Combinatorial Optimization and Graph Theory ORCO Matroids

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Matroids

Planning

1. Examples, Definitions, Constructions
2. Algorithmic aspects,
3. Matroid intersection,
4. Minimum Weight Spanning Arborescences

Matroids

Examples

1. Set of linearly independent vectors in a vector space,
2. Set of edge sets of forests in a graph,
3. Set of subsets of cardinality at most k in a set of cardinality $n \geq k$.

What common properties do the above examples have ?

(I_1) $\emptyset \in \mathcal{I}$,

(I_2) If $Y \subseteq X \in \mathcal{I}$ then $Y \in \mathcal{I}$,

(I_3) If $X, Y \in \mathcal{I}$ and $|X| > |Y|$ then $\exists x \in X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{I}$.

Matroids

Properties of independent sets

(I_1) $\emptyset \in \mathcal{I}$,

(I_2) If $Y \subseteq X \in \mathcal{I}$ then $Y \in \mathcal{I}$,

(I_3) If $X, Y \in \mathcal{I}$ and $|X| > |Y|$ then $\exists x \in X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{I}$.

Definition:

Let S be a set and $\mathcal{I}$ a set of subsets of S that satisfies (I_1), (I_2) and (I_3).

1. $\mathcal{M} = (S, \mathcal{I})$ is called matroid.
2. The elements of $\mathcal{I}$ are called independent sets.
3. The maximal elements of $\mathcal{I}$ are called bases.
4. The rank of $X \subseteq S$ is $r(X) := \max\{|Z| : Z \subseteq X \text{ and } Z \in \mathcal{I}\}$.

Bases

Exercise: Properties of bases

The set $\mathcal{B}$ of bases of a matroid satisfies:

(B_1) $\mathcal{B} \neq \emptyset$,

(B_2) $|X| = |Y|$ for all $X, Y \in \mathcal{B}$,

(B_3) If $X, Y \in \mathcal{B}$ and $x \in X \setminus Y$ then $\exists y \in Y \setminus X$ s.t. $(X \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

Exercise:

1. Let us suppose that $\mathcal{B}$ satisfies (B_1), (B_2) and (B_3).
2. Then $\mathcal{I} := \{X \subseteq S : \exists Y \in \mathcal{B}, X \subseteq Y\}$ satisfies (I_1), (I_2) and (I_3).
3. Deduce from (2) that there exists a matroid $\mathcal{M}$ such that $\mathcal{B}$ is the set of bases of $\mathcal{M}$.

Rank function

Exercise: Properties of the rank function r of a matroid $\mathcal{M} = (\mathcal{S}, \mathcal{I})$

(R_1) $r(X)$ is an integer,

(R_2) $0 \leq r(X) \leq |X|$,

(R_3) If $X \subseteq Y$ then $r(X) \leq r(Y)$,

(R_4) $r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y)$ $\forall X, Y \subseteq S$. (Submodularity)

Proof:

1. (R_1), (R_2) and (R_3) are trivially satisfied. To prove (R_4) let $X, Y \subseteq S$.
2. Let $A \in \mathcal{I}, A \subseteq X \cap Y, |A| = r(X \cap Y)$, $B \in \mathcal{I}, A \subseteq B \subseteq X \cup Y$, $|B| = r(X \cup Y)$, $(\exists B$ by $(I_3))$, $X' := B \cap X$ and $Y' := B \cap Y$.
3. By (I_2), $X', Y' \in \mathcal{I}$, so $r(X) \geq |X'|$ and $r(Y) \geq |Y'|$.
4. By definition of A, B, and (I_2): $X' \cap Y' = A$, $X' \cup Y' = B$.
5. $r(X) + r(Y) \geq |X'| + |Y'| = |X' \cap Y'| + |X' \cup Y'| = |A| + |B| = r(X \cap Y) + r(X \cup Y)$.

Rank function

Properties of the rank function r of a matroid $\mathcal{M} = (\mathcal{S}, \mathcal{I})$

(R_1) $r(X)$ is an integer,

(R_2) $0 \leq r(X) \leq |X|$,

(R_3) If $X \subseteq Y$ then $r(X) \leq r(Y)$,

(R_4) $r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y)$ $\forall X, Y \subseteq S$.

Exercise:

1. Let us suppose that a function r satisfies (R_1), (R_2), (R_3) and (R_4).
2. Prove that $\mathcal{I} := \{X \subseteq S : r(X) = |X|\}$ satisfies (I_1), (I_2) and (I_3).
3. Deduce from (2) that there exists a matroid $\mathcal{M}$ such that r is the rank function of $\mathcal{M}$.

Rank function

Previous exercise in an exam

1. Deduce from (R_2) that $\mathcal{I}$ satisfies (I_1).
2. Let $Y \subseteq X \in \mathcal{I}$. Deduce from (R_4) and (R_2) that $\mathcal{I}$ satisfies (I_2).
3. Let $X, Y \in \mathcal{I}$ such that $|X| > |Y|$. Let Y' be a maximal subset of $X \cup Y$ such that $Y \subseteq Y'$ and $r(Y) = r(Y')$.
    1. Deduce from (R_3) that there exists $x \in X \setminus Y'$.
    2. Prove that $r(Y' \cup x) > r(Y)$.
    3. Deduce from (R_4) and 3.2 that $r(Y \cup x) > r(Y)$.
    4. Deduce from (R_2), 3.3, (R_1) and $Y \in \mathcal{I}$ that $r(Y \cup x) = |Y \cup x|$.
    5. Conclude that $\mathcal{I}$ satisfies (I_3).
4. Deduce that $(S, \mathcal{I})$ is a matroid with rank function r'.
5. Let $X \subseteq S$, $Y \subseteq X$ be a maximal independent set in $(S, \mathcal{I})$ and $Y \subseteq Z \subseteq X$ a maximal set with $r(Y) = r(Z)$.
6. Prove that $Z = X$.
7. Conclude that $r' = r$.

Solution

1. By (R_2), $0 \leq r(\emptyset) \leq |\emptyset| = 0$, so $r(\emptyset) = |\emptyset|$ that is $\emptyset \in \mathcal{I}$, and $\mathcal{I}$ satisfies (I_1).
2. $|X| = |Y| + |X \setminus Y| \geq r(Y) + r(X \setminus Y) \geq r(\emptyset) + r(X) = |X|$ by (R_2), (R_4), (I_1). Thus $|Y| = r(Y)$ and $\mathcal{I}$ satisfies (I_2).
3.
    1. Otherwise, $X \subseteq Y'$, so by (R_3), $r(X) \leq r(Y') = r(Y) < r(X)$, contradiction.
    2. By the definition of Y', $r(Y' \cup x) > r(Y)$.
    3. By (R_4) and 3.2, $r(Y \cup x) + r(Y') \geq r(Y) + r(Y' \cup x) > r(Y') + r(Y)$.
    4. By (R_2), 3.3, (R_1), $Y \in \mathcal{I}$: $|Y \cup x| \geq r(Y \cup x) \geq r(Y) + 1 = |Y \cup x|$.
    5. By 3.4, $|Y \cup x| = r(Y \cup x)$ so $Y \cup x \in \mathcal{I}$ and $\mathcal{I}$ satisfies (I_3).
4. By 1,2 and 3, $\mathcal{I}$ satisfies (I_1), (I_2) and (I_3) so $(S, \mathcal{I})$ is a matroid whose rank function is r'.
6. If $\exists x \in X \setminus Z$, then $r(Y) \leq r(Y \cup x) \leq |Y| = r(Y)$ so $2r(Y) = r(Y \cup x) + r(Z) \geq r(Y) + r(Z \cup x) > 2r(Y)$, contradiction.
7. $r'(X) = |Y| = r(Y) = r(Z) = r(X)$.

Constructions

Exercise:

1. Let S be a set, $S_1 \cup \cdots \cup S_k$ a partition of S and $a_1, \ldots, a_k \in \mathbb{Z}_+$.
2. Let $\mathcal{I} := \{X \subseteq S : |X \cap S_i| \leq a_i \text{ for } 1 \leq i \leq k\}$.
3. Prove that $\mathcal{I}$ satisfies (I_1), (I_2) and (I_3).
4. The matroid $(S, \mathcal{I})$ is called partition matroid.

Exercise:

1. Let $\mathcal{B}$ be the set of bases of a matroid $\mathcal{M}$ on S and
2. $\mathcal{B}^* := \{X \subseteq S : S \setminus X \in \mathcal{B}\}$.
3. Prove that $\mathcal{B}^*$ satisfies (B_1), (B_2) and (B_3).
4. The matroid $\mathcal{M}^*$ whose set of bases is $\mathcal{B}^*$ is called the dual of $\mathcal{M}$.
5. Prove that the rank function of the dual $\mathcal{M}^*$ is
   $r^*(X) = |X| + r(S \setminus X) - r(S)$.

Constructions

Exercise:

1. Let $\mathcal{M}_i := (S, \mathcal{I}_i)$ be k matroids on S with rank functions r_i.
2. Let $\mathcal{I}_\cup := \{X \subseteq S : \exists$ a partition $X_1 \cup \dots \cup X_k$ of $X, X_i \in \mathcal{I}_i \ \forall i\}$.
3. Prove that $\mathcal{I}_\cup$ satisfies (I_1), (I_2) and (I_3).
4. The matroid $\mathcal{M}_\cup := (S, \mathcal{I}_\cup)$ is called sum of the k matroids $\mathcal{M}_i$.
5. Prove that in $\mathcal{M}_\cup$: $r_\cup(Z) = \min_{X \subseteq Z}\{|Z \setminus X| + \sum_1^k r_i(X)\}$.

Corollary:

Given a connected graph G and $k \in \mathbb{Z}_+$, we can decide using matroid theory whether G contains k edge-disjoint spanning trees.

Exercise: Given a graph $G := (V, E)$,

1. $\mathcal{I} := \{X \subseteq V : \exists$ a matching M of G s.t. $X \subseteq M$-saturated vertices$\}$.
2. Prove that $\mathcal{M} := (V, \mathcal{I})$ is a matroid.

Constructions

Exercise:

1. Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and s an element of S and
2. $\mathcal{I}_{\setminus s} := \{X \subseteq S \setminus s : X \in \mathcal{I}\}$.
3. Prove that $\mathcal{I}'$ satisfies (I_1), (I_2) and (I_3).
4. The operation is called deletion of the element s.
5. The matroid $(S \setminus s, \mathcal{I}_{\setminus s})$ is denoted by $\mathcal{M} \setminus s$.

Exercise:

1. Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and $s \in S$.
2. Let $\mathcal{I}_{/s} := \{X \subseteq S \setminus \{s\} : X \cup \{s\} \in \mathcal{I}\}$.
3. Prove that $\mathcal{I}_{/s}$ satisfies (I_1), (I_2) and (I_3).
4. The operation is called contraction of the element s.
5. The matroid $(S \setminus \{s\}, \mathcal{I}_{/s})$ is denoted by $\mathcal{M}/s$.
6. Prove that the rank function of $\mathcal{M}/s$ is $r_{/s}(X) = r(X \cup s) - 1$.

Algorithmic aspects

Greedy algorithm

Input: A matroid $\mathcal{M} = (S, \mathcal{I})$ and non-negative weight function c on S.
Output: A base of $\mathcal{M}$ of maximum weight.

Step 0: *Initialization.* $J := \emptyset$.
Step 1: *Augmentation.*
While $\exists s \in S \setminus J : J \cup s \in \mathcal{I}$ do
choose such an s of maximum c-weight,
$J := J \cup s$.
Step 2: *End of algorithm.* STOP.

Theorem

Greedy algorithm finds a maximum weight base of $\mathcal{M}$.

Corollary

A maximum weight spanning tree can be found by the greedy algorithm.

Algorithmic aspects

Proof:

1. By (I_1) and (I_3), the greedy algorithm stops with a base B_a of $\mathcal{M}$.
2. Suppose indirect that there exists a base B_b of $\mathcal{M} : c(B_b) > c(B_a)$.
3. By (B_2), $|B_a| = |B_b|$, let n be this value.
4. Let us sort the elements of B_a and B_b the following way :
   $c(a_1) \geq c(a_2) \geq \cdots \geq c(a_n)$ and $c(b_1) \geq c(b_2) \geq \cdots \geq c(b_n)$.
5. Since $\sum_{i=1}^{n} c(b_i) = c(B_b) > c(B_a) = \sum_{i=1}^{n} c(a_i), \exists k : c(b_k) > c(a_k)$.
6. Let $A := \{a_1, \ldots, a_{k-1}\}$ and $B := \{b_1, \ldots, b_k\}$.
7. Since, by (I_2), $A, B \in \mathcal{I}$ and $|B| > |A|$, there exists, by (I_3),
   $b_j \in B \setminus A$ such that $A \cup b_j \in \mathcal{I}$.
8. Since, by the order of B_b and by assumption, $c(b_j) \geq c(b_k) > c(a_k)$,
9. the greedy algorithm should have chosen b_j and not a_k,
10. which is a contradiction.

Algorithmic aspects

Theorem

1. Suppose that $(S, \mathcal{I})$ satisfies (I_1) and (I_2) but does not satisfy (I_3).
2. Prove that there exists a non-negative weight function on S such that the greedy algorithm finds a set of $\mathcal{I}$ that is not of maximum weight.

Proof:

1. By assumption, $\exists X, Y \in \mathcal{I}, |X| > |Y|, \forall x \in X \setminus Y, Y \cup x \notin \mathcal{I}$.
2. By $|X \setminus Y| > |Y \setminus X|, \exists a/b \in \mathbb{R}_+ : 1 > a/b > |Y \setminus X|/|X \setminus Y|$.
3. Let $c(s) := a$ if $s \in X \setminus Y$, b if $s \in Y$, 0 otherwise.
4. By (I_1), (I_2), $b > a > 0$, (1), greedy finds $Y \subseteq Z \subseteq S \setminus (X \setminus Y)$.
5. $c(Z) = |Y|b = |X \cap Y|b + |Y \setminus X|b < |X \cap Y|b + |X \setminus Y|a = c(X)$,
6. Thus the set $Z \in \mathcal{I}$ found by greedy is not of maximum weight.

Matroid intersection

Examples

1. Matching in a bipartite graph $G := (V_1, V_2; E)$.
   1. $\mathcal{M}_i := (E, \mathcal{I}_i), \mathcal{I}_i := \{F \subseteq E : d_F(v) \leq 1 \ \forall v \in V_i\}$.
   2. $M \subseteq E$ is a matching of G if and only if $M \in \mathcal{I}_1 \cap \mathcal{I}_2$.
2. Spanning s-arborescence in a digraph $\vec{G} := (V, A)$.
   1. $\mathcal{M}_1 := (A, \mathcal{B}_1)$, $\mathcal{B}_1 := \{F \subseteq A$: (V, F) is a spanning tree of $G\}$,
   2. $\mathcal{M}_2 := (A, \mathcal{B}_2)$, $\mathcal{B}_2 := \{F \subseteq A$: $d_F^-(v) = 1 \ \forall v \in V \setminus \{s\}, = 0$ for $s\}$.
   3. (V, F) is a spanning s-arborescence of $\vec{G}$ if and only if $F \in \mathcal{B}_1 \cap \mathcal{B}_2$.

Matroid intersection

Theorem (Edmonds)

Given two matroids $\mathcal{M}_i = (S, \mathcal{I}_i)$ with rank functions r_i,
$\max\{|Y| : Y \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min\{r_1(X) + r_2(S \setminus X) : X \subseteq S\}$.

Proof of max $\leq$ min

1. Let $\hat{Y} \in \mathcal{I}_1 \cap \mathcal{I}_2$ and $\hat{X} \subseteq S$ be the sets that provide max and min.
2. By (I_2), $\hat{Y} \cap \hat{X} \in \mathcal{I}_1$ and $\hat{Y} \cap (S \setminus \hat{X}) \in \mathcal{I}_2$.
3. $\max = |\hat{Y}| = |\hat{Y} \cap \hat{X}| + |\hat{Y} \cap (S \setminus \hat{X})| \leq r_1(\hat{X}) + r_2(S \setminus \hat{X}) = \min$.

Matroid intersection

Proof of max $\geq$ min

1. By induction on $|S|$.
2. Let k be the minimum.

Case: $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$.

1. For all $s \in S$, either $r_1(s) = 0$ or $r_2(s) = 0$.
2. Let $X := \{s \in S : r_1(s) = 0\}$.
3. Thus $r_2(s) = 0\ \forall s \in S \setminus X$.
4. By (I_2), $r_1(X) = 0$ and $r_2(S \setminus X) = 0$.
5. Then $r_1(X) + r_2(S \setminus X) = 0$ and hence
6. $\min = 0 \leq \max$, so we are done.

Matroid intersection

Case: $\exists s \in \mathcal{I}_1 \cap \mathcal{I}_2$.

1. We delete first s from both matroids: $\mathcal{M}'_i := \mathcal{M}_i \setminus s$.
2. If the minimum concerning $\mathcal{M}'_1$ and $\mathcal{M}'_2$ is at least k then there exists, by induction, a common independent set of $\mathcal{M}'_1$ and $\mathcal{M}'_2$ (and hence of $\mathcal{M}_1$ and $\mathcal{M}_2$) of size k; and we are done.
3. We may hence suppose that there exists $A \subseteq S \setminus \{s\}$ such that $r_1(A) + r_2((S \setminus \{s\}) \setminus A) \leq k - 1$.

Matroid intersection

Case: $s \in \mathcal{I}_1 \cap \mathcal{I}_2$.

1. We contract now s in both matroids: $\mathcal{M}''_i := \mathcal{M}_i/s$.
2. If the minimum concerning $\mathcal{M}''_1$ and $\mathcal{M}''_2$ is at least $k-1$ then there exists, by induction, a common independent set Y'' of $\mathcal{M}''_1$ and $\mathcal{M}''_2$ of size $k-1$. Thus $Y'' \cup s$ is a common independent set of $\mathcal{M}_1$ and $\mathcal{M}_2$ of size k; and we are done.
3. We may hence suppose that there exists $B \subseteq S \setminus \{s\}$ such that $r''_1(B) + r''_2((S \setminus \{s\}) \setminus B) \leq k-2$, that is, by (b) of the exercise on contraction, $r_1(B \cup s) - 1 + r_2(((S \setminus \{s\}) \setminus B) \cup s) - 1 \leq k-2$.

Matroid intersection

Case: $s \in \mathcal{I}_1 \cap \mathcal{I}_2$.

1. Using (R_4) for r_1 and r_2 and adding the inequalities on A and B:
2. $r_1(A \cup B \cup s) + r_1(A \cap B) + r_2(S \setminus (A \cap B)) + r_2(S \setminus (A \cup B \cup s)) \leq r_1(A) + r_2((S \setminus \{s\}) \setminus A) + r_1(B \cup s) + r_2(((S \setminus \{s\}) \setminus B) \cup s) \leq 2k - 1.$
3. Then for $X = A \cap B$ or $A \cup B \cup s$:
4. $k = \min \leq r_1(X) + r_2(S \setminus X) \leq k - 1,$ which is a contradiction.

Matroid intersection

Theorem (Edmonds)

1. Given two matroids on S and a non-negative weight function on S, one can find in polynomial time a common independent set of the two matroids of maximum weight.
2. Intersection of three matroids is NP-complet.

Proof of (2)

1. For a digraph $\vec{G} = (V, A), s \in V$ and $i = 1, 2, 3,$
   $\mathcal{M}_i := (A, \mathcal{I}_i)$ where
   $\mathcal{I}_1 := \{F \subseteq A : (V, F)$ is a forest of $G\}$,
   $\mathcal{I}_2 := \{F \subseteq A : d_F^-(v) \leq 1\ \forall v \in V \setminus \{s\}$ and $= 0$ for $s\}$,
   $\mathcal{I}_3 := \{F \subseteq A : d_F^+(v) \leq 1\ \forall v \in V\}$.
2. $F \in \mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3$ with $|F| = |V| - 1 \iff (V, F)$ is a Hamiltonian path of $\vec{G}$ whose first vertex is s, which is an NP-complet problem.

Matroid intersection

Edmonds implies Kőnig

1. Matching in a bipartite graph $G := (V_1, V_2; E)$.
   1. $\mathcal{M}_i := (E, \mathcal{I}_i), \mathcal{I}_i := \{F \subseteq E : d_F(v) \leq 1 \ \forall v \in V_i\}$.
   2. $M \subseteq E$ is a matching of G if and only if $M \in \mathcal{I}_1 \cap \mathcal{I}_2$.
2. By Edmonds, $\max\{|M| : M \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min\{r_1(F) + r_2(E \setminus F) : F \subseteq E\}$.
3. Let $\hat{M}$ and $\hat{F}$ be the sets that provide max and min .
4. $\hat{M}$ is a matching of size max.
5. $T := (V_1 \cap V(\hat{F})) \cup (V_2 \cap V(E \setminus \hat{F}))$.
6. For any $v_1 v_2 \in E$, either $v_1 v_2 \in \hat{F}$ or $v_1 v_2 \in E \setminus \hat{F}$ and hence $v_1 \in T$ or $v_2 \in T$, that is T is a transversal of G.
7. $|T| = |V_1 \cap V(\hat{F})| + |V_2 \cap V(E \setminus \hat{F})| = r_1(\hat{F}) + r_2(E \setminus \hat{F}) = \min$.
8. $\nu(G) \geq |\hat{M}| = \max = \min = |T| \geq \tau(G) \geq \nu(G)$.
9. Hence $\nu(G) = \tau(G)$, and the theorem of Kőnig is proven.

Minimum weight spanning arborescence

Minimum weight spanning arborescence algorithm (Edmonds)

Input: Digraph $G = (V, A)$, $s \in V$, non-negative weight function c on A.
Output: A spanning s-arborescence F of G of minimum c-weight.

Ideas

1. If a spanning s-arborescence of G is of c-weight 0, then it is optimal.
2. If we decrease the weight of all arcs entering a vertex by the same value, then we get an equivalent problem.
3. If we contract the arcs of a circuit of c-weight 0, then we get an equivalent problem.

Minimum weight spanning arborescence

Step 0: *Initialization.*

$G_1 := G, c_1 := c, i := 1.$

Step 1: *Modification of the weight.*

For all $v \in V(G_i) \setminus \{s\}$,

$\varepsilon_i(v) := \min\{c_i(uv) : uv \in A(G_i)\}$,

$c_{i+1}(uv) := c_i(uv) - \varepsilon_i(v) \ \forall uv \in A(G_i)$.

Step 2: *Searching.*

Let G_i^0 be the subgraph of G_i induced by the arcs of c_{i+1}-weight 0.

Execute any searching algorithm on G_i^0 and s to obtain

- the set S_i of vertices that can be attained from s by a path in G_i^0,
- a set F_i of arcs of G_i^0 such that (S_i, F_i) is an s-arborescence.

Minimum weight spanning arborescence

Step 3: *Contraction of a circuit.*

If $S_i \neq V(G_i)$ then

$C_i :=$ an elementary circuit of $G_i^0 - S_i$,

$G_{i+1} := G_i / C_i, i := i + 1,$

go to Step 1.

Step 4: *Expand the contracted circuits in reverse order.*

$A_i := F_i.$

While $i > 1$ do:

$v_{C_{i-1}} \in V(G_i)$ obtained by contracting C_{i-1} in G_{i-1},

$e_i \in A_i$ the unique arc entering $v_{C_{i-1}}$,

$w_{i-1} :=$ the head of e_i in G_{i-1},

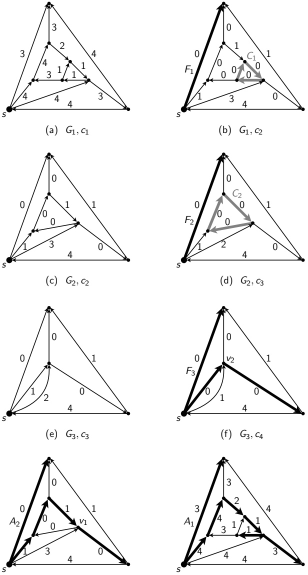
$f_{i-1} :=$ the arc entering w_{i-1} in C_{i-1},

$A_{i-1} := A_i \cup (A(C_{i-1}) \setminus \{f_{i-1}\}),$

$i := i - 1.$

Step 5: *End of algorithm.* $F := (V, A_1)$, STOP.

Minimum weight spanning arborescence



(a) G_1, c_1 (b) G_1, c_2

(c) G_2, c_2 (d) G_2, c_3

(e) G_3, c_3 (f) G_3, c_4

(g) G_2, c_2 (h) G_1, c_1