RIMS Preprint No. 1636

A Note on Disjoint Arborescences

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July 2008

Abstract

Recently Kamiyama, Katoh, and Takizawa have shown a theorem on packing arc-disjoint arborescences that is a proper extension of Edmonds' theorem on disjoint spanning branchings. We show a further extension of their theorem, which makes clear an essential rôle of a reachability condition played in the theorem. The right concept required for the further extension is "convexity" instead of "reachability."

1. Introduction: a theorem of Kamiyama, Katoh, and Takizawa

Recently Kamiyama, Katoh, and Takizawa [3] have shown a theorem (KKT theorem for short in the sequel) on packing arc-disjoint arborescences that is a proper extension of Edmonds' theorem [2] on disjoint spanning branchings, which is described as follows. (The precise definitions of terms used here will be given later.)

Let $G = (V, A)$ be a directed graph with a vertex set V and an arc set A. For any vertex $v \in V$ we denote by $R^+_G(v)$ the set of vertices reachable from v by directed paths in G. Given a set of roots r_i $(i \in I)$, KKT theorem gives a characterization of the existence of a set of arc-disjoint arborescences H_i $(i \in I)$ such that for each $i \in I$ arborescence H_i has a root r_i and exactly spans $R^+_G(r_i)$.

In this note we show a further extension of KKT theorem, which makes clear an essential rôle played by a reachability condition in the theorem. The right concept required for the further extension is "convexity" instead of "reachability."

For more information about disjoint arborescences, their extensions, and related topics see [4, Part V] and [1].

2. An extension of KKT theorem

Let $G = (V, A)$ be a directed graph with a vertex set V and an arc set A. Each arc $a \in A$ has a *tail* denoted by $\partial^+ a$ and a *head* denoted by $\partial^- a$. For any vertex v the *in-degree* of v is equal to the number of arcs that have v as their heads. A *branching* in G is a subgraph $H = (U, B)$ of G without any cycle such that every vertex u in U has in-degrees at most one in H. Each connected component of branching H has a unique vertex, called a *root*, that has the in-degree equal to zero in H. A connected branching is called an *arborescence*, which has a single root.

For any vertex $v \in V$ we denote by $R^+_G(v)$ the set of vertices reachable from v by directed paths in G and by $R^-_G(v)$ the set of vertices from which v is reachable by a directed path in G. Also define for any $W \subseteq V$

$$R^+_G(W) = \bigcup\{R^+_G(v) \mid v \in W\}, \quad R^-_G(W) = \bigcup\{R^-_G(v) \mid v \in W\}. \tag{2.1}$$

A vertex subset W is called a *convex set* in G if we have $W = R^+_G(W) \cap R^-_G(W)$, i.e., for every directed path P from a vertex in W to a vertex in W all the intermediate vertices of P also lie in W. The concept of convexity plays an essential rôle in our result, which replaces the rôle of reachability from roots in KKT theorem [3]. It should be noted that for any convex set U in G and the vertex set W of any strongly connected component of G that satisfy $U \cap W \neq \emptyset$, we must have $U \supseteq W$.

Suppose that we are given a finite index set I and, for each $i \in I$, a specified vertex $r_i \in V$. Here we may allow $r_i = r_j$ for some distinct $i, j \in I$. For each $i \in I$ we are also given a convex set $U_i \subseteq V$ such that $r_i \in U_i$. For any $v \in V$ define

$$I(v) = \{i \in I \mid v \in U_i\}. \tag{2.2}$$

We assume that $I(v) \neq \emptyset$ for all $v \in V$.

Now we are ready to state our main theorem, which is an extension of KKT theorem. It should be noted that replacing U_i by $R^+_G(r_i)$ for all $i \in I$ in our theorem yields KKT theorem. Our proof employs KKT theorem recursively. For any vertex subset $Z \subseteq V$ denote by $G[Z]$ the subgraph of G induced by Z.

Theorem 2.1: *The following two statements are equivalent.*

(a) *There exist arc-disjoint arborescences $H_i = (U_i, B_i)$ $(i \in I)$ such that for each $i \in I$ arborescence H_i has a root r_i.*

(b) *For each $v \in V$ there exist arc-disjoint directed paths P_i $(i \in I(v))$ such that for each $i \in I(v)$ path P_i is from r_i to v.*

(Proof) ((a) $\Rightarrow$ (b)): This implication is easy.

((b) $\Rightarrow$ (a)): Suppose (b) holds.

Consider the decomposition of graph G into strongly connected components, which defines a partial order $\preceq$ on the set of strongly connected components as follows. For two strongly connected components H and H' we have $H \preceq H'$ if and only if there exists a directed path from H' to H. Let $W \subseteq V$ be the vertex set of a strongly connected component that is minimal with respect to the partial order $\preceq$. In other words, W is the vertex set of a strongly connected component in G such that $R^+_G(W) = W$.

Define

$$I(W) = \bigcup \{I(v) \mid v \in W\} (= \{i \in I \mid W \subseteq U_i\}), \tag{2.3}$$
$$U_i(W) = U_i \cap R^-_G(W) \qquad (i \in I(W)), \tag{2.4}$$
$$V(W) = \bigcup \{U_i(W) \mid i \in I(W)\}. \tag{2.5}$$

Then consider the subgraph $\hat{G} = G[V(W)]$ of G induced by $V(W)$. Because of the convexity of U_i $(i \in I)$, definitions (2.3)–(2.5), and assumption (b) we can show the following two facts.

Fact 1: For each $i \in I(W)$ $U_i(W)$ is exactly the set of vertices that can be reached from r_i by directed paths in $\hat{G}$, i.e., $R^+_{\hat{G}}(r_i) = U_i(W)$.

Fact 2: For any $v \in V(W)$ and any directed path P in G from r_i $(i \in I(W))$ to v all the intermediate vertices of P lie in $U_i(W)$.

It follows from these two facts that assumption (b) (appropriately modified) also holds for graph $\hat{G}$ with index set $I(W)$ and convex (reachable) sets $R^+_{\hat{G}}(r_i) = U_i(W)$ $(i \in I(W))$. More precisely, the following (*) holds.

(*) for each $v \in V(W)$ there exist arc-disjoint directed paths P_i $(i \in I(v) \cap I(W))$ such that for each $i \in I(v) \cap I(W)$ path P_i is from r_i to v in $\hat{G}$.

Hence from KKT theorem there exist arc-disjoint arborescences $\hat{H}_i = (U_i(W), \hat{B}_i)$ $(i \in I(W))$ such that each arborescence $\hat{H}_i$ $(i \in I(W))$ has a root r_i.

Define

$$B^W_i = \hat{B}_i \cap \delta^- W \qquad (i \in I(W)), \tag{2.6}$$

where $\delta^- W$ is the set of arcs $a \in A$ with $\partial^- a \in W$. (Here note that we may have $\partial^+ a \in W$.) For all $i \in I \setminus I(W)$ define $B^W_i = \emptyset$. Then put

$$G \leftarrow G \setminus W, \tag{2.7}$$
$$U_i \leftarrow U_i \setminus W \quad (i \in I), \tag{2.8}$$
$$I \leftarrow I \setminus \{i \in I \mid r_i \in W\}, \tag{2.9}$$

where $G \setminus W$ is the graph obtained by removing from G the vertices of W and the arcs incident to W. Note that if $G \setminus W$ has desired arc-disjoint arborescences $H'_i = (U_i \setminus W, B'_i)$ $(i \in I)$ restricted on $G \setminus W$, then $H_i = (U_i, B'_i \cup B_i^W)$ $(i \in I)$ are desired ones for G. It should also be noted that $U_i \setminus W$ $(i \in I)$ are convex sets in the original graph G and hence in the new G as well. Since $U_i \setminus W$ $(i \in I)$ are convex sets in the original graph G, directed paths within $U_i \setminus W$ in the original G are also directed path in the new G. Hence assumption (b) also holds for the new G, I, U_i $(i \in I)$, and r_i $(i \in I)$.

Repeat this process until G becomes empty. Let $W_1, \cdots, W_k$ be the sequence of Ws chosen in the repeated above-mentioned process.

Define for each $i \in I$

$$B_i = \bigcup \{B_i^{W_\ell} \mid \ell = 1, \cdots, k\}, \tag{2.10}$$

where $B_i^{W_\ell}$ is defined to be B_i^W for $W = W_\ell$. We can easily see that $H_i \equiv (U_i, B_i)$ $(i \in I)$ are desired arborescences with roots r_i $(i \in I)$, one for each corresponding H_i. □

We can also show the following. Define $I'(v) = \{i \in I(v) \mid r_i \neq v\}$ for all $v \in V$.

Theorem 2.2: *The following two statements are equivalent to* (a) (*and* (b)) *in Theorem* 2.1.

(c) *For any vertex subset $Z \subset V$*

$$|\Delta^- Z| \geq |\{i \in I(Z) \mid r_i \notin Z\}|, \tag{2.11}$$

where $\Delta^- Z$ denotes the set of arcs $a \in A$ such that $\partial^+ a \notin Z$ and $\partial^- a \in Z$.

(d) *There exist spanning trees $T_i = (U_i, E_i)$ of $G[U_i]$ $(i \in I)$ such that E_i $(i \in I)$ are pairwise disjoint and every vertex $v \in V$ has in-degree equal to $|I'(v)|$ in the union of T_i $(i \in I)$ (as a subgraph $H = (V, \cup_{i \in I} E_i)$ of G).*

(Proof) We show the implications (c) ⇒ (b) ((a)) ⇒ (d) ⇒ (c).

((c) ⇒ (b)): Let v be any vertex in V. Consider any $Z \subset V$ with $v \in Z$ in (c). Then it follows from (c) (with any such Z) and the max-flow min-cut theorem that (b) for v holds.

((b) ⇒ (d)): This is easy since (a) and (b) are equivalent.

((d) ⇒ (c)): Let Z be any subset of V. Denote by $A_H[Z]$ the set of arcs a in H with $\partial^+ a, \partial^- a \in Z$. Then we have

$$|\Delta^- Z| \geq \sum_{v \in Z} |I'(v)| - |A_H[Z]| \geq |\{i \in I(Z) \mid r_i \notin Z\}|, \tag{2.12}$$

where the second inequality follows from the fact that $|E_i \cap A_H[Z]| \leq |U_i \cap Z| - 1$ for all $i \in I(Z)$. Hence (2.11) holds. □

Acknowledgments

I am grateful to Naoyuki Kamiyama for pointing out an error in statement (d) of Theorem 2.2 in an earlier version of this note. Thanks are also due to András Frank for useful discussions on disjoint arborescences. The present research was supported by a Grant-in-Aid from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

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