SIAM J. DISCRETE MATH.
Vol. 30, No. 4, pp. 2107–2114

ON MAXIMAL INDEPENDENT ARBORESCENCE PACKING*

CSABA KIRÁLY†

Abstract. By generalizing the results of [N. Kamiyama, N. Katoh, and A. Takizawa, *Combinatorica*, 29 (2009), pp. 197–214], we solve the following problem. Given a digraph $D = (V, A)$ and a matroid on a set $\mathsf{S} = \{\mathsf{s}_1, \ldots, \mathsf{s}_k\}$ along with a map $\pi : \mathsf{S} \to V$, find k edge-disjoint arborescences $T_1, \ldots, T_k$ with roots $\pi(\mathsf{s}_1), \ldots, \pi(\mathsf{s}_k)$, respectively, such that, for any $v \in V$, the set $\{\mathsf{s}_i : v \in T_i\}$ is independent and its rank reaches the theoretical maximum. We also give a simplified proof for a result of [S. Fujishige, *Combinatorica*, 30 (2010), pp. 247–252].

Key words. arborescence, packing, matroid

AMS subject classifications. 05C70, 05C40, 05B35

DOI. 10.1137/130938396

1. Introduction.

Recent research in rigidity theory has motivated new extensions of the well-known results of Tutte [13] and Nash-Williams [12, 11] on packing and covering with trees. Katoh and Tanigawa [8] recently proved that minimal rigidity of "bar-slider frameworks" is equivalent to some colored rooted-forest packing properties. This result inspired extensive research on the possible extensions of Tutte's and Nash-Williams' results. Katoh and Tanigawa [9] proved a theorem on the existence of colored rooted-forest packings. In [10], they generalized this result so as to include matroidal constraints on the roots and they also showed an overview of possible applications in rigidity theory.

Frank [4] showed how to derive Nash-Williams' [11] result from the weak form of Edmonds' theorem [3] on arborescence packings. Following this idea, Durand de Gevigney, Nguyen, and Szigeti [2] generalized Edmonds' weak theorem so as to obtain an alternative proof of the packing part of [10]. Actually, [2] also generalizes the strong form of the result of Edmonds. This raises the question whether earlier extensions of [3] such as the one of Kamiyama, Katoh, and Takizawa [7] can be generalized to such a form. We answer the question positively by extending [7]. In addition, it is also shown how another extension due to Fujishige [6] can easily be derived from [7]. (For a survey on tree and arborescence packing, see [1] and [5, Chapter 10].)

The following definitions are used throughout the paper. In a digraph $D = (V, A)$, $\varrho_D(X)$ and $\delta_D(X)$ denote the in-degree and the out-degree of a set $X \subseteq V$, respectively. For a nonempty set $R \subseteq V$, $B = (V, A')$ is said to be an *R-branching* if it consists of $|R|$ node-disjoint arborescences whose roots are in R. Let $D = (V, A)$ be a digraph. Then an R-branching is said to be *spanning* if it spans the node set V and it is said to be *maximal* if it spans all the nodes that are reachable from R in D. For nonempty sets $X, Z \subseteq V$, let $Z \mapsto X$ denote that X and Z are disjoint and X is reachable from Z, that is, there is a directed path from Z to X. For simplicity, we will denote the set $\{v\}$ by v.

*Received by the editors September 24, 2013; accepted for publication (in revised form) August 23, 2016; published electronically November 1, 2016.
http://www.siam.org/journals/sidma/30-4/93839.html
Funding: Research was supported by the Hungarian Scientific Research Fund – OTKA, CK80124 and K109240, and by the MTA-ELTE Egerváry Research Group.
†Department of Operations Research, Eötvös Loránd University, Pázmány Péter sétány 1/C, Budapest, 1117, Hungary (cskiraly@cs.elte.hu).

Throughout the paper, $D = (V, A)$ is a digraph, $\mathcal{M}$ is a matroid on a set S with rank function $r_{\mathcal{M}}$, and $\pi : \mathsf{S} \to V$ is a (not necessarily injective) map. For $\mathsf{P} \subseteq \mathsf{S}$, $\mathrm{Span}_{\mathcal{M}}(\mathsf{P})$ denotes the subset of S spanned by P, that is, the maximal set $\mathsf{X} \subseteq \mathsf{S}$ for which $\mathsf{P} \subseteq \mathsf{X}$ and $r_{\mathcal{M}}(\mathsf{X}) = \mathsf{r}_{\mathcal{M}}(\mathsf{P})$. For related definitions and properties of matroids, we refer to [5]. We say that the quadruple $(D, \mathcal{M}, \mathsf{S}, \pi)$ is a *matroid-based rooted digraph.* As in [2], π is called *$\mathcal{M}$-independent* if $\pi^{-1}(v)$ is independent in $\mathcal{M}$ for each $v \in V$. For $X \subseteq V$, we will denote by S_X the set $\pi^{-1}(X)$.

2. Preliminaries.

The weak form of the result of Edmonds [3] asserts the following.

THEOREM 1 (Edmonds' weak arborescence theorem). *In a digraph $D = (V, A)$, there are k edge-disjoint spanning arborescences with root r_0 if and only if*

$$\varrho_D(X) \geq k \tag{1}$$

holds for every $\emptyset \neq X \subset V - r_0$.

The strong form of Edmonds' theorem considers a more general problem when we want to find k edge-disjoint branchings in D.

THEOREM 2 (Edmonds [3]). *In a digraph $D = (V, A)$, let $\mathcal{R} := \{R_1, \ldots, R_k\}$ be a family of nonempty subsets of V. There are edge-disjoint spanning R_i-branchings in D for $i = 1, \ldots, k$ if and only if*

$$\varrho_D(X) \geq p_{\mathcal{R}}(X) \tag{2}$$

holds for every $\emptyset \neq X \subseteq V$, where $p_{\mathcal{R}}(X)$ denotes the number of the members of $\mathcal{R}$ disjoint from X.

When one wants to extend this result, it is natural to ask the following.

PROBLEM 2.1. *Given two families $\mathcal{R} = \{R_1, \ldots, R_k\}$ and $\mathcal{U} = \{U_1, \ldots, U_k\}$ of nonempty subsets of nodes in a digraph D such that $R_i \subseteq U_i$, find edge-dijoint R_i-branchings in D spanning U_i for $i = 1, \ldots, k$.*

This problem is NP-hard generally but it is polynomially solvable for some special cases. When $U_1 = \cdots = U_k = V$, the problem is solved by Theorem 2. When each U_i is the set of nodes reachable from R_i on a directed path in D for $i = 1, \ldots, k$ (that is, we want to find edge-disjoint maximal R_i-branchings for $i = 1, \ldots k$), then the problem is solved by Kamiyama, Katoh, and Takizawa [7], as follows.

THEOREM 3 (Kamiyama, Katoh, and Takizawa [7]). *In a digraph $D = (V, A)$, let $\mathcal{R} := \{R_1, \ldots, R_k\}$ be a family of nonempty subsets of V. There are edge-disjoint maximal R_i-branchings in D for $i = 1, \ldots, k$ if and only if*

$$\varrho_D(X) \geq p'_{\mathcal{R}}(X) \tag{3}$$

holds for every $\emptyset \neq X \subseteq V$, where $p'_{\mathcal{R}}(X)$ denotes the number of R_i's for which $R_i \mapsto X$.

A set of nodes U is called *convex* if there is no node $v \in V - U$ for which $v \mapsto U$ and $U \mapsto v$. Fujishige [6] used Theorem 3 to solve Problem 2.1 for the case where each U_i is convex. We present here a simplified proof for Fujishige's theorem which shows that, in fact, it follows more easily from Theorem 3.

THEOREM 4 (Fujishige [6]). *In a digraph $D = (V, A)$, let $\mathcal{R} := \{R_1, \ldots, R_k\}$ be a family of nonempty subsets of V and let $U_i \subseteq V$ be convex sets with $R_i \subseteq U_i$. There are edge-disjoint R_i-branchings spanning U_i in D for $i = 1, \ldots, k$ if and only if*

$$\varrho_D(X) \geq p_{\mathcal{R}}^{\{U_1,\ldots,U_k\}}(X) \tag{4}$$

holds for every $\emptyset \neq X \subseteq V$, where $p_{\mathcal{R}}^{\{U_1,\ldots,U_k\}}(X)$ denotes the number of U_i's for which $U_i \cap X \neq \emptyset$ and $X \cap R_i = \emptyset$.

Proof. As the proof of necessity of (4) is straightforward, only sufficiency will be proved.

It is easy to see that (4) ensures the the existence of an R_i-branching spanning U_i for each $i \in \{1, \ldots, k\}$. Let Z_i be the set of nodes reachable from R_i on a directed path in D and let $R'_i := Z_i - (U_i - R_i)$ for $i = 1, \ldots, k$. Observe that $\delta_D(Z_i) = 0$ by definition, thus $\delta_D(R'_i - R_i) = \delta_D(Z_i - U_i) = 0$ by convexity of U_i as $U_i \mapsto v$ for each $v \in R'_i - R_i$. Thus a maximal R'_i-branching consists of the single nodes as roots in $R'_i - \{R_i\}$ and an R_i-branching spanning U_i for $i = 1, \ldots, k$. Therefore, the existence of edge-disjoint maximal R'_i-branchings for $i = 1, \ldots, k$ is equivalent to the existence of edge-disjoint R_i-branchings spanning U_i for $i = 1, \ldots, k$.

$p_{\mathcal{R}}^{\{U_1,\ldots,U_k\}}(X) \geq p'_{\{R'_1,\ldots,R'_k\}}(X)$ holds for $X \subseteq V$ since $U_i \cap X \neq \emptyset$ and $R_i \cap X = \emptyset$ if $R'_i \mapsto X$ $(i = 1, \ldots, k)$. Thus, for $X \subseteq V$, if (4) holds, then (3) also holds. Therefore, the statement follows by Theorem 3. ∎

Next we present a recent result of Durand de Gevigney, Nguyen, and Szigeti [2] that generalizes Edmonds' results [3] in another direction. An *$\mathcal{M}$-based packing of arborescences* in $(D, \mathcal{M}, \mathsf{S}, \pi)$ is a set $\{T_1, \ldots, T_{|\mathsf{S}|}\}$ of pairwise edge-disjoint arborescences in D such that T_i has root $\pi(\mathsf{s}_i)$ for $i = 1, \ldots, |\mathsf{S}|$ and also the set $\{\mathsf{s}_j \in \mathsf{S} : v \in V(T_j)\}$ forms a base of $\mathcal{M}$ for each $v \in V$. The result of [2] is the following.

THEOREM 5 (Durand de Gevigney, Nguyen, and Szigeti [2]). *Let $(D, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted digraph. There exists an $\mathcal{M}$-based packing of arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$ if and only if π is $\mathcal{M}$-independent and*

$$\varrho_D(X) \geq r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X) \tag{5}$$

holds for every $\emptyset \neq X \subseteq V$.

Let $\mathcal{R} = \{R_1, \ldots, R_k\}$ be a family of nonempty subsets of V. If $\mathsf{S} := \bigcup \mathcal{R}$ (as a multiset), π maps each occurrence of r in S to the node $r \in V$, and $\mathcal{M}$ is the partition matroid on S given by $\mathcal{R}$, where a set $\mathsf{P} \subseteq \mathsf{S}$ is independent if and only if $|\mathsf{P} \cap R_i| \leq 1$ for $i = 1, \ldots, k$, then the problem of $\mathcal{M}$-based packing of arborescences and that of edge-disjoint spanning R_i-branchings for $i = 1, \ldots k$ coincide. Moreover, in this case π is always $\mathcal{M}$-independent and (5) is equivalent to (2). Therefore, Theorem 2 follows from Theorem 5. In the next section, we will extend Theorem 5 to a theorem from which Theorem 3 follows.

In our proof, we will use the following technical lemma pointed out in [2].

LEMMA 6 (see [2]). *Let $\mathcal{M}$ be a matroid on S with rank function $r_{\mathcal{M}}$ and let $\mathsf{P}, \mathsf{Q} \subseteq \mathsf{S}$ such that $r_{\mathcal{M}}(\mathsf{P}) + r_{\mathcal{M}}(\mathsf{Q}) = r_{\mathcal{M}}(\mathsf{P} \cap \mathsf{Q}) + r_{\mathcal{M}}(\mathsf{P} \cup \mathsf{Q})$. Then $Span_{\mathcal{M}}(\mathsf{P}) \cap Span_{\mathcal{M}}(\mathsf{Q}) \subseteq Span_{\mathcal{M}}(\mathsf{P} \cap \mathsf{Q})$.*

3. The main result. Let $P(X) := X \cup \{v \in V - X : v \mapsto X\}$. We call a *maximal $\mathcal{M}$-independent packing of arborescences* a set $\{T_1, \dots, T_{|\mathsf{S}|}\}$ of pairwise edge-disjoint arborescences for which T_i has root $\pi(\mathsf{s}_i)$ for $i = 1, \dots, |\mathsf{S}|$, the set $\{\mathsf{s}_j \in \mathsf{S} : v \in V(T_j)\}$ is independent in $\mathcal{M}$, and $|\{\mathsf{s}_j \in \mathsf{S} : v \in V(T_j)\}| = r_{\mathcal{M}}(\mathsf{S}_{P(v)})$ for each $v \in V$. (We will also say that s_i is the root of T_i.) Our main result is the following.

THEOREM 7. *Let $(D, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted digraph. There exists a maximal $\mathcal{M}$-independent packing of arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$ if and only if π is $\mathcal{M}$-independent and*

$$\varrho_D(X) \geq r_{\mathcal{M}}(\mathsf{S}_{P(X)}) - r_{\mathcal{M}}(\mathsf{S}_X) \tag{6}$$

holds for each $X \subseteq V$.

One can see that Theorem 3 follows from this theorem in the same way as Theorem 2 did from Theorem 5.

Before proving Theorem 7, we prove some lemmas that will be useful in the proof. For $X \subseteq V$, let $p(X) := r_{\mathcal{M}}(\mathsf{S}_{P(X)}) - r_{\mathcal{M}}(\mathsf{S}_X)$. X is called *tight* if $p(X) = \varrho_D(X)$. Two sets X and Y are called *intersecting* if $X - Y, Y - X$, and $X \cap Y$ are nonempty sets. Although p is not crossing supermodular, in general, we will prove the supermodular inequality for specific pairs in the next lemma.

LEMMA 8. *Let X and Y be two intersecting tight subsets of V for which $v \mapsto X \cap Y$ for every $v \in Y - X$. Then*

$$p(X) + p(Y) \leq p(X \cup Y) + p(X \cap Y). \tag{7}$$

Proof. As $v \mapsto X \cap Y$ for every $v \in Y - X$ and the reachability is transitive, we get $P(Y) \subseteq P(X \cap Y)$. Furthermore, $P(X) \subseteq P(X \cup Y)$ is obvious. Thus by the monotonicity of the rank function,

$$r_{\mathcal{M}}(\mathsf{S}_{P(X)}) + r_{\mathcal{M}}(\mathsf{S}_{P(Y)}) \leq r_{\mathcal{M}}(\mathsf{S}_{P(X \cup Y)}) + r_{\mathcal{M}}(\mathsf{S}_{P(X \cap Y)}). \tag{8}$$

Clearly, $\mathsf{S}_X \cap \mathsf{S}_Y = \mathsf{S}_{X \cap Y}$ and $\mathsf{S}_X \cup \mathsf{S}_Y = \mathsf{S}_{X \cup Y}$. Thus by the submodularity of the rank function,

$$r_{\mathcal{M}}(\mathsf{S}_X) + r_{\mathcal{M}}(\mathsf{S}_Y) \geq r_{\mathcal{M}}(\mathsf{S}_{X \cup Y}) + r_{\mathcal{M}}(\mathsf{S}_{X \cap Y}). \tag{9}$$

Subtracting (9) from (8) we get (7). ∎

Next we use Lemma 8 to prove that the intersection of two tight sets is tight in this special case.

LEMMA 9. *Let X and Y be two intersecting tight subsets of V. If $v \mapsto X \cap Y$ for every $v \in Y - X$, then $X \cap Y$ is tight and $\mathit{Span}_{\mathcal{M}}(\mathsf{S}_X) \cap \mathit{Span}_{\mathcal{M}}(\mathsf{S}_Y) \subseteq \mathit{Span}_{\mathcal{M}}(\mathsf{S}_{X \cap Y})$.*

Proof. The tightness of X and Y, along with inequalities (7), (6), and the submodularity of ϱ_D implies

$$\varrho_D(X) + \varrho_D(Y) = p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$$

$$\leq \varrho_D(X \cap Y) + \varrho_D(X \cup Y) \leq \varrho_D(X) + \varrho_D(Y). \tag{10}$$

Hence $p(X \cap Y) + p(X \cup Y) = \varrho_D(X \cap Y) + \varrho_D(X \cup Y)$. Thus $X \cap Y$ and $X \cup Y$ are tight. Moreover, $p(X) + p(Y) = p(X \cap Y) + p(X \cup Y)$. Therefore, in (9) equality must hold. Thus by Lemma 6 and by $\mathsf{S}_X \cap \mathsf{S}_Y = \mathsf{S}_{X \cap Y}$, we get $\mathrm{Span}_{\mathcal{M}}(\mathsf{S}_X) \cap \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_Y) \subseteq \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_{X \cap Y})$. ∎

The following statement follows easily from Lemma 9.

COROLLARY 3.1. *Let X and Y be two intersecting subsets of V such that X is tight and $\varrho_D(Y) = 0$. If $v \mapsto X \cap Y$ for every $v \in Y - X$, then $X \cap Y$ is tight and $Span_{\mathcal{M}}(\mathsf{S}_X) \cap Span_{\mathcal{M}}(\mathsf{S}_Y) \subseteq Span_{\mathcal{M}}(\mathsf{S}_{X\cap Y})$.*

Proof. By (6) and the monotonicity of the rank function, we get

$$0 = \varrho_D(Y) \geq r_{\mathcal{M}}(\mathsf{S}_{P(Y)}) - r_{\mathcal{M}}(\mathsf{S}_Y) \geq 0.$$

Thus Y is tight and the claim follows by Lemma 9. □

Following [2], we introduce some definitions. For $X, Y \subseteq V$, Y *dominates* X if $\mathsf{S}_X \subseteq \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_Y)$. It is easy to see that domination is a transitive relation. An edge $uv \in A$ is said to be *bad* if v dominates u, otherwise it is *good*. A good edge is called s-*good* if $\pi(\mathsf{s}) = u$ and $\mathsf{s} \notin \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_v)$. Note that a good edge is s-good for at least one $\mathsf{s} \in \mathsf{S}$. For $\mathsf{s} \in \mathsf{S}$, a tight set $X \subseteq V$ is s-*critical* if $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_X)$ and there exists a node v for which $\pi(\mathsf{s})v$ is an s-good edge of D that enters X. We call a set *critical* if there exists an $\mathsf{s} \in \mathsf{S}$ for which it is s-critical.

Next we prove a useful property of minimal s-critical sets. Let $D[X]$ denote the subgraph of $D = (V, A)$ induced by $X \subseteq V$.

LEMMA 10. *Let $(D, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-based rooted digraph for which* (6) *holds. For $\mathsf{s} \in \mathsf{S}$, let $\pi(\mathsf{s})v$ be an s-good edge and let X be a minimal s-critical set with $v \in X$. Then $X \subseteq P(v)$. Moreover, v is reachable from all points of X in $D[X]$.*

Proof. Assume for a contradiction that X is not a subset of $Y := P(v)$. Then X and Y are intersecting sets, $\varrho_D(Y) = 0$, and $v \in X \cap Y$ is reachable from all elements of $Y - X$. As X is tight, $X \cap Y$ is also tight by Corollary 3.1. Moreover, X is s-critical, hence, $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_X)$; furthermore, $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_Y)$ as $\pi(\mathsf{s}) \in Y$. Thus $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_{X\cap Y})$ also holds by Corollary 3.1. Therefore, $X \cap Y \subset X$ is an s-critical set such that $\pi(\mathsf{s})v$ enters it, contradicting the minimality of X.

To prove the second part, assume for a contradiction that v is not reachable from all elements of X in $D[X]$. Let Y' denote the subset of X from which v is reachable in $D[X]$. Then $\varrho_D(Y') \leq \varrho_D(X)$. Furthermore, $P(Y') = P(X) = Y$ by the first part of the lemma. As Y' is not s-critical by the minimality of X, $\mathsf{s} \notin \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_{Y'})$ and thus $r_{\mathcal{M}}(\mathsf{S}_{Y'}) < r_{\mathcal{M}}(\mathsf{S}_X)$. Therefore,

$$\varrho_D(Y') \leq \varrho_D(X) = r_{\mathcal{M}}(\mathsf{S}_{P(X)}) - r_{\mathcal{M}}(\mathsf{S}_X) < r_{\mathcal{M}}(\mathsf{S}_{P(Y')}) - r_{\mathcal{M}}(\mathsf{S}_{Y'}),$$

contradicting (6). □

Now we are ready to prove Theorem 7.

Proof of Theorem 7. As necessity of (6) and $\mathcal{M}$-independency are straightforward, only sufficiency will be proved. We use induction on $|A|$. Consider first the case when no good edge exists (this includes the case $A = \emptyset$).

CLAIM 3.2. *If there are only bad edges, then the arborescences $T_i^{root} = (\{\pi(\mathsf{s}_i)\}, \emptyset)$ consisting only of their roots for all $i \in \{1, \ldots, |\mathsf{S}|\}$ form a maximal $\mathcal{M}$-independent packing of arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$.*

Proof. By transitivity of the domination, if there are only bad edges, then $\mathsf{S}_{P(v)} \subseteq \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_v)$ for every $v \in V$. Thus $r_{\mathcal{M}}(\mathsf{S}_v) \leq r_{\mathcal{M}}(\mathsf{S}_{P(v)}) \leq r_{\mathcal{M}}(\mathsf{S}_v)$ holds for every $v \in V$. By the definition of the arborescences T_i^{root}'s, $\{\mathsf{s}_j \in \mathsf{S} : v \in V(T_j^{\mathrm{root}})\} = \mathsf{S}_v$. Moreover the $\mathcal{M}$-independency of π ensures the independency of S_v. Therefore, the set $\{\mathsf{s}_j \in \mathsf{S} : v \in V(T_j^{\mathrm{root}})\}$ is independent for every $v \in V$ and

$$|\{\mathsf{s}_j \in \mathsf{S} : v \in V(T_j^{\text{root}})\}| = |\mathsf{S}_v| = r_{\mathcal{M}}(\mathsf{S}_v) = r_{\mathcal{M}}(\mathsf{S}_{P(v)}). \qquad \square$$

Assume now that there is a good edge in $(D, \mathcal{M}, \mathsf{S}, \pi)$. For induction, we need the following stronger property.

CLAIM 3.3. *If there exists a good edge in $(D, \mathcal{M}, \mathsf{S}, \pi)$, then there exists an $\mathsf{s} \in \mathsf{S}$ and an s-good edge of the form $\pi(\mathsf{s})v$ such that it enters no s-critical set.*

Proof. Suppose that for all $\mathsf{s} \in \mathsf{S}$ and for each s-good edge, there exists an s-critical set entered by $\pi(\mathsf{s})v$. Choose a minimal critical set X. We can assume that, say, X is s-critical for $\mathsf{s} \in \mathsf{S}$ and $\pi(\mathsf{s})v$ is an s-good edge entering X.

If X induces no good edges, then $\mathrm{Span}_{\mathcal{M}}(\mathsf{S}_X) \subseteq \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_v)$ by the minimality of X and Lemma 10. Moreover, as X is s-critical, $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}\mathsf{S}_X$. Thus $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}\mathsf{S}_v$, that is, v dominates $\pi(\mathsf{s})$ contradicting that $\pi(\mathsf{s})v$ is an s-good edge.

Thus there is a good edge $u'v'$ spanned by X. By our assumption, there is an s'-critical set X' for any $\mathsf{s}' \in \mathsf{S}_{u'} - \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_{v'})$ such that $u'v'$ enters X'. Let us use the notation $Z := P(v')$. Then, by Corollary 3.1, $Z \cap X$ is tight as $\varrho_D(Z) = 0$ and $v' \in Z \cap X$ is reachable from all elements of $Z - X$. Moreover, $Z \cap X \cap X'$ is also tight by Lemma 9 since $(Z \cap X) - X' \subseteq Z - v' = P(v') - v'$ and $v' \in Z \cap X \cap X'$ is reachable fom each element of $P(v') - v'$. Thus $Z \cap X \cap X'$ is s'-critical and $Z \cap X \cap X' \subset X$ because $u' \in X - X'$ and $v' \in Z \cap X \cap X'$. Therefore, $Z \cap X \cap X'$ is a proper critical subset of X, contradicting the minimality of X. $\square$

Let $\pi(\mathsf{s})v$ be an s-good edge entering no s-critical set, where $\mathsf{s} \in \mathsf{S}$. Let $D' := D - \pi(\mathsf{s})v$, $\mathsf{S}' := \mathsf{S} \cup \{\mathsf{s}'\}$, where $\mathsf{s}' \notin \mathsf{S}$, $\pi' : \mathsf{S}' \to V$ such that $\pi'|_{\mathsf{S}} \equiv \pi$, and $\pi'(\mathsf{s}') = v$ and let $\mathcal{M}'$ be the matroid on S' that is obtained from $\mathcal{M}$ by considering s' as an element parallel to s. For $X \subseteq V$, let $P'(X) := X \cup \{v \in V : v \mapsto_{D'} X\}$ and $\mathsf{S}'_X := (\pi')^{-1}(X)$.

The content of the next claim is that the conditions of Theorem 7 remain valid for $(D', \mathcal{M}', \mathsf{S}', \pi')$.

CLAIM 3.4. *π' is $\mathcal{M}'$-independent and*

$$\varrho_{D'}(X) \geq r_{\mathcal{M}'}(\mathsf{S}'_{P'(X)}) - r_{\mathcal{M}'}(\mathsf{S}'_X) \tag{11}$$

holds for every $X \subseteq V$.

Proof. π' is $\mathcal{M}'$-independent since $\pi(\mathsf{s})v$ was an s-good edge.

Take an arbitrary set $X \subseteq V$. To prove (11), first observe that $P'(X) \subseteq P(X)$ and they are not equal if and only if $v \in P'(X)$ and $\pi(\mathsf{s}) \notin P'(X)$ both hold. Thus $r_{\mathcal{M}'}(\mathsf{S}'_{P'(X)}) \leq r_{\mathcal{M}}(\mathsf{S}_{P(X)})$ by the definition of $\mathcal{M}'$. Also by definition, $r_{\mathcal{M}}(\mathsf{S}'_X) \geq r_{\mathcal{M}}(\mathsf{S}_X)$. Thus the right-hand side of (6) is at most that of (11). Therefore, if $X \subseteq V$ is not tight, then (11) holds trivially as $\varrho_{D'}(X) + 1 \geq \varrho_D(X)$ and (6) holds with strict inequality; if $X \subseteq V$ is tight but $\pi(\mathsf{s})v$ does not enter X, then (11) holds trivially as $\varrho_{D'}(X) = \varrho_D(X)$. If X is tight and $\pi(\mathsf{s})v$ enters X, then $r_{\mathcal{M}}(\mathsf{S}'_X) > r_{\mathcal{M}}(\mathsf{S}_X)$ because $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}'}(\mathsf{S}'_X)$ as $s' \in \mathsf{S}'_X$ but $\mathsf{s} \notin \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_X)$ since $\pi(\mathsf{s})v$ enters no s-critical set. Thus in this case, $\varrho_D(X) = \varrho_{D'}(X) + 1$ and $r_{\mathcal{M}}(\mathsf{S}_{P(X)}) - r_{\mathcal{M}}(\mathsf{S}_X) \geq r_{\mathcal{M}'}(\mathsf{S}'_{P'(X)}) - r_{\mathcal{M}'}(\mathsf{S}'_X) + 1$, hence, (11) is again a consequence of (6). $\square$

Claim 3.4 ensures that there exists a maximal $\mathcal{M}'$-independent packing of arborescences $\mathcal{P}'$ in $(D', \mathcal{M}', \mathsf{S}', \pi')$ by induction.

Since s and s' are parallel in $\mathcal{M}'$, the arborescences $T, T' \in \mathcal{P}'$ rooted at s and s' are node disjoint. Therefore, $T \cup T' \cup \pi(\mathsf{s})v$ is an arborescence rooted at $\pi(\mathsf{s})$ and $\mathcal{P} = \mathcal{P} - \{T, T'\} \cup \{T \cup T' \cup \pi(\mathsf{s})v\}$ is a packing of arborescences rooted at S in D.

CLAIM 3.5. *$\mathcal{P}$ is a maximal $\mathcal{M}$-independent packing of arborescences in $(D, \mathcal{M}, \mathsf{S}, \pi)$.*

Proof. First observe that the $\mathcal{M}$-rank of the root set of the arborescences in $\mathcal{P}$ covering an arbitrary node u is the same as the $\mathcal{M}'$-rank of the root set of the arborescences in $\mathcal{P}'$ covering u by the definitions of $\mathcal{M}'$ and $\mathcal{P}'$. As $\mathcal{P}'$ is a maximal $\mathcal{M}'$-independent packing of arborescences, this latter value is equal to $r_{\mathcal{M}'}(\mathsf{S}'_{P'(u)})$. Hence the only thing we need to show is that

$$r_{\mathcal{M}'}(\mathsf{S}'_{P'(u)}) = r_{\mathcal{M}}(\mathsf{S}_{P(u)}) \text{ for all } u \in V.$$

As $r_{\mathcal{M}'}(\mathsf{S}'_{P'(u)}) \leq r_{\mathcal{M}}(\mathsf{S}_{P(u)})$ is obvious, we only prove that $r_{\mathcal{M}'}(\mathsf{S}'_{P'(u)}) \geq r_{\mathcal{M}}(\mathsf{S}_{P(u)})$. Suppose to the contrary that $r_{\mathcal{M}'}(\mathsf{S}'_{P'(u)}) < r_{\mathcal{M}}(\mathsf{S}_{P(u)})$ for a given $u \in V$. Since $r_{\mathcal{M}'}(\mathsf{S}'_Q) \geq r_{\mathcal{M}}(\mathsf{S}_Q)$ holds for any $Q \subseteq V$, $P'(u) \neq P(u)$ in this case. Thus $v \in P'(u)$ follows but $\pi(\mathsf{s}) \notin P'(u)$ because D and D' differ only on the edge $\pi(\mathsf{s})v$. Therefore, $\pi(\mathsf{s})v$ is the single edge of D that enters $P'(u)$. Thus inequality (6) for $X = P'(u)$ transforms to

$$1 = \varrho_D(P'(u)) \geq r_{\mathcal{M}}(\mathsf{S}_{P(P'(u))}) - r_{\mathcal{M}}(\mathsf{S}_{P'(u)}) = r_{\mathcal{M}}(\mathsf{S}_{P(u)}) - r_{\mathcal{M}}(\mathsf{S}_{P'(u)})$$

and hence, by our assumption that $r_{\mathcal{M}'}(\mathsf{S}'_{P'(u)}) < r_{\mathcal{M}}(\mathsf{S}_{P(u)})$,

$$r_{\mathcal{M}}(\mathsf{S}_{P'(u)}) + 1 \geq r_{\mathcal{M}}(\mathsf{S}_{P(u)}) \geq r_{\mathcal{M}'}(\mathsf{S}'_{P'(u)}) + 1 \geq r_{\mathcal{M}}(\mathsf{S}_{P'(u)}) + 1.$$

Therefore, equality must hold throughout. From $r_{\mathcal{M}}(\mathsf{S}_{P'(u)}) + 1 = r_{\mathcal{M}}(\mathsf{S}_{P(u)})$, we get that $P'(u)$ is tight, and by $r_{\mathcal{M}'}(\mathsf{S}'_{P'(u)}) = r_{\mathcal{M}}(\mathsf{S}_{P'(u)})$, we get that $\mathsf{s} \in \mathrm{Span}_{\mathcal{M}}(\mathsf{S}_{P'(u)})$. Thus $P'(u)$ is s-critical, a contradiction to the assumption that $\pi(s)v$ is an s-good edge that enters no s-critical sets. ∎

This completes the proof of Theorem 7. ∎

4. Concluding remarks. The proof of Theorem 7 gives rise to an algorithm if the matroid is given by an oracle for the rank function. Durand de Gevigney, Nguyen, and Szigeti [2] gave an algorithm also for the weighted case of their problem using polyhedral techniques. This, along with their proof for the undirected case, uses the fact that the right-hand side function of (5) is supermodular. As we noted, p is not crossing supermodular, in general, though the supermodular inequality holds for specific pairs of sets. It would be interesting to prove some more properties of this function. Developing such a property could help to give an algorithm for the weighted case as well and to prove an undirected version of Theorem 7. These problems remain open.

Acknowledgments. The author is grateful to Kristóf Bérczi and Erika R. Bérczi-Kovács for the inspiring discussions and to Tamás Király and András Frank for their comments.

REFERENCES

[1] K. Bérczi and A. Frank, *Packing arborescences*, in Combinatorial Optimization and Discrete Algorithms, S. Iwata, ed., RIMS Kokyuroku Bessatsu B23 RIMS, Kyoto, 2010, pp. 1–31.

[2] O. Durand de Gevigney, V.-H. Nguyen, and Z. Szigeti, *Matroid-based packing of arborescences*, SIAM J. Discrete Math., 27 (2013), pp. 567–574.

[3] J. Edmonds, *Edge-disjoint branchings*, in Combinatorial Algorihms, B. Rustin, ed., Academic, New York, 1973, pp. 91–96.

[4] A. Frank, *On disjoint trees and arborescences*, in Algebraic Methods in Graph Theory, Colloq. Math. Soc. J. Bolyai 25, Norh-Holland, Amsterdam 1981, pp. 59–169.
[5] A. Frank, *Connections in Combinatorial Optimization*, Oxford University Press, Oxford, 2011.
[6] S. Fujishige, *A note on disjoint arborescences*, Combinatorica, 30 (2010), pp. 247–252.
[7] N. Kamiyama, N. Katoh, and A. Takizawa, *Arc-disjoint in-trees in directed graphs*, Combinatorica, 29 (2009), pp. 197–214.
[8] N. Katoh and S. Tanigawa, *On the infinitesimal rigidity of bar-and-slider frameworks*, in Algorithms and Computation Springer, Berlin, 2009, pp. 524–533.
[9] N. Katoh and S. Tanigawa, *A rooted-forest partition with uniform vertex demand*, J. Comb. Optim., 24 (2012), pp. 67–98.
[10] N. Katoh and S. Tanigawa, *Rooted-tree decompositions with matroid constraints and the infinitesimal rigidity of frameworks with boundaries*, SIAM J. Discrete Math., 27 (2013), pp. 155–185.
[11] C. St. J. A. Nash-Williams, *Decomposition of finite graphs into forests*, J. Lond. Math. Soc. (2), 39 (1961), p. 12.
[12] C. St. J. A. Nash-Williams, *Edge-disjoint spanning trees of finite graphs*, J. Lond. Math. Soc. (2), 36 (1961), pp. 445–450.
[13] W. T. Tutte, *On the problem of decomposing a graph into n connected factors*, J. Lond. Math. Soc. (2), 36 (1961), pp. 221–230.