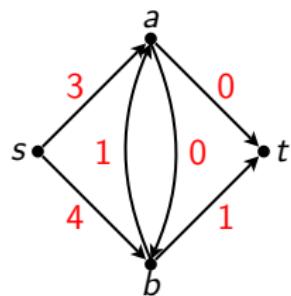
## Remark

- **Advantage:** local operation, not like flow augmentation.
- **Disadvantage:** does not preserve flow conservation.
- We thus need the following more general framework.

# Push

Definition: function  $x$  on the arcs of  $D$ ,

- $x$ -excess at  $v$ :  $f_x(v) := d_x^-(v) - d_x^+(v)$ ,
- $(s, t)$ -preflow:  $f_x(v) \geq 0 \quad \forall v \neq s, t$ ,
- $x$ -active vertex:  $v \neq s, t$  such that  $f_x(v) > 0$ .



- $f_x(a) = 4 - 0 = 4 > 0 \implies a$  is active,
- $f_x(b) = 4 - 2 = 2 > 0 \implies b$  is active,
- $x$  is an  $(s, t)$ -preflow.

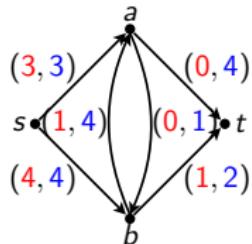
## Remark

An  $(s, t)$ -preflow  $x$  is an  $(s, t)$ -flow if and only if no  $x$ -active vertex exists.

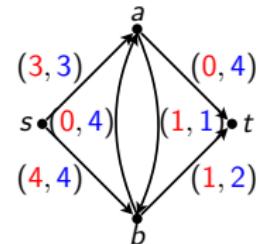
# Push

## Operation PUSH:

- Suppose that
  - $x$  is a  $g$ -feasible  $(s, t)$ -preflow in  $D$ ,
  - $uv \in A_x$  such that  $u$  is  $x$ -active.
- Modification of  $x$ :
  - $\varepsilon \quad := \min\{\bar{g}(uv), f_x(u)\}$ ,
  - $\varepsilon' \quad := \min\{x(vu), \varepsilon\}$ ,
  - $x'(vu) := x(vu) - \varepsilon'$ ,
  - $x'(uv) := x(uv) + \varepsilon - \varepsilon'$ .
  - $x'(e) \quad := x(e) \quad \forall e \in A \setminus \{uv, vu\}$ .



$$\begin{aligned}f_x(a) &= d_x^-(a) - d_x^+(a) = 4 - 0 > 0 \\ \bar{g}(ab) &= g(ab) - x(ab) + x(ba) = 1 - 0 + 1 = 2 \\ \varepsilon &= \min\{\bar{g}(ab), f_x(a)\} = \min\{2, 4\} = 2 \\ \varepsilon' &= \min\{x(ba), \varepsilon\} = \min\{1, 2\} = 1\end{aligned}$$



# Push

## Lemma

- ①  $\varepsilon > 0$ ,
- ②  $\varepsilon' \geq 0$ .
- ③  $x'$  is an  $(s, t)$ -preflow,
- ④  $x'$  is  $g$ -feasible.

## Proof of (1)

- $uv \in A_x \implies \bar{g}(uv) > 0$ .
- $u$  is  $x$ -active  $\implies f_x(u) > 0$ .
- $\varepsilon = \min\{\bar{g}(uv), f_x(u)\} > 0$ .

## Proof of (2)

- $x$  is  $g$ -feasible  $\implies x(vu) \geq 0$ .
- $\varepsilon' = \min\{x(vu), \varepsilon\} \geq 0$ .

# Push

## Proof of (3)

- For  $w \in V \setminus \{s, t, u, v\}$  :  $f_{x'}(w) = f_x(w) \geq 0$ ,
- For  $u$  : By definition,  $\varepsilon = \min\{\bar{g}(uv), f_x(u)\} \leq f_x(u)$ . Thus

$$\begin{aligned}f_{x'}(u) &= d_{x'}^-(u) - d_{x'}^+(u) \\&= (d_x^-(u) - \varepsilon') - (d_x^+(u) + \varepsilon - \varepsilon') \\&= f_x(u) - \varepsilon \geq 0.\end{aligned}$$

- For  $v$  : Since  $x$  is an  $(s, t)$ -preflow and by (1),

$$\begin{aligned}f_{x'}(v) &= d_{x'}^-(v) - d_{x'}^+(v) \\&= (d_x^-(v) + \varepsilon - \varepsilon') - (d_x^+(v) - \varepsilon') \\&= f_x(v) + \varepsilon \geq 0.\end{aligned}$$



# Push

## Proof of (4)

- For  $e \in A \setminus \{uv, vu\}$  :  $0 \leq x(e) = x'(e) = x(e) \leq g(e)$ ,
- For  $uv$  :  $0 \leq x(uv) \leq x(uv) + \varepsilon - \varepsilon' = x'(uv)$ .
  - if  $\varepsilon' = x(vu)$  :  $x'(uv) = x(uv) + \varepsilon - \varepsilon' \leq x(uv) + \bar{g}(uv) - x(vu) = g(uv)$ .
  - if  $\varepsilon' = \varepsilon$  :  $x'(uv) = x(uv) + \varepsilon - \varepsilon' = x(uv) \leq g(uv)$ .
- For  $vu$  : by (2),  $0 \leq x(vu) - \varepsilon' = x'(vu) = x(vu) - \varepsilon' \leq g(vu)$ .

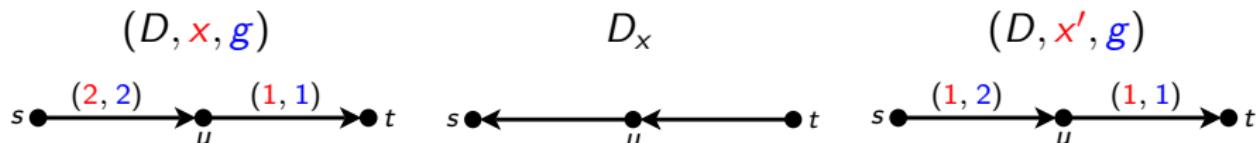
## Reminder

- $\varepsilon' = \min\{x(vu), \varepsilon\}$
- $\varepsilon = \min\{\bar{g}(uv), f_x(u)\}$
- $x'(uv) = x(uv) + \varepsilon - \varepsilon'$
- $x'(vu) = x(vu) - \varepsilon'$
- $\bar{g}(uv) = g(uv) - x(uv) + x(vu)$

# Push

## Example

When one can not push forward, only backward.



- $f_x(u) = d_x^-(u) - d_x^+(u) = 2 - 1 = 1 > 0$
- $\bar{g}(us) = g(us) - x(us) + x(su) = 0 - 0 + 2 = 2$
- $\varepsilon = \min\{\bar{g}(us), f_x(u)\} = \min\{2, 1\} = 1$
- $\varepsilon' = \min\{x(su), \varepsilon\} = \min\{2, 1\} = 1$
- $x'(us) = x(us) + \varepsilon - \varepsilon' = 0 + 1 - 1 = 0$
- $x'(su) = x(su) - \varepsilon' = 2 - 1 = 1$

# Labelling

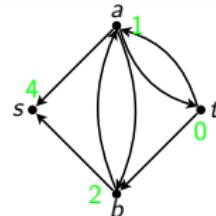
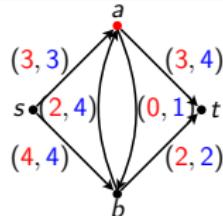
## Question

- In the previous example we could not push forward.
- What does **forward** mean ?

## Definition

For the auxiliary graph  $D_x = (V, A_x)$  of a  $g$ -feasible  $(s, t)$ -preflow  $x$ ,  $x$ -valid  $(s, t)$ -labelling:  $\ell : V \rightarrow \mathbb{Z}_+$  such that

- ①  $\ell(s) = |V|$ ,
- ②  $\ell(t) = 0$ ,
- ③  $\ell(u) \leq \ell(v) + 1 \quad \forall uv \in A_x$ .



# Initialization

## INITIALIZATION

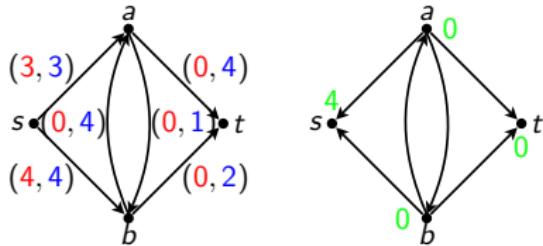
Given a network  $(D, g, s, t)$ :

- ①  $(s, t)$ -preflow  $x$ :

- ①  $x(sv) := g(sv) \quad \forall sv \in A,$
- ②  $x(uv) := 0 \quad \forall uv \in A, u \neq s.$

- ②  $(s, t)$ -Labelling  $\ell$ :

- ①  $\ell(s) := |V|,$
- ②  $\ell(v) := 0 \quad \forall v \neq s.$



## Lemma

- ①  $x$  is a  $g$ -feasible  $(s, t)$ -preflow,
- ②  $\ell$  is a  $x$ -valid  $(s, t)$ -labelling.

# Initialization

## Lemma

- ①  $x$  is a  $g$ -feasible  $(s, t)$ -preflow,
- ②  $\ell$  is a  $x$ -valid  $(s, t)$ -labelling.

## Proof

- ① By  $g \geq 0$ ,
  - ①  $f_x(v) = d_x^-(v) - d_x^+(v) \geq 0 - 0 = 0 \quad \forall v \in V \setminus \{s, t\}$ ,  
thus  $x$  is an  $(s, t)$ -preflow.
  - ②  $0 \leq \min\{0, g(uv)\} \leq x(uv) \leq \max\{0, g(uv)\} \leq g(uv)$ ,  
thus  $x$  is  $g$ -feasible.
- ② By initialization,
  - ①  $\ell(s) = |V|$ ,
  - ②  $\ell(t) = 0$ ,
  - ③  $x(sv) = g(sv)$  and  $x(us) = 0$ , hence there exists no  $sv \in A_x$ . Thus  
 $\ell(u) = 0 \leq 1 \leq \ell(v) + 1 \quad \forall uv \in A_x$ .

# Labelling

## Example

There exists  $g$ -feasible  $(s, t)$ -preflows with no  $x$ -valid  $(s, t)$ -labelling.

- ① Suppose that  $\ell$  is an  $x$ -valid  $(s, t)$ -labelling.
- ②  $2 = |V| = \ell(s) \leq \ell(t) + 1 = 0 + 1 = 1$ , contradiction.



# Saturated cut

## Lemma

- Let  $x$  be a  $g$ -feasible  $(s, t)$ -preflow and  $\ell$  an  $x$ -valid  $(s, t)$ -labelling.
- There exists a saturated  $(s, t)$ -cut  $Z$ :  
 $d_x^+(Z) - d_x^-(Z) = d_g^+(Z) - d_0^-(Z) (= \text{cap}_g(Z))$ .

## Proof

- Since the  $n$  vertices of  $V$  can not take the  $n + 1$  values between 0 and  $n$ , there exists a value  $0 < k < n$  such that  $\ell(v) \neq k \forall v \in V$ .
- Let  $Z := \{v \in V : \ell(v) > k\}$ .
- $Z$  is an  $(s, t)$ -cut:  $s \in Z, t \notin Z$  since  $\ell(s) = n > k > 0 = \ell(t)$ .
- $d_x^+(Z) - d_x^-(Z) = d_g^+(Z) - d_0^-(Z)$ ; otherwise,  $\exists uv \in \delta_{D_x}^+(Z)$  so, by  $u \in Z, \ell$  is  $x$ -valid,  $v \notin Z$  and  $\ell(v) \neq k$ , we have a contradiction:  
 $k < \ell(u) \leq \ell(v) + 1 \leq (k - 1) + 1 = k$ .

# Stopping rule

## Remark

- If  $x$  is a  $g$ -feasible  $(s, t)$ -flow and  $\ell$  is a  $x$ -valid  $(s, t)$ -labelling,
- then  $x$  is of **maximum value**.

## Remark

- ① Edmonds-Karp:
  - ① In each iteration we have a  $g$ -feasible  $(s, t)$ -flow,
  - ② we stop when an  $(s, t)$ -cut becomes saturated.
- ② Push-Relabel:
  - ① In each iteration we have a  $g$ -feasible  $(s, t)$ -preflow and a saturated  $(s, t)$ -cut,
  - ② we stop when the  $(s, t)$ -preflow becomes an  $(s, t)$ -flow.

# Distance and labelling

## Lemma

- If  $x$  is a  $g$ -feasible  $(s, t)$ -preflow and  $\ell$  is a  $x$ -valid  $(s, t)$ -labelling,
- then  $dist_{D_x}(u, v) \geq \ell(u) - \ell(v) \quad \forall u, v \in V.$

## Proof

- ① It is true if  $dist_{D_x}(u, v) = \infty$ .
- ② Otherwise, there exists a  $(u, v)$ -path  $w_0 = u, \dots, w_k = v$  in  $D_x$  such that  $dist_{D_x}(u, v) = k$ .
- ③ Since  $\ell$  is  $x$ -valid,

$$\ell(u) - \ell(v) = \sum_{i=0}^{k-1} (\ell(w_i) - \ell(w_{i+1})) \leq \sum_{i=0}^{k-1} 1 = k = dist_{D_x}(u, v).$$

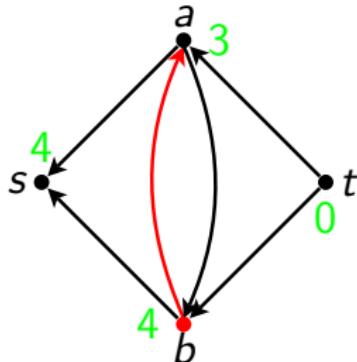
# Distance and labelling

## Corollary

$$dist_{D_x}(u, t) \geq \ell(u) \quad \forall u \in V.$$

## Idea

- ① Push the preflow on  $uv \in A_x$  in the direction of  $t$ : if  $\ell(v) < \ell(u)$ .
- ② But, since  $\ell$  is  $x$ -valid, we have  $\ell(u) \leq \ell(v) + 1$ .
- ③ Push the preflow on  $uv \in A_x$  if it is  **$\ell$ -tight**:  $\ell(u) = \ell(v) + 1$ .



# Pushing on tight arcs

## Lemma

- Suppose that
  - ➊  $x$  is a  $g$ -feasible  $(s, t)$ -preflow in  $D$ ,
  - ➋  $\ell$  is a  $x$ -valid  $(s, t)$ -labelling,
  - ➌  $u$  is an  $x$ -active vertex and
  - ➍  $uv$  is an  $\ell$ -tight arc in  $D_x$ .
- After executing Operation PUSH on  $uv$ , for the new  $(s, t)$ -preflow  $x'$ ,  
 $\ell$  is  $x'$ -valid.

## Proof

- ➊ We have to check the inequality if  $vu$  becomes an arc of  $D_{x'}$ :
- ➋ Since  $uv$  is  $\ell$ -tight,  $\ell(v) = \ell(u) - 1 < \ell(u) + 1$ , thus  $\ell$  is  $x'$ -valid.

# No tight arcs

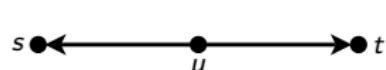
## Example

- ① Active vertex exists but no tight arc exists.
- ②  $f_x(u) = d_x^-(u) - d_x^+(u) = 2 - 0 = 2 > 0 \Rightarrow u$  is  $x$ -active.
- ③  $\ell(s) = 3, \ell(u) = 0, \ell(t) = 0 \Rightarrow$ 
  - $\ell(u) = 0 \neq 4 = \ell(s) + 1$
  - $\ell(u) = 0 \neq 1 = \ell(t) + 1$
  - $us, ut$  are not  $\ell$ -tight.

$(D, \textcolor{red}{x}, \textcolor{blue}{g})$



$D_x$



# Relabel

## Operation RELABEL

① Suppose that

- ①  $x$  is a  $g$ -feasible  $(s, t)$ -preflow in  $D$ ,
- ②  $\ell$  is a  $x$ -valid  $(s, t)$ -labelling,
- ③  $u$  is an  $x$ -active vertex and
- ④ no  $\ell$ -tight arc leaves  $u$ .

② Modification of  $\ell$ :

- ①  $\ell'(u) := \min\{\ell(v) + 1 : uv \in A_x\}$ ,
- ②  $\ell'(w) := \ell(w) \quad \forall w \neq u$ .

③ An arc  $uv$  becomes  $\ell'$ -tight !



$$\begin{aligned}\ell(s) &= 3, \ell(u) = 0, \ell(t) = 0 \\ \ell'(s) &= 3, \ell'(u) = 1, \ell'(t) = 0\end{aligned}$$

## Lemma

After executing Operation RELABEL at  $u$ , the new labelling  $\ell'$  is  $x$ -valid.

## Proof

- ① We have to check  $\ell'(a) \leq \ell'(b) + 1 \quad \forall ab \in A_x$ .
- ②  $\ell'(v) = \ell(v) \quad \forall v \neq u$  and, by  $\ell$  is  $x$ -valid,  
$$\ell'(u) = \min\{\ell(v) + 1 : uv \in A_x\} = \ell(w) + 1 \geq \ell(u).$$
- ③ If  $a = u$ , then  $\ell'(u) \leq \ell(b) + 1 = \ell'(b) + 1$ ,
- ④ If  $a \neq u$ , then, by  $\ell$  is  $x$ -valid,  $\ell'(a) = \ell(a) \leq \ell(b) + 1 \leq \ell'(b) + 1$ .
- ⑤ Thus  $\ell'$  is indeed  $x$ -valid.

# Push-Relabel

## ALGORITHM OF GOLDBERG

INPUT : A network  $(D, g \geq 0, s, t)$ .

OUTPUT: A  $g$ -feasible  $(s, t)$ -flow  $x$  of maximum value.

Step 0: INITIALIZE  $x$  and  $\ell$ .

Step 1: If  $x$  is a  $(s, t)$ -flow, then STOP with  $x$ .

Step 2: Otherwise, let  $u$  be an  $x$ -active vertex.

Step 3: While  $u$  is  $x$ -active and  $\ell$ -tight  $uv \in A_x$  exists, PUSH on  $uv$ .

Step 4: If  $u$  is not  $x$ -active then go to Step 1.

Step 5: Otherwise RELABEL  $u$  and go to Step 3.

## Remark

Complexity (without proof): Faster than Edmonds - Karp:

- ① Goldberg:  $O(n^2m)$ ,
- ② Edmonds - Karp:  $O(nm^2)$ .