Combinatorial Optimization and Graph Theory
ORCO
Applications of submodular functions

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Applications of submodular functions

Planning

1. Definitions, Examples,
2. Uncrossing technique,
3. Splitting off technique,
4. Constructive characterization,
5. Orientation,
6. Augmentation,
7. Submodular function minimization.

Submodular functions

Definitions

1. A function $m : 2^S \to \mathbb{R}$ is modular if for all $X, Y \subseteq S$,
$$m(X) + m(Y) = m(X \cap Y) + m(X \cup Y).$$
2. A function $b : 2^S \to \mathbb{R} \cup \{+\infty\}$ is submodular if for all $X, Y \subseteq S$,
$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y).$$
3. A function $p : 2^S \to \mathbb{R} \cup \{-\infty\}$ is supermodular if for all $X, Y \subseteq S$,
$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y).$$

Modular functions

Examples for Modular functions

1. $m(X) = k$: constant function, where $X \subseteq S, k \in \mathbb{R}$,
2. $m(X) = |X|$: cardinality function on a set S,
3. $m(X) = m(\emptyset) + \sum_{v \in X} m(v)$: where $X \subseteq S, m(\emptyset), m(v) \in \mathbb{R} \quad \forall v \in S$.

Submodular functions

Examples for Submodular functions

1. $d_G(X)$: degree function of an undirected graph G,
   (by $d_G(X)+d_G(Y)=d_G(X\cap Y)+d_G(X\cup Y)+2d_G(X\setminus Y, Y\setminus X)$)
2. $d_D^+(X)$: out-degree function of a directed graph D,
3. $d_g^+(X)$: capacity function of a network (D,g),
4. $|\Gamma(X)|$: number of neighbors of X in a bipartite graph, (by modularity of $|\cdot|$, $\Gamma(X)\cup\Gamma(Y)=\Gamma(X\cup Y)$ and $\Gamma(X)\cap\Gamma(Y)\supseteq\Gamma(X\cap Y)$),
5. $r(X)$: rank function of a matroid,
6. $r_1(X)+r_2(S\setminus X)$: for rank functions r_1 and r_2 of two matroids on S,
7. $g(|X|)$: for a concave function $g:\mathbb{R}_+\to\mathbb{R}_+$.

Supermodular functions

Examples for Supermodular functions

1. $|E(X)|$: where $E(X)$ is the set of edges of G inside $X \subseteq V$,
(by $|E(X)| = \frac{1}{2}(\sum_{v \in X} d_G(v) - d_G(X))$,
2. $c_G(F)$: the number of connected components of the subgraph of $G = (V, E)$ induced by $F \subseteq E$.
(by $c_G(F) = |V| - r_G(F)$, where r_G is the rank function of the forest matroid of G).

Uncrossing technique: Flows

Theorem

In a network (D, g), the intersection and the union of two minimum capacity (s, t)-cuts are minimum capacity (s, t)-cuts.

Proof:

1. Let X and Y be two (s, t)-cuts of capacity min.
2. Then $d_g^+(X) = \text{min}$ and $d_g^+(Y) = \text{min}$.
3. Since $X \cap Y$ and $X \cup Y$ are (s, t)-cuts,
4. $d_g^+(X \cap Y) \geq \text{min}$ and $d_g^+(X \cup Y) \geq \text{min}$.
5. $\text{min} + \text{min} = d_g^+(X) + d_g^+(Y) \geq d_g^+(X \cap Y) + d_g^+(X \cup Y)$
   $\geq \text{min} + \text{min}$ by (2), submodularity and (4).
6. Hence equality holds everywhere: $d_g^+(X \cap Y) = \text{min}$ and $d_g^+(X \cup Y) = \text{min}$.

Uncrossing technique: Matchings

Theorem (Frobenius)

A bipartite graph $B = (U, V; E)$ has a matching covering U if and only if $(*)$ $|\Gamma(X)| \geq |X|$ for all $X \subseteq U$.

Proof:

1. We show only the difficult direction.
2. We call a set $X \subseteq U$ tight if $|\Gamma(X)| = |X|$.
3. If X and Y are tight, then $X \cap Y$ and $X \cup Y$ are also tight:
   1. By the tightness of X and Y, the submodularity of $|\Gamma(\cdot)|$, $(*)$ and the modularity of $|\cdot|$, we have
      $$\begin{aligned} |X| + |Y| &= |\Gamma(X)| + |\Gamma(Y)| \\ &\geq |\Gamma(X \cap Y)| + |\Gamma(X \cup Y)| \\ &\geq |X \cap Y| + |X \cup Y| \\ &= |X| + |Y|, \end{aligned}$$
   2. hence equality holds everywhere and $X \cap Y$ and $X \cup Y$ are tight.

Uncrossing technique: Matchings

Proof:

4. We may suppose that after deleting any edge of B, $(*)$ doesn't hold anymore.
5. Then every edge uv of B enters a tight set X_{uv} such that u is the only neighbor of v in X_{uv}:
   1. Since after deleting uv from B, $(*)$ doesn't hold,
   2. $\exists X_{uv} \subseteq U : |X_{uv}| - 1 \geq |\Gamma_{B-uv}(X_{uv})|$.
   3. Moreover, $|\Gamma_{B-uv}(X_{uv})| \geq |\Gamma_B(X_{uv})| - 1$, and
   4. by $(*)$, $|\Gamma_B(X_{uv})| - 1 \geq |X_{uv}| - 1$,
   5. hence equality holds everywhere, that is
   6. X_{uv} is tight and u is the only neighbor of v in X_{uv}.

Uncrossing technique: Matchings

Proof:

6. We show that every vertex of U is of degree 1 in B.
   1. Suppose that $u \in U$ is incident to two edges uv and uw in B.
   2. By (5), $X := X_{uv} \cap X_{uw}$ is tight, u is unique neighbor of v (of w) in X.
   3. Then, by $(*)$ and the tightness of X, we have a contradiction: $|X| - 1 = |X \setminus u| \leq |\Gamma_B(X \setminus u)| \leq |\Gamma_B(X)| - 2 = |X| - 2.$
7. Two vertices u and u' in U can not have a common neighbor since $|\Gamma_B(\{u, u'\})| \geq 2.$
8. By (6) and (7), E is a matching of B covering U.

Uncrossing technique: General lemma

Definitions:

1. A graph G covers a function p on V if $d_G(X) \geq p(X)$ for all $X \subseteq V$.
2. $X \subseteq V$ is tight if $d_G(X) = p(X)$.
3. Two sets X and Y of V are crossing if none of $X \setminus Y, Y \setminus X, X \cap Y$ and $V \setminus (X \cup Y)$ is empty.
4. A function is crossing supermodular if the supermodular inequality holds for any crossing sets X and Y.

Uncrossing technique: General lemma

Uncrossing Lemma

If G covers a crossing supermodular function p then the intersection and the union of crossing tight sets are tight.

Proof:

1. Let X and Y be two crossing tight sets of V.
2. Since they are tight, $d_G(\cdot)$ is submodular, G covers p and p is crossing supermodular, we have
$$\begin{aligned} p(X)+p(Y) &= d_G(X)+d_G(Y) \\ &\geq d_G(X\cap Y)+d_G(X\cup Y) \\ &\geq p(X\cap Y)+p(X\cup Y) \\ &\geq p(X)+p(Y), \end{aligned}$$
3. hence equality holds everywhere and the lemma follows.

Uncrossing technique for minimum tight sets

Definitions:

1. A graph G is called *k-edge-connected* if $d_G(X) \geq k \quad \forall \emptyset \neq X \subset V(G)$.
2. G is *minimally k-edge-connected* if
   1. G is k-edge-connected and
   2. for each edge e of G, $G - e$ is not k-edge-connected anymore.

Uncrossing technique for minimum tight sets

Theorem (Mader)

A minimally k-edge-connected graph G has a vertex of degree k.

Proof:

1. Let $p(X) := k$ if $\emptyset \neq X \subset V$ and 0 otherwise.
2. Then p is crossing supermodular.
3. Since G is k-edge-connected, $d_G(X) \geq k = p(X)$, so G covers p.
4. By minimality of G, each edge of G enters a tight set.
5. Let X be a minimal non-empty tight set.
6. Suppose that X is not a vertex.
7. By minimality of X, there exists an edge uv in X.
8. Let Y be a tight set that uv enters.

Uncrossing technique for minimum tight sets

Proof:

9. By minimality of X, X and Y are crossing.
   1. Since uv enters Y, we may suppose that $u \in X \cap Y$ and $v \in X \setminus Y$.
   2. By the minimality of $X, X \cap Y$ is not tight, so $Y \setminus X \neq \emptyset$.
   3. By the minimality of $X, X \setminus Y$ is not tight, so $V \setminus (X \cup Y) \neq \emptyset$.
10. Then, by the Uncrossing Lemma, $X \cap Y$ is a tight set that contradicts the minimality of X.
11. Then $X = v$ and $d_G(v) = p(v) = k$.

Uncrossing technique for minimum tight sets

Definitions:

1. A directed graph D is k-arc-connected if $d^+_D(X) \geq k \ \forall \emptyset \neq X \subset V$.
2. D is minimally k-arc-connected if
   1. D is k-arc-connected and
   2. for each arc e of D, $D - e$ is not k-arc-connected anymore.

Theorem (Mader)

A minimally k-arc-connected directed graph has a vertex of in- and out-degree k.

Remark

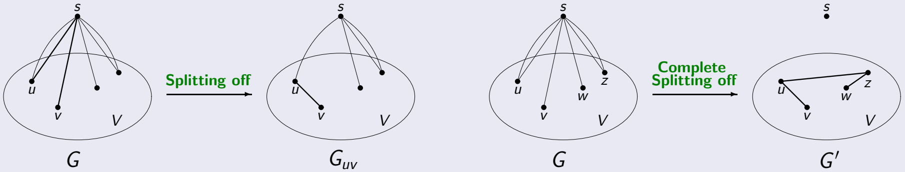
1. One can easily show that there exists a vertex of in-degree k and a vertex of out-degree k but
2. it is not so easy to see that there exists a vertex with both in- and out-degree k.

Splitting off technique

Definitions: for $G := (V \cup s, E)$

1. Operation splitting off at s: for $su, sv \in E$, we replace su, sv by an edge uv, that is $G_{uv} := (V \cup s, (E \setminus \{su, sv\}) \cup \{uv\})$.
2. Operation complete splitting off at s:
   1. $d_G(s)$ is even,
   2. $\frac{d_G(s)}{2}$ splitting off at s and
   3. deleting the vertex s.
3. The graph G is k-edge-connected in V if $d_G(X) \geq k \ \forall \emptyset \neq X \subset V$.

Definitions



Splitting off technique

Theorem (Lovász)

If $G = (V \cup s, E)$ is k-edge-connected in V $(k \geq 2)$ and $d_G(s)$ is even, then there is a complete splitting off at s preserving k-edge-connectivity.

Proof:

1. We show that for every edge su there exists an edge sv so that G_{uv} is k-edge-connected in V.
2. Then the theorem follows by induction on $d_G(s)$.
3. If not, then, for every edge sv, there exists a dangerous set $X \subset V$ such that $d_G(X) \leq k+1$ and $u, v \in X$.
   1. Indeed, if G_{uv} is not k-edge-connected in V, then there exists $X \subset V$ such that $k-1 \geq d_{G_{uv}}(X)$.
   2. Since $d_{G_{uv}}(X) \geq d_G(X) - 2$ and $d_G(X) \geq k$, X is dangerous.

Splitting off technique

Proof:

4. By (3), there exists a minimal set $\mathcal{M}$ of dangerous sets such that
   1. $u \in \bigcap_{X \in \mathcal{M}} X$ and
   2. $N_G(s) \subseteq \bigcup_{X \in \mathcal{M}} X$.
5. Any set X of $\mathcal{M}$ contains at most $\frac{d_G(s)}{2}$ neighbors of s.
   Indeed, $k+1 \geq d_G(X)$
   $$= d_G(V \setminus X) - d_G(s, V \setminus X) + d_G(s, X)$$
   $$\geq k - d_G(s) + 2d_G(s, X).$$

Splitting off technique

Proof:

6. By $u \in \bigcap_{X \in \mathcal{M}} X, N_G(s) \subseteq \bigcup_{X \in \mathcal{M}} X$ and (5), $\exists$ $A, B, C \in \mathcal{M}$.
7. By the minimality of $\mathcal{M}$, $A \setminus (B \cup C), B \setminus (A \cup C), C \setminus (A \cup B) \neq \emptyset$.
8. Since A, B, C are dangerous, this inequality holds, G is k-edge-connected, $u \in A \cap B \cap C, su \in E$ and $k \geq 2$, we have a contradiction:

$$
\begin{aligned}
3(k+1) &\geq d_G(A) + d_G(B) + d_G(C) \\
&\geq d_G(A \setminus (B \cup C)) + d_G(B \setminus (A \cup C)) + d_G(C \setminus (A \cup B)) \\
&\quad + d_G(A \cap B \cap C) + 2d_G(A \cap B \cap C, (V \cup s) \setminus (A \cup B \cup C)) \\
&\geq k + k + k + k + 2.
\end{aligned}
$$

Splitting off technique

Theorem (Mader)

If $D = (V \cup s, A)$ is k-arc-connected $(k \geq 1)$ and $d_D^+(s) = d_D^-(s)$, then there is a complete directed splitting off at s preserving k-arc-connectivity.

Proof

Similar to previous one.

Constructive characterization

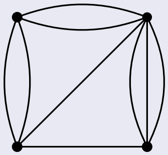
Theorem (Lovász)

A graph is $2k$-edge-connected if and only if it can be obtained from K_2^{2k} by a sequence of the following two operations:

(a) *adding a new edge,*

(b) *pinching k edges: subdivide each of the k edges by a new vertex and identify these new vertices.*

Example



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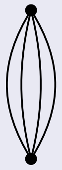

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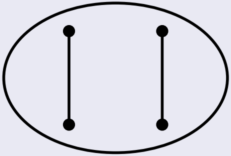

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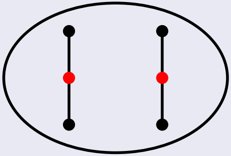

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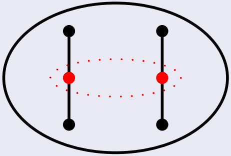

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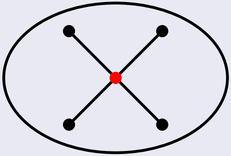

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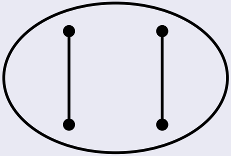

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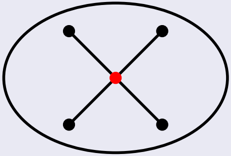

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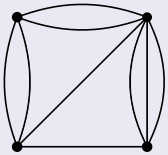

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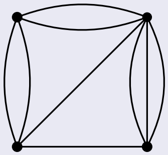

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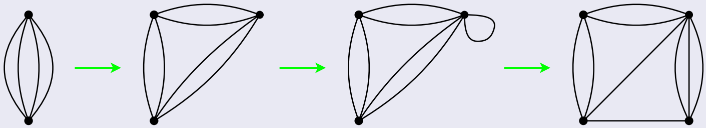
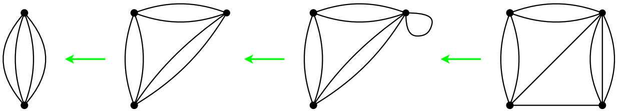


Constructive characterization

Proof:

1. We show that G can be reduced to K_2^{2k} via $2k$-edge-connected graphs by the inverse operations:
   1. deleting an edge and
   2. complete splitting off at a vertex of degree $2k$.
2. While $G \neq K_2^{2k}$ repeat the following.
   1. By deleting edges we get a minimally $2k$-edge-connected graph.
   2. By Theorem of Mader, it contains a vertex of degree $2k$.
   3. By Theorem of Lovász, there exists a complete splitting off at that vertex that preserves $2k$-edge-connectivity.
   4. Let G be the graph obtained after this complete splitting off.



Constructive characterization

Theorem (Mader)

For $k \geq 1$, a graph is k-arc-connected if and only if it can be obtained from $K_2^{k,k}$, the directed graph on 2 vertices with k arcs between them in both directions, by a sequence of the following two operations:

1. adding a new arc,
2. pinching k arcs.

Proof

Similar to previous one, by applying Mader's results on

1. minimally k-arc-connected graphs and,
2. complete directed splitting off.

Orientation

Theorem (Nash-Williams)

G has a k-arc-connected orientation if and only if G is $2k$-edge-connected.

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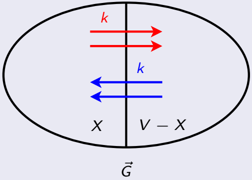

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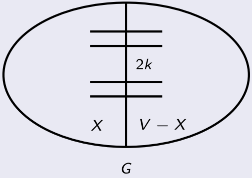

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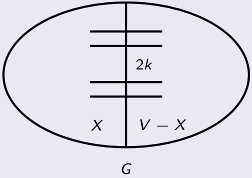
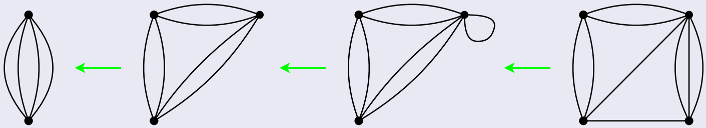


Sufficiency :



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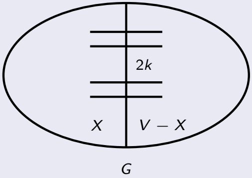

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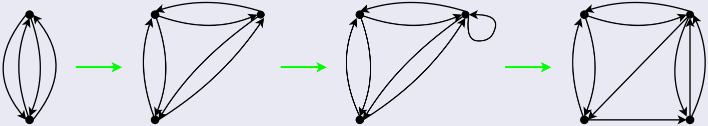


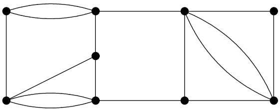
Augmentation

Edge-connectivity augmentation problem:

Given a graph $G = (V, E)$ and $k \in \mathbb{Z}_+$, what is the minimum number γ of new edges whose addition results in a k-edge-connected graph?

Theorem (Watanabe-Nakamura)

Let $G = (V, E)$ be a graph and $k \geq 2$ an integer.
$\min\{|F| : (V, E \cup F) \text{ is } k\text{-edge-conn.}\} = \left\lceil \frac{1}{2} \max \left\{ \sum_{X \in \mathcal{X}} (k - d_G(X)) \right\} \right\rceil,$
where $\mathcal{X}$ is a subpartition of V.



Graph G and $k = 4$

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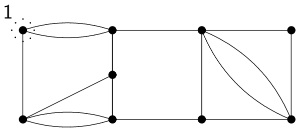

Deficient sets, deficiency $= 4 - d_G(X)$

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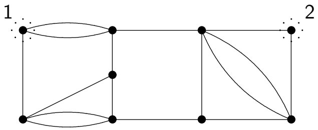

Deficient sets, deficiency $= 4 - d_G(X)$

Augmentation

Edge-connectivity augmentation problem:

Given a graph $G = (V, E)$ and $k \in \mathbb{Z}_+$, what is the minimum number γ of new edges whose addition results in a k-edge-connected graph?

Theorem (Watanabe-Nakamura)

Let $G = (V, E)$ be a graph and $k \geq 2$ an integer.
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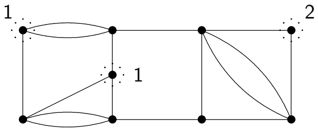

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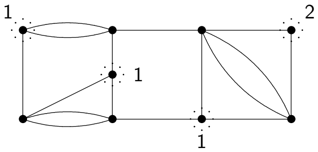

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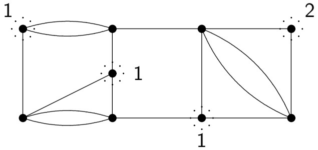

$$\text{Opt} \geq \lceil \tfrac{5}{2} \rceil = 3$$

Augmentation

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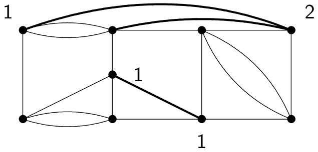

Graph $G + F$ is 4-edge-connected and $|F| = 3$

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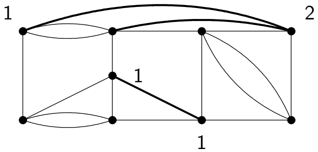


Opt $= \lceil \frac{1}{2}$maximum deficiency of a subpartition of $V\rceil$

Augmentation

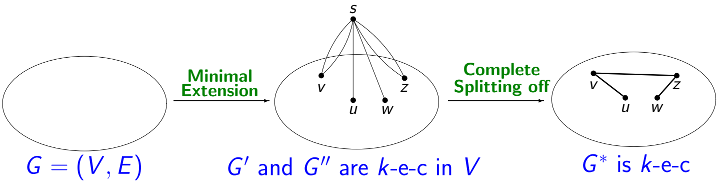
Proof:

1. First we provide the lower bound on γ.
2. Suppose that G is not k-edge-connected.
3. This is because there is a set X of degree $d_G(X)$ less than k.
4. Then the deficiency of X is $k - d_G(X)$, that is, we must add at least $k - d_G(X)$ edges between X and $V \setminus X$.
5. Let $\{X_1, \dots, X_\ell\}$ be a subpartition of V.
6. The deficiency of $\{X_1, \dots, X_\ell\}$ is the sum of the deficiencies of X_i's.
7. By adding a new edge we may decrease the deficiency of at most two X_i's so we may decrease the deficiency of $\{X_1, \dots, X_\ell\}$ by at most 2,
8. hence we obtain the following lower bound:
   $\gamma \geq \alpha :=$ ⌈half of the maximum deficiency of a subpartition of V⌉.

Augmentation

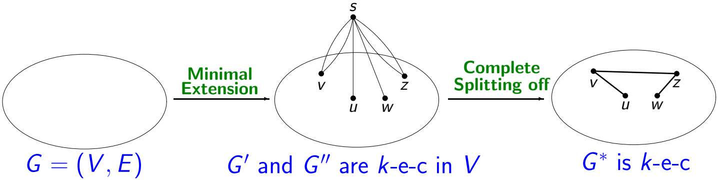
Frank's algorithm

1. Minimal extension,
   1. Add a new vertex s,
   2. Add a minimum number of new edges incident to s to satisfy the edge-connectivity requirements,
   3. If the degree of s is odd, then add an arbitrary edge incident to s.
2. Complete splitting off preserving the edge-connectivity requirements.



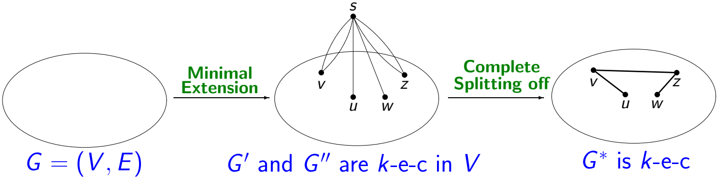
Minimal extension:

1. Add a new vertex s to G and connect it to each vertex of G by k edges. The resulting graph is k-edge-connected in V.
2. Delete as many new edges as possible preserving k-edge-connectivity in V to get $G' = (V \cup s, E \cup F')$.
3. If $d_{G'}(s)$ is odd, then add an arbitrary new edge incident to s to get $G'' = (V \cup s, E \cup F'')$ that is k-edge-connected in V and $d_{G''}(s)$ is even.



Splitting off:

1. By Theorem of Lovász, there exists in G'' a complete splitting off at s that preserves k-edge-connectivity.
2. This way we obtain a k-edge-connected graph $G^* = (V, E \cup F)$ with $|F| = \frac{|F''|}{2} = \lceil\frac{|F'|}{2}\rceil$.



Augmentation

Optimality:

1. In G', no edge incident to s can be deleted without violating k-edge-connectivity in V, so each edge $e \in F'$ enters a maximal proper subset X_e in V of degree k, that is, $d_G(X_e) + d_{F'}(X_e) = k$.
2. By Uncrossing Lemma, these sets form a subpartition $\{X_1, \dots, X_\ell\}$ of V.
   1. Suppose that $X_i \cap X_j \neq \emptyset$.
   2. Then, by Uncrossing Lemma and the maximality of X_i, $X_i \cup X_j = V$.
   3. By $k + k = d_{G'}(X_i) + d_{G'}(X_j)$
      $= d_{G'}(X_i \setminus X_j) + d_{G'}(X_j \setminus X_i) + 2d_{G'}(X_i \cap X_j, \overline{X_i \cup X_j})$
      $\geq k + k + 0,$
   4. $d_{G'}(X_i \setminus X_j) = k = d_{G'}(X_j \setminus X_i)$ and every edge incident to s enters either $X_i \setminus X_j$ or $X_j \setminus X_i$, that is $\{X_i \setminus X_j, X_j \setminus X_i\}$ is the required subpartition.
3. $\gamma \leq |F| = \lceil \frac{|F'|}{2} \rceil = \lceil \frac{1}{2} \sum_1^\ell d_{F'}(X_i) \rceil = \lceil \frac{1}{2} \sum_1^\ell (k - d_G(X_i)) \rceil \leq \alpha \leq \gamma.$

Augmentation

Theorem (Frank)

Let $D = (V, A)$ be a directed graph and $k \geq 1$ an integer.

$$\min\{|F| : (V, A \cup F) \text{ is } k\text{-arc-connected}\} =$$
$$\max\{\sum_{X\in\mathcal{X}}(k - d_D^+(X)), \sum_{X\in\mathcal{X}}(k - d_D^-(X))\}$$

where $\mathcal{X}$ is a subpartition of V.

Proof

Similar to previous one, by applying Mader's directed splitting off theorem.

Generalizations

1. local edge-connectivity; polynomially solvable,
2. hypergraphs; polynomially solvable,
3. partition constrained; polynomially solvable,
4. weighted; NP-complete even for $k = 2$.

Submodular function minimization

Theorem (Grötschel-Lovász-Schrijver, Fujishige-Fleicher-Iwata, Schrijver)

The minimum value of a submodular function can be found in poly. time.

Corollary: One can decide in polynomial time whether

1. a graph G is k-edge-connected
   (by minimizing $d_G(X \cup u)$ $X \subseteq V - v$ $\forall u, v \in V$),
2. a network (D, g) has a feasible flow of value k
   (by minimizing $d_g^+(Z \cup s)$ $Z \subseteq V \setminus \{s, t\}$),
3. a bipartite graph G has a perfect matching
   (by minimizing $|\Gamma(X)| - |X|$),
4. two matroids have a common independent set of size k
   (by minimizing $r_1(X) + r_2(S - X)$),
5. a digraph D has a packing of k spanning s-arborescences
   (by minimizing $d_D^-(X \cup u)$ $X \subseteq V - s$ $\forall u \in V - s$).