Common Root Functions of Two Digraphs

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ABSTRACT

Let D_1 and D_2 be finite digraphs, both with vertex set V, let a and b be given functions from V to Z_+, and let k be a positive integer. In this paper we give a necessary and sufficient condition for the existence of k arc-disjoint arborescences in each of D_1 and D_2 satisfying the condition that for each v in V

$$a(v) \leqslant r_1(v) = r_2(v) \leqslant b(v),$$

where $r_i(v)$ denotes the number of the arborescences in D_i rooted at v, $i = 1, 2$.

Let $D = (V, A)$ be a finite digraph with vertex set V and arc set A. Multiple arcs are allowed but loops are not. For $V' \subseteq V$, the indegree $d^-(V')$ is the number of arcs in D entering V', and $\overline{V}' = V - V'$.

An arborescence of D is defined as a spanning tree directed in such a way that each vertex of D, except one called the root of the arborescence, has one arc entering it.

If f is a rational function defined on V, and $V' \subseteq V$, we write $f(V') = \sum_{v \in V'} f(v)$, and set $f(\emptyset) = 0$. Let Z_+ denote the set of nonnegative integers.

A function r: $V \rightarrow Z_+$ is called a root function of a digraph $D = (V, A)$ if D contains $r(V)$ arc-disjoint arborescences such that exactly $r(v)$ of them are rooted at v for each $v \in V$.

Two subsets S and T of V are intersecting if none of $S \cap T$, and $S - T$ and $T - S$ is empty. A family F of subsets of V is called intersecting if $S \cap T$ and $S \cup T$ belong to F for all intersecting pairs S, T of F. Let $|S|$ denote the cardinality of set S.

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A rational function f defined on an intersecting family F is called submodular on intersecting pairs if $f(S) + f(T) \geq f(S \cap T) + f(S \cup T)$ for all intersecting members S and T of F.

A polymatroid P in the space R_+^V is a compact nonempty subset of R_+^V such that (a) $0 \leq y \leq x \in P \to y \in P$, and (b) for every $a \in R_+^V$, every maximal $x \in P$ with $x \leq a$ has the same sum $x(V)$.

For real number q, let $\lfloor q \rfloor$ denote the largest integer less than or equal to q.

The present paper is a natural continuation of [1]. Its purpose is to further generalize the following theorem of Edmonds, which will be used in the proof of the main theorem of this paper:

Theorem 1 [3]. A function r: $V \to Z_+$ is a root function of a diagraph $D = (V, A)$ if and only if for each $V' \subset V$

$$r(V') \leq d^-(\overline{V}'). \tag{1}$$

Let us consider the following:

Problem. Let $D_1 = (V, A_1)$ and $D_2 = (V, A_2)$ be two digraphs, both with vertex set V, let a and b be given functions: $V \to Z_+$ such that $a \leq b$, and let k be a positive integer. What conditions are needed for D_1 and D_2 to have a common root function r satisfying

$$r(V) = k \quad \text{and} \quad a \leq r \leq b? \tag{2}$$

Remark. The problem without lower and upper bounds on root function r is due to A. Schrijver.

Let $d_i^-(V')$ denote the indegree of subset V' in D_i, $i = 1, 2$, and F^* the family of all nonempty subsets of V. Obviously, F^* is intersecting. The problem can be formulated with the help of the following linear programming (Q).

$$\begin{aligned} &\min x(V) \\ (Q)\colon\ &x(S) \geq k - d_i^-(S) \quad \text{for all } S \in F^*,\ i = 1, 2, \\ &a \leq x \leq b. \end{aligned} \tag{3}$$

The main result of this paper is

Theorem 2. Let D_1, D_2, a, b, and k be given as in the problem. Then the following statements are equivalent:

(i) D_1 and D_2 have a common root function r satisfying (2).
(ii) (Q) has an integral optimum solution x with $x(V) = k$.
(iii) For any two families F_1 and F_2 (possibly empty) of disjoint nonempty subsets of V

$$\sum_{S\in F_1} [k - d_1^-(S)] + \sum_{T\in F_2} [k - d_2^-(T)] - b(V_1 \cap V_2) + a(\overline{V_1 \cup V_2}) \leq k , \tag{4}$$

where $V_i = \bigcup_{R\in F_i} R$, $i = 1,2$.

In order to prove the theorem we need the following result that is well-known in polymatroid theory, due to Edmonds [2].

Polymatroid Intersection Theorem. For $i = 1,2$, let function f_i: $F^* \to Z_+$ be submodular on intersecting pairs, let $P_i = \{y \in R_+^V: y(S) \leq f_i(S)$ for every $S \in F^*\}$ be the polymatroid associated, and let $h \in Z_+$. Then there exists an integral vector $y \in P_1 \cap P_2$ with $y(V) = h$ if and only if for every choice of $S_1, \ldots, S_s, T_1, \ldots, T_t \in F^*$ such that $S_1, \ldots, S_s$ are pairwise disjoint, $T_1, \ldots, T_t$ are pairwise disjoint, and $S_1 \cup \ldots \cup S_s \cup T_1 \cup \ldots \cup T_t = V$, one has

$$\sum_{j=1}^{s} f_1(S_j) + \sum_{j=1}^{t} f_2(T_j) \geq h . \tag{5}$$

Proof of Theorem 2. First note that, by taking $S = V$ in (3), $x(V) \geq k$ for any feasible solution x of (Q). Therefore a feasible solution x of (Q) with $x(V) = k$ is optimum.

It is easy to show the equivalence of (i) and (ii). Indeed, if D_1 and D_2 have a common root function r satisfying (2), it follows from (1) that for every $R \in F^*$, $r(R) = k - r(\overline{R}) \geq k - d_i^-(R)$, $i = 1,2$. So r is an integral optimum solution of (Q) with $r(V) = k$. Conversely, let x be an integral optimum solution with $x(V) = k$. Then $x(\overline{R}) \geq k - d_i^-(\overline{R})$ for every $\overline{R} \in F^*$, $i = 1,2$. Hence $x(R) = k - x(\overline{R}) \leq d_i^-(\overline{R})$ for every $R \subset V$, $i = 1,2$. By Theorem 1, x is a common root function satisfying (2).

It is easy to prove the implication (ii) $\to$ (iii). Indeed, let x be an integral optimum solution of (Q) with $x(V) = k$. Then, for any two families F_1 and F_2 as in (iii) (Recall $V_i = \bigcup_{R\in F_i} R$),

$$\begin{aligned}
&\sum_{S\in F_1} [k - d_1^-(S)] + \sum_{T\in F_2} [k - d_2^-(T)] - b(V_1 \cap V_2) + a(\overline{V_1 \cup V_2}) \\
&\qquad \leq \sum_{S\in F_1} x(S) + \sum_{T\in F_2} x(T) - x(V_1 \cap V_2) + x(\overline{V_1 \cup V_2}) \\
&\qquad = x(V_1) + x(V_2) - x(V_1 \cap V_2) + x(\overline{V_1 \cup V_2}) \\
&\qquad = x(V) \\
&\qquad = k .
\end{aligned}$$

Now let us show the converse. We first deduce from (4) the inequalities

$$b(R) \geq k - d_i^-(R) \quad \text{for every } R \in F^*,\ i = 1,2\,.$$

Indeed, by taking $F_1 = \{R\}$ and $F_2 = \{V\}$, (4) yields

$$b(R) \geq k - d_1^-(R)\,.$$

Similarly, $b(R) \geq k - d_2^-(R)$.

Set

$$f_i(S) = \begin{cases} b(S) + d_i^-(S) - k & (S \subseteq V,\ |S| \geq 2)\,, \\ \min\{b(S) + d_i^-(S) - k,\ b(S) - a(S)\} & (S \subseteq V,\ |S| = 1)\,, \end{cases}$$

for $i = 1,2$, and $h = b(V) - k$. Then f_i is nonnegative, integral, and submodular on intersecting pairs. The submodularity of f_i follows from that of d_i^-.

Let P_i be the polymatroid associated with f_i, $i = 1,2$. Then clearly there exists an integral vector $y \in P_1 \cap P_2$ with $y(V) = h$ if and only if (Q) has an integral optimum solution x with $x(V) = k$ (indeed, $y = b - x$). So, by the Polymatroid Intersection Theorem, to complete the proof it suffices to show the implication (4) $\rightarrow$ (5).

Let $S_1, \ldots, S_s, T_1, \ldots, T_t$ be chosen as in Polymatroid Intersection Theorem. Put

$$\begin{aligned} F_1 &= \{S_j\colon 1 \leq j \leq s,\ |S_j| \geq 2, \text{ or } |S_j| = 1 \text{ and} \\ &\qquad b(S_j) + d_1^-(S_j) - k \leq b(S_j) - a(S_j)\}\,, \\ F_2 &= \{T_j\colon 1 \leq j \leq t,\ |T_j| \geq 2, \text{ or } |T_j| = 1 \text{ and} \\ &\qquad b(T_j) + d_2^-(T_j) - k \leq b(T_j) - a(T_j)\}\,. \end{aligned}$$

Then

$$\begin{aligned} \sum_{j=1}^{s} f_1(S_j) + \sum_{j=1}^{t} f_2(T_j) &\geq \sum_{S\in F_1} [b(S) + d_1^-(S) - k] + \sum_{T\in F_2} [b(T) + d_2^-(T) - k] \\ &\quad + (b - a)(\overline{V_1 \cup V_2}) \\ &= \sum_{S\in F_1} [d_1^-(S) - k] + \sum_{T\in F_2} [d_2^-(T) - k] + b(V_1 \cap V_2) \\ &\quad + b(V) - a(\overline{V_1 \cup V_2}) \\ &\geq h\,. \end{aligned}$$

The last inequality follows from (4). This is the end of the proof for Theorem 2.

By taking $a = 0$, $b = k$ in Theorem 2, an immediate consequence is

Corollary 1. Two digraphs $D_1 = (V, A_1)$ and $D_2 = (V, A_2)$ have a common root function r with $r(V) = k$ if and only if for every family F of disjoint nonempty subsets of V

$$\sum_{S \in F} \max\{k - d_1^-(S),\, k - d_2^-(S)\} \leqslant k\,. \tag{6}$$

Proof. For F, define

$$F_1 = \{S \in F\colon d_1^-(S) \leqslant d_2^-(S)\}, \qquad F_2 = F - F_1\,.$$

It follows from (4) that

$$\sum_{S \in F} \max\{k - d_1^-(S),\, k - d_2^-(S)\} = \sum_{S \in F_1} [k - d_1^-(S)] + \sum_{T \in F_2} [k - d_2^-(T)] \leqslant k\,.$$

Conversely, let F_1 and F_2 be two families given in (iii) of Theorem 2. Define

$$F_2' = \{S \in F_2\colon S \cap V_1 = \varnothing\}, \qquad F = F_1 \cup F_2' \qquad \left(V_i = \bigcup_{S \in F_i} S\right).$$

Using $a = 0$ and $b = k$, one deduces from (6) that

$$\begin{aligned}
&\sum_{S \in F_1} [k - d_1^-(S)] + \sum_{T \in F_2} [k - d_2^-(T)] - b(V_1 \cap V_2) + a(\overline{V_1 \cup V_2}) \\
&\quad = \sum_{S \in F_1} [k - d_1^-(S)] + \sum_{T \in F_2} [k - d_2^-(T) - k|T \cap V_1|] \\
&\quad \leqslant \sum_{S \in F_1} [k - d_1^-(S)] + \sum_{T \in F_2'} [k - d_2^-(T)] \\
&\quad \leqslant \sum_{S \in F} \max\{k - d_1^-(S),\, k - d_2^-(S)\} \\
&\quad \leqslant k\,.
\end{aligned}$$

Applying Theorem 2 yields the required result.

The corollary can be stated in a min–max form.

Corollary 1′. Let D_1 and D_2 be digraphs, both with vertex set V. The maximum value k' of common root functions of D_1 and D_2 is equal to

$$k'' = \min \left\lfloor \left[\sum_{S \in F} \min\{d_1^-(S),\, d_2^-(S)\} \right] \Big/ (|F| - 1) \right\rfloor, \tag{7}$$

where the minimum ranges over all families F of disjoint nonempty subsets of V with $|F| \geqslant 2$.

Proof. First note that (6) holds for any nonnegative integer k when $|F| = 0$ or 1. So we only consider the families F in (7) with $|F| \geqslant 2$.

As k' satisfies (6), we obtain

$$k'(|F| - 1) \leqslant \sum_{S \in F} \min\{d_1^-(S),\, d_2^-(S)\},$$

which yields

$$k' \leqslant k''.$$

On the other hand, it follows from (7) that k'' satisfies (6). By Corollary 1, D_1 and D_2 have a common root function r with $r(V) = k''$. By the definition of k',

$$k'' \leqslant k'.$$

This concludes the proof.

Another consequence of Theorem 2 is

Corollary 2. Let a, b, and k be given as in the problem. Then a diagraph $D = (V, A)$ has a root function r satisfying (2) if and only if

(a) $b(S) \geqslant k - d^-(S)$ for every $S \in F^*$ and
(b) $\sum_{S \in F'} \lfloor k - d^-(S) \rfloor + a(\overline{V_1'}) \leqslant k$ for every family F' (possibly empty) of disjoint nonempty subsets of V, where $V_1' = \bigcup_{S \in F'} S$.

Proof. By taking $F_1 = \{S\}$ and $F_2 = \{V\}$, (4) yields (a). Setting $F_2 = \varnothing$ in (4), one gets (b).

Conversely, let $D_1 = D$ and D_2 be a k-strongly connected digraph in Theorem 2. We show that (4) follows from (a) and (b).

If $F_2 = \varnothing$, (4) becomes (b).

If $F_2 = \{V\}$, then it follows from (a) that

$$\sum_{S \in F_1} [k - d^-(S)] + \sum_{T \in F_2} [k - d_2^-(T)] - b(V_1 \cap V_2) + a(\overline{V_1 \cup V_2})$$

$$= \sum_{S \in F_1} [k - d^-(S)] + k - b(V_1)$$

$$= \sum_{S \in F_1} [k - d^-(S) - b(S)] + k$$
$$\leq k.$$

If $F_2 \neq \varnothing$, $\{V\}$, we deduce from (b) and the assumption of D_2 that

$$\sum_{S \in F_1} [k - d^-(S)] + \sum_{T \in F_2} [k - d_2^-(T)] - b(V_1 \cap V_2) + a(\overline{V_1 \cup V_2})$$
$$\leq \sum_{S \in F_1} [k - d^-(S)] + a(\overline{V_1})$$
$$\leq k.$$

The proof is completed.

Corollary 2 has been stated in the following way:

Corollary 2′ [1]. Let a, b, and k be given as above. Then a diagraph $D = (V, A)$ has a root function r satisfying (2) if and only if for any partition $P = \{S_1, \ldots, S_n\}$ of V,

$$\sum_{i=1}^{n} \max\{a(S_i),\, k - b(\overline{S_i}),\, k - d^-(S_i)\} \leq k. \tag{8}$$

Proof. For $P = \{S, \overline{S}\}$ and $F' \cup \{\overline{V'_1}\}$, (8) yields (a) and (b), respectively.

Conversely, for a partition $P = \{S_1, \ldots, S_n\}$, define

$$F' = \{S_i \in P\colon a(S_i) \leq k - d^-(S_i) \geq k - b(\overline{S_i})\},$$
$$F'' = \{S_i \in P\colon a(S_i) \leq k - b(\overline{S_i}) > k - d^-(S_i)\},$$
$$V' = \bigcup_{S \in F'} S, \qquad V'' = \bigcup_{S \in F''} S.$$

If $F'' = \varnothing$, then (b) yields (8); otherwise, using (b), we obtain

$$\sum_{i=1}^{n} \max\{a(S_i),\, k - b(\overline{S_i}),\, k - d^-(S_i)\}$$
$$= \sum_{S \in F'} [k - d^-(S)] + \sum_{S \in F''} [k - b(\overline{S})] + a(\overline{V' \cup V''})$$
$$\leq \sum_{S \in F'} b(S) + \sum_{S \in F''} [k - b(\overline{S})] + a(\overline{V' \cup V''})$$

$$
\begin{aligned}
&= b(V') + |F''|[k - b(V)] + b(V'') + a(\overline{V' \cup V''}) \\
&\leqslant b(V') + k - b(V) + b(V'') + a(\overline{V' \cup V''}) \\
&= k - b(\overline{V' \cup V''}) + a(\overline{V' \cup V''}) \\
&\leqslant k .
\end{aligned}
$$

The equivalence of their conditions is proved.

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