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On packing time-respecting arborescences

Romain Chapoullié [a], Zoltán Szigeti [b,*]

[a] *Ensimag, Grenoble, France*
[b] *Univ. Grenoble Alpes, Grenoble INP, CNRS, G-SCOP, 46 Avenue Félix Viallet, Grenoble, France, 38000*

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ABSTRACT

We present a slight generalization of the result of Kamiyama and Kawase (2015) on packing time-respecting arborescences in acyclic pre-flow temporal networks. Our main contribution is to provide the first results on packing time-respecting arborescences in non-acyclic temporal networks. As negative results, we prove the NP-completeness of the decision problem of the existence of 2 arc-disjoint spanning time-respecting arborescences and of a related problem proposed in this paper.

1. Introduction

Temporal networks were introduced to model the exchange of information in a network or the spread of a disease in a population. We are given a directed graph D and a time label function τ on the arcs of D, the pair (D, τ) is called a temporal network. Intuitively, for an arc a of D, $\tau(a)$ is the time when the end-vertices of a communicate, that is when the tail of a can transmit a piece of information to the head of a. Then the information can propagate through a path P if it is time-respecting, meaning that the time labels of the arcs of P in the order they are passed are non-decreasing. For a nice introduction to temporal networks, see [1].

Problems about packing arborescences in temporal networks were investigated in [2]. An arborescence is called time-respecting if all the directed paths it contains are time-respecting. The main result of [2] provides a packing of time-respecting arborescences, each vertex belonging to many of them, if the network is pre-flow and acyclic. Here pre-flow means intuitively that each vertex different from the root has at least as many arcs entering as leaving, while acyclic means that no directed cycle exists. Kamiyama and Kawase [2] presented examples to show that these conditions cannot be dropped.

Two questions naturally arise from these results: Must all kinds of directed cycles be forbidden? Does high time-respecting root-connectivity imply the existence of 2 arc-disjoint spanning time-respecting arborescences in a non-pre-flow temporal network?

* Corresponding author.
E-mail address: zoltan.szigeti@grenoble-inp.fr (Z. Szigeti).

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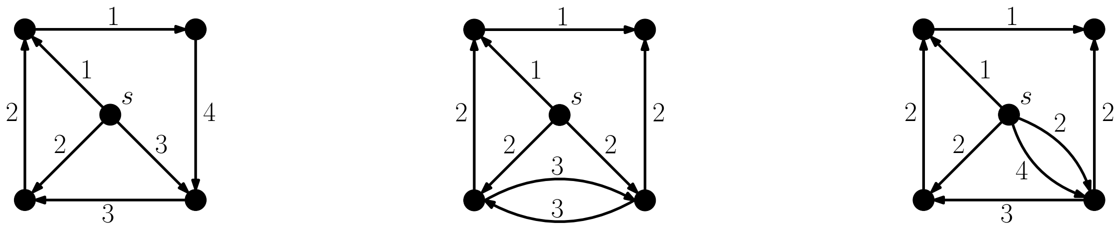


Fig. 1. Three temporal networks N where the τ-value of an arc is presented on the arc. The first two are non-acyclic pre-flow, the second one is consistent. The third one is acyclic but not pre-flow. They contain no 2 arc-disjoint τ-respecting s-arborescences such that each vertex v belongs to $\min\{2, \lambda_N(s, v)\}$ of them.

Let us now present our contributions that give an answer to those questions.

We first propose a generalized version of the result of [2] with a simplified proof in Theorem 2.

Our main result, Theorem 4, is about packing time-respecting arborescences in pre-flow temporal networks that may contain directed cycles. The condition in Theorem 4 is that the arcs in the same strongly connected component must have the same τ-value. If this condition holds then our intuition would be to use regular arborescences in the strongly connected components and then to try to extend them to obtain a packing of time-respecting arborescences in the temporal network. This idea is a step in the right direction, however the exact process used in the proof is a bit more complex, see Section 4.

By the famous result of Edmonds [3], we know that k-root-connectivity implies the existence of a packing of k spanning s-arborescences. The authors of [1] show that for any positive integer k, time-respecting k-root-connectivity does not imply the existence of 2 arc-disjoint spanning time-respecting arborescences in a temporal network. To explain this construction (or more precisely, a slightly modified version of it), we point out and recall in Section 5 the close relation between packings of spanning time-respecting arborescences, packings of Steiner arborescences and proper 2-colorings of hypergraphs. We remark in Theorem 12 that the decision problem, whether there exist 2 arc-disjoint spanning time-respecting arborescences, is NP-complete.

We show in Theorem 11 that time-respecting $(n-1)$-root-connectivity implies the existence of a packing of 2 spanning time-respecting s-arborescences in an arbitrary temporal network on n vertices. This result becomes more interesting if we note that the examples of Fig. 1 show that time-respecting $(n-3)$-root-connectivity is not enough.

Finally, in Theorem 13, we show that in an acyclic temporal network (D, τ), it is NP-complete to decide whether there exists a spanning arborescence whose directed paths consist of arcs of the same τ-value.

2. Definitions

Let $D = (V \cup s, A)$ be a directed graph with a special vertex s, called *root*, such that no arc enters s. The set of arcs entering, leaving a vertex set X of D is denoted by $\rho_D(X)$, $\delta_D(X)$, respectively. Sometimes we use $\rho_A(X)$ for $\rho_D(X)$ and similarly $\delta_A(X)$ for $\delta_D(X)$. We denote $|\rho_D(X)|$ and $|\delta_D(X)|$ by $d_D^-(X)$ and $d_D^+(X)$, respectively. We call the directed graph D *acyclic* if D contains no directed cycle. If $d_D^-(v) = d_D^+(v)$ for all $v \in V$, then D is called *Eulerian*. We say that D is *pre-flow* if $d_D^-(v) \geq d_D^+(v)$ for all $v \in V$. A subgraph $F = (V' \cup s, A')$ of D is called an *s-arborescence* if F is acyclic and $d_F^-(v) = 1$ for all $v \in V'$. We say that F is *spanning* if $V' = V$. For $U \subseteq V$, F is called a *Steiner s-arborescence* or an *(s, U)-arborescence* if F is an s-arborescence and it contains all the vertices in U. A *packing* of arborescences means a set of arc-disjoint arborescences. For $v \in V$, a path from s to v is called an *(s, v)-path* and $\lambda_D(s, v)$ denotes the maximum number of arc-disjoint (s, v)-paths in D. For some $k \in \mathbb{N}$, we say that D is *k-root-connected* if $\lambda_D(s, v) \geq k$ for all $v \in V$. For some $U \subseteq V$ and $k \in \mathbb{N}$, we say that D is *Steiner k-root-connected* if $\lambda_D(s, v) \geq k$ for all $v \in U$. We call a directed graph $D' = (V \cup \{s, t\}, A')$ *almost Eulerian* if $d_{D'}^-(v) = d_{D'}^+(v)$ for all $v \in V$ and $d_{D'}^-(s) = 0 = d_{D'}^+(t)$.

For a function $\tau : A \to \mathbb{N}$, $N = (D, \tau)$ is called a *temporal network*. For $i \in \mathbb{N}$, let $\rho^i_N(v) := \{a \in \rho_D(v) : \tau(a) \le i\}$ and $\delta^i_N(v) := \{a \in \delta_D(v) : \tau(a) \le i\}$. We call the temporal network N *acyclic* if D is acyclic. We say that N is *pre-flow* if $|\rho^i_N(v)| \ge |\delta^i_N(v)|$ for all $i \in \mathbb{N}$ and for all $v \in V$. Note that if a temporal network (D, τ) is pre-flow, then the directed graph D is pre-flow. We say that (D, τ) is *consistent* if arcs of different τ-values cannot belong to the same strongly connected component of D. In this case in each strongly connected component Q of D that contains at least one arc, each arc has the same τ-value, that we denote by $\tau(Q)$. A directed path P of D, consisting of the arcs $a_1, \dots, a_\ell$ in this order, is called *time-respecting* or *τ-respecting* if $\tau(a_i) \le \tau(a_{i+1})$ for $1 \le i \le \ell - 1$. An s-arborescence F of D is called *time-respecting* or *τ-respecting* if for every vertex v of F, the unique (s, v)-path in F is τ-respecting. For $v \in V$, $\lambda_N(s, v)$ denotes the maximum number of arc-disjoint τ-respecting (s, v)-paths in D. We say that N is *time-respecting k-root-connected* if $\lambda_N(s, v) \ge k$ for all $v \in V$. If $N' = (D', \tau')$ is a temporal network where $D' = (V \cup \{s, t\}, A')$ is almost Eulerian, then for a vertex $v \in V$, we call a bijection μ'_v from $\delta_{D'}(v)$ to $\rho_{D'}(v)$ *τ'-respecting* if $\tau'(\mu'_v(f)) \le \tau'(f)$ for all $f \in \delta_{D'}(v)$.

A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is defined by its vertex set V and its hyperedge set $\mathcal{E}$ where a hyperedge is a subset of V. For some $r \in \mathbb{N}$, the hypergraph $\mathcal{H}$ is called *r-uniform* if each hyperedge in $\mathcal{E}$ is of size r and *r-regular* if each vertex in V belongs to exactly r hyperedges. A 2-coloring of the vertex set V is called *proper* if each hyperedge in $\mathcal{E}$ contains vertices of both colors, in other words no monochromatic hyperedge exists in $\mathcal{E}$. We call $\mathcal{E}' \subseteq \mathcal{E}$ an *exact cover* of $\mathcal{H}$ if each vertex in V belongs to exactly one hyperedge in $\mathcal{E}'$.

3. Packing time-respecting arborescences in acyclic pre-flow temporal networks

The aim of this section is to generalize the following result of Kamiyama and Kawase [2] on packing time-respecting arborescences in acyclic pre-flow temporal networks.

Theorem 1 (*[2]*)**.** *Let $N = ((V \cup s, A), \tau)$ be an acyclic pre-flow temporal network and $k \in \mathbb{N}$. There exists a packing of k τ-respecting s-arborescences such that each vertex v in V belongs to $\min\{k, \lambda_N(s, v)\}$ of them.*

Note that Theorem 1 implies that in a time-respecting k-root-connected acyclic pre-flow temporal network there exists a packing of k spanning time-respecting s-arborescences.

We now present our first result, a slight extension of Theorem 1.

Theorem 2. *Let $N = ((V \cup s, A), \tau)$ be an acyclic temporal network and $k \in \mathbb{N}$ such that*

$$\min\{k, |\rho^i_N(v)|\} \ge \min\{k, |\delta^i_N(v)|\} \quad \textit{for all } i \in \mathbb{N}, \textit{ for all } v \in V. \tag{1}$$

There exists a packing of k τ-respecting s-arborescences such that each vertex v in V belongs to $\min\{k, d^-_A(v)\}$ of them.

We will partially follow the proof of [2] but we will point out that Lemmas 3 and 4 in [2] are not needed to prove Theorem 2. Hence the proof of Theorem 2 is simpler than that of Theorem 1. The following algorithm is a slightly modified version of the algorithm of Kamiyama and Kawase [2]. Its input is an acyclic temporal network $N = ((V \cup s, A), \tau)$ and $k \in \mathbb{N}$ such that (1) is satisfied. Its output is a packing of τ-respecting s-arborescences $T_1, \dots, T_k$ such that each vertex v in V belongs to $\min\{k, d^-_A(v)\}$ of them. For every $v \in V$, let $I(v)$ be a set of arcs of smallest τ-values entering v of size $\min\{k, d^-_A(v)\}$. The algorithm will use arcs only in $\bigcup_{v \in V} I(v)$. The algorithm heavily relies on the fact that the network is acyclic. It is well-known that a directed graph D is acyclic if and only if a *topological ordering* $v_1, \dots, v_n$ of its vertex set exists, that is if $v_i v_j$ is an arc of D then $i > j$. Since no arc enters s, we may suppose that in a topological ordering $v_n = s$.

Algorithm "PACKING TIME-RESPECTING ARBORESCENCES"

Let $\boldsymbol{v_n} = s, \ldots, \boldsymbol{v_1}$ be a topological ordering of $V \cup s$.
Let $\boldsymbol{A_i} = \emptyset$ for all $1 \le i \le k$.
For $j = 1$ to $n-1$, let
  $\boldsymbol{I} = \{1 \le i \le k : \delta_{A_i}(v_j) \ne \emptyset\}$,
  $\boldsymbol{a_i}$ be an arc in $\delta_{A_i}(v_j)$ of minimum τ-value for all $i \in I$,
  $\{\bar{\boldsymbol{a}}_{\boldsymbol{1}}, \ldots, \bar{\boldsymbol{a}}_{\boldsymbol{|I|}}\}$ be an ordering of $\{a_i : i \in I\}$ such that $\tau(\bar{a}_1) \le \cdots \le \tau(\bar{a}_{|I|})$,
  $\boldsymbol{\pi} : I \to \{1, \ldots, |I|\}$ be the bijection such that $a_i = \bar{a}_{\pi(i)}$ for all $i \in I$,
  $\boldsymbol{J}$ be a subset of $\{1, \ldots, k\} \setminus I$ of size $|I(v_j)| - |I|$,
  $\boldsymbol{\sigma} : J \to \{1, \ldots, |J|\}$ be a bijection,
  $\{\boldsymbol{e_1, \ldots, e_{|I|}, f_1, \ldots, f_{|J|}}\}$ be an ordering of $I(v_j)$ such that
    $\tau(e_1) \le \cdots \le \tau(e_{|I|}) \le \tau(f_1) \le \cdots \le \tau(f_{|J|})$,
  $\boldsymbol{A_i} = A_i \cup e_{\pi(i)}$ for all $i \in I$,
  $\boldsymbol{A_i} = A_i \cup f_{\sigma(i)}$ for all $i \in J$.
Let $\boldsymbol{T_i} = (V_i, A_i)$ where $\boldsymbol{V_i}$ is the vertex set of the arc set A_i for all $1 \le i \le k$.
Stop.

Theorem 3. *Given an acyclic temporal network $N = ((V \cup s, A), \tau)$ and $k \in \mathbb{N}$ such that (1) is satisfied, Algorithm* PACKING TIME-RESPECTING ARBORESCENCES *outputs a packing of k τ-respecting s-arborescences such that each vertex v in V belongs to $\min\{k, d_A^-(v)\}$ of them.*

Proof. For all $1 \le j \le n-1$, in the jth iteration of the algorithm, by the definition of I, (1) and the definition of $I(v_j)$, we have $|I| \le \min\{k, d_A^+(v_j)\} \le \min\{k, d_A^-(v_j)\} = |I(v_j)|$. This implies that J exists. By construction, the digraphs $T_1, \ldots, T_k$ are pairwise arc-disjoint and the in-degree of each vertex $v_j \in V_i - s$ is 1 in T_i. Then, since N is acyclic, T_i is an s-arborescence for all $1 \le i \le k$. Moreover, $|\{1 \le i \le k : v_j \in V_i\}| = |I| + |J| = |I(v_j)| = \min\{k, d_A^-(v_j)\}$ for all $1 \le j \le n-1$. To see that T_i is time-respecting for all $1 \le i \le k$, let v_j be a vertex in $V_i - s$ and $a \in \delta_{A_i}(v_j)$. Then $e_{\pi(i)} \in \rho_{A_i}(v_j)$. Suppose on the contrary that $\tau(e_{\pi(i)}) > \tau(a)$. Since $\tau(g) \ge \tau(e_{\pi(i)}) > \tau(a)$ for all $g \in \rho_A(v_j) \setminus \{e_1, \ldots, e_{\pi(i)-1}\}$, we have $|\rho_N^{\tau(a)}(v_j)| \le |\{e_1, \ldots, e_{\pi(i)-1}\}| = \pi(i) - 1$. Since $\tau(a) \ge \tau(a_i) = \tau(\bar{a}_{\pi(i)}) \ge \tau(\bar{a}_\ell)$ for all $1 \le \ell \le \pi(i)$ and $\pi(i) \le |I| \le k$, we have $\pi(i) = |\{\bar{a}_1, \ldots, \bar{a}_{\pi(i)}\}| \le \min\{|\delta_N^{\tau(a)}(v_j)|, k\}$. Thus $|\rho_N^{\tau(a)}(v_j)| < \min\{|\delta_N^{\tau(a)}(v_j)|, k\}$ that contradicts (1). This contradiction completes the proof. ■

Note that Theorem 3 implies Theorem 2. Note also that Theorem 2 implies Theorem 1. Indeed, if N is pre-flow, then (1) is satisfied, so, by Theorem 2, there exists a packing of k τ-respecting s-arborescences such that each vertex v in V belongs to exactly $\min\{k, d_A^-(v)\}$ of them. This implies that $\min\{k, \lambda_N(s, v)\} = \min\{k, d_A^-(v)\}$ and hence Theorem 1 follows.

4. Packing time-respecting arborescences in non-acyclic pre-flow temporal networks

In [2], Kamiyama and Kawase provide an example of 7 vertices and 12 arcs that shows that in Theorem 1 one cannot delete the condition that D is acyclic. Here we provide a smaller example with 5 vertices and 7 arcs, see the first temporal network in Fig. 1. Note that this temporal network contains a directed cycle whose arcs are not of the same τ-values and hence the temporal network is not consistent.

The second temporal network in Fig. 1 is another example that shows that in Theorem 1 one cannot delete the condition that D is acyclic. Here the temporal network contains one directed cycle C and all the arcs of C are of the same τ-values and hence the temporal network is consistent. Note that in this example there exists a packing of three τ-respecting s-arborescences such that each vertex v belongs to exactly $\lambda_N(s,v)$ of them.

Kamiyama and Kawase [2] also provide an example of 7 vertices and 12 arcs that shows that in Theorem 1 one cannot delete the condition that D is pre-flow. Here we provide a smaller example with 5 vertices and 8 arcs, see the third temporal network in Fig. 1.

We now present the main result of this paper on packing of time-respecting arborescences in consistent pre-flow temporal networks where only the natural upper bound is given on the number of arborescences.

Theorem 4. *Let $N = (D = (V \cup s, A), \tau)$ be a consistent pre-flow temporal network. There exists a packing of $d^+_D(s)$ τ-respecting s-arborescences, each vertex v in V belonging to $\lambda_N(s,v)$ of them.*

To prove Theorem 4, we need an easy observation on almost Eulerian acyclic pre-flow temporal networks. A similar result has already been presented in [2].

Proposition 1. *If $N = (D = (V \cup \{s,t\}, A), \tau)$ is an almost Eulerian acyclic temporal network and μ_v is a τ-respecting bijection from $\delta_D(v)$ to $\rho_D(v)$ for all $v \in V$, then D decomposes into $d^+_D(s)$ τ-respecting (s,t)-paths such that each vertex $v \in V$ belongs to $d^-_D(v)$ of them.*

Proof. We prove the claim by induction on $d^+_D(s)$. If $d^+_D(s) = 0$, then, since D is almost Eulerian and acyclic, we have $d^-_D(v) = 0$ for all $v \in V$ and we are done. Otherwise, there exists an arc leaving s. Then, using the bijections μ_v^{-1} and the facts that D is acyclic and μ_v is a τ-respecting, we find a τ-respecting directed (s,t)-path P. By deleting the arcs of P and applying the induction, the claim follows. ■

We also need the following result of Bang-Jensen, Frank, Jackson [4].

Theorem 5 ([4]). *Let $D = (V \cup s, A)$ be a pre-flow directed graph. There exists a packing of s-arborescences, each vertex $v \in V$ belonging to $\lambda_D(s,v)$ of them.*

We are ready to prove our main result.

Proof of Theorem 4. First we transform the instance into another one $N' = (D', \tau')$ as follows. The directed graph $D' = (V \cup \{s,t\}, A \cup A')$ is obtained from D by adding a new vertex t and $d^-_D(v) - d^+_D(v)$ parallel arcs from v to t for all $v \in V$ and we define $\tau'(a)$ to be equal to $\tau(a)$ if $a \in A$ and to M if $a \in A'$, where $M = \max\{\tau(a) : a \in A\}$. Since N is pre-flow, so is D, that is $d^-_D(v) - d^+_D(v) \geq 0$ for all $v \in V$ and hence the construction is correct. This way we get an instance which remains consistent ($\{t\}$ is a new strongly connected component) and pre-flow (by the definition of M) and D' is almost Eulerian.

For each vertex $v \in V$, let us fix orderings of $\rho_{D'}(v)$ and $\delta_{D'}(v)$ such that $\tau'(e_1) \leq \cdots \leq \tau'(e_{d^-_{D'}(v)})$ and $\tau'(f_1) \leq \cdots \leq \tau'(f_{d^+_{D'}(v)})$, respectively. Then $\mu'_v(f_j) = e_j$ for all $1 \leq j \leq d^+_{D'}(v)$ is a τ'-respecting bijection for all $v \in V$. Indeed, if there exists j such that $\tau'(e_j) = \tau'(\mu'_v(f_j)) > \tau'(f_j) =: i$, then $|\rho^i_{N'}(v)| \leq j-1 < j \leq |\delta^i_{N'}(v)|$ that contradicts the fact that N' is pre-flow.

To reduce the problem to an easy acyclic problem that can be treated by Proposition 1 and some problems that can be treated by Theorem 5, let us denote the strongly connected components of D' by $Q'_1, \ldots, Q'_\ell$. Let U_j denote the vertex set of Q'_j for all $1 \leq j \leq \ell$. Then the directed graph D'' obtained from D' by contracting each Q'_j into a vertex q''_j is acyclic. By changing the indices if it is necessary, we may suppose that $q''_\ell = s, \ldots, q''_1 = t$ is a topological ordering of the vertices of D''. Let $N'' = (D'', \tau'')$ be the temporal network where $\tau''(a) = \tau'(a)$ for all $a \in A(D'')$. Note that since D' is almost Eulerian, so is D''. Indeed, we have $d^-_{D''}(q''_j) - d^+_{D''}(q''_j) = d^-_{D'}(U_j) - d^+_{D'}(U_j) = \sum_{v \in U_j}(d^-_{D'}(v) - d^+_{D'}(v)) = 0$ for all $2 \leq j \leq \ell - 1$. Note also that $d^+_D(s) = d^+_{D'}(s) = d^+_{D''}(s)$.

To define a convenient τ''-respecting bijection μ''_j from $\delta_{D''}(q''_j) = \delta_{D'}(U_j)$ to $\rho_{D''}(q''_j) = \rho_{D'}(U_j)$ for all $2 \le j \le \ell - 1$, let us fix such a j and let us define the following sets:
$R_j^1 = \{vw \in \delta_{D'}(U_j) : \tau'(\mu'_v(vw)) > \tau'(Q'_j)\}$,
$R_j^2 = \{vw \in \delta_{D'}(U_j) : \tau'(vw) < \tau'(Q'_j)\}$,
$R_j^3 = \delta_{D'}(U_j) \setminus (R_j^1 \cup R_j^2)$,
$S_j^1 = \{\mu'_v(vw) : vw \in R_j^1\}$,
$S_j^2 = \{\mu'_v(vw) : vw \in R_j^2\}$ and
$S_j^3 = \rho_{D'}(U_j) \setminus (S_j^1 \cup S_j^2)$.

Claim 1. *$\{R_j^1, R_j^2, R_j^3\}$ is a partition of $\delta_{D'}(U_j)$ and $\{S_j^1, S_j^2, S_j^3\}$ is a partition of $\rho_{D'}(U_j)$.*

Proof. If $vw \in R_j^1$, $v'w' \in R_j^2$, $uv = \mu'_v(vw) \in S_j^1$ and $u'v' = \mu'_{v'}(v'w') \in S_j^2$, then, since μ'_v and $\mu'_{v'}$ are τ'-respecting bijections, we have $\tau'(vw) \ge \tau'(\mu'_v(vw)) = \tau'(uv) > \tau'(Q'_j) > \tau'(v'w') \ge \tau'(\mu'_v(v'w')) = \tau'(u'v')$. Thus $vw \ne v'w'$ and $uv \ne u'v'$, so $R_j^1 \cap R_j^2 = \emptyset$ and $S_j^1 \cap S_j^2 = \emptyset$. By the definition of R_j^1 and R_j^2, we have $R_j^1 \cup R_j^2 \subseteq \delta_{D'}(U_j)$. If $vw \in R_j^1$, then $\tau'(\mu'_v(vw)) > \tau'(Q'_j)$. If $vw \in R_j^2$, then, since μ'_v is a τ'-respecting bijection, we get $\tau'(\mu'_v(vw)) \le \tau'(vw) < \tau'(Q'_j)$. Then, using that each arc in Q'_j has τ'-value $\tau'(Q'_j)$, we have $S_j^1 \cup S_j^2 \subseteq \rho_{D'}(U_j)$. By the definition of R_j^3 and S_j^3, Claim 1 follows. ■

We now start to define μ''_j. For $vw \in R_j^1 \cup R_j^2$, let $\mu''_j(vw) = \mu'_v(vw)$. Since each μ'_v is τ'-respecting, we have $\tau''(vw) = \tau'(vw) \ge \tau'(\mu'_v(vw)) = \tau''(\mu''_v(vw))$. Note that for all $xy \in R_j^3$ and for all $uv \in S_j^3$, $\tau'(xy) \ge \tau'(Q'_j) \ge \tau'(uv)$. However, we cannot take an arbitrary bijection from R_j^3 to S_j^3 because we have to guarantee that the vertices in Q'_j also belong to the required number of arborescences. In order to do this, let us define the temporal network $N'_j = (D'_j, \tau'_j)$ where the directed graph D'_j is obtained from D' by contracting $\bigcup_{i>j} U_i$ into a vertex s_j, contracting $\bigcup_{i<j} U_i$ into a vertex t_j and deleting the arcs from s_j to t_j and $\tau'_j(a) = \tau'(a)$ for all $a \in A(D'_j)$.

Claim 2. *N'_j satisfies the following.*

(a) D'_j is almost Eulerian,
(b) $\lambda_{D'_j}(s_j, t_j) = d^-_{D'_j}(t_j)$,
(c) $\lambda_{N'_j}(s_j, v) \ge \lambda_{N'}(s, v)$ for all $v \in U_j$.

Proof. (a) Since D' is almost Eulerian, so is D'_j. Indeed, we have $d^-_{D'_j}(v) = d^-_{D'}(v) = d^+_{D'}(v) = d^+_{D'_j}(v)$ for all $v \in U_j$.

(b) By (a) and $d^-_{D'_j}(s_j) = 0 = d^+_{D'_j}(t_j)$, (b) easily follows. Indeed, let $r_j = d^-_{D'_j}(t_j)$ and let us define D^*_j by adding r_j arcs $\{h_1, \dots, h_{r_j}\}$ from t_j to s_j in D'_j. Then, by (a), D^*_j is Eulerian. Thus it decomposes into directed cycles. Let $C_1, \dots, C_{r_j}$ be the arc-disjoint directed cycles that contain the arcs $h_1, \dots, h_{r_j}$. Then $P_1 = C_1 - h_1, \dots, P_{r_j} = C_{r_j} - h_{r_j}$ are arc-disjoint directed (s_j, t_j)-paths. Hence $r_j \le \lambda_{D'_j}(s_j, t_j) \le r_j$, and we have (b).

(c) For all $v \in U_j$, any τ'-respecting (s, v)-path in N' provides a τ'_j-respecting (s_j, v)-path in N'_j, and (c) follows. ■

To be able to use normal arborescences (not time-respecting ones), we have to modify D'_j. No τ-respecting directed path in D may contain an arc in S_j^1 and an arc in Q'_j, hence the corresponding arcs in R_j^1 and S_j^1 will be deleted from D'_j. A τ-respecting s-arborescence in D may contain an arc $\mu'_v(vw)$ in S_j^2 (where $vw \in R_j^2$) and an arc in Q'_j, but this arborescence must contain vw. To guarantee this property we use a trick: we replace the corresponding two arcs in R_j^2 and S_j^2 in D'_j by two convenient arcs. More precisely, let H_j be obtained from D'_j by deleting s_jv and vt_j that correspond to $\mu'_v(vw)$ and vw for all $vw \in R^1_j$ and replacing s_jv and vt_j that correspond to $\mu'_v(vw)$ and vw for all $vw \in R^2_j$ by $e_{vw} = s_jt_j$ and $f_{vw} = t_jv$. Let $E_j = \{e_{vw} : vw \in R^2_j\}$ and $F_j = \{f_{vw} : vw \in R^2_j\}$.

Claim 3. *H_j satisfies the following.*

(a) H_j is pre-flow,
(b) $\lambda_{H_j}(s_j, t_j) = d^-_{H_j}(t_j)$,
(c) $\lambda_{H_j}(s_j, v) \geq \lambda_{N'_j}(s_j, v) - d^-_{S^1_j}(v)$ for all $v \in U_j$.

Proof. (a) By Claim 2(a), D'_j is almost Eulerian. Then, by $\delta_{D'_j}(t_j) = \emptyset$, D'_j is pre-flow. By deleting from D'_j the arcs s_jv and vt_j that correspond to $\mu'_v(vw)$ and vw for all $vw \in R^1_j$, we decreased the in-degree and the out-degree of each vertex by the same value so the directed graph obtained this way remained pre-flow. By replacing s_jv and vt_j that correspond to $\mu'_v(vw)$ and vw for all $vw \in R^2_j$ by s_jt_j and t_jv, we may decrease the out-degrees of the vertices in Q'_j but the in-degrees remained unchanged. Further, $d^+_{H_j}(t_j) = d^+_{D'_j}(t_j) + |F_j| = |E_j| \leq d^-_{H_j}(t_j)$. It follows that H_j is pre-flow.

(b) Note that for all $t_j \in X \subseteq U_j \cup t_j$, $d^-_{H_j}(X) = d^-_{D'_j}(X) - |R^1_j|$. Then, by Claim 2(b), we have $d^-_{H_j}(t_j) \geq \lambda_{H_j}(s_j, t_j) \geq \lambda_{D'_j}(s_j, t_j) - |R^1_j| = d^-_{D'_j}(t_j) - |R^1_j| = d^-_{H_j}(t_j)$ and (b) follows.

(c) On the one hand, by deleting the arcs corresponding to $\rho_{S^1_j}(v)$, we destroyed at most $d^-_{S^1_j}(v)$ τ'_j-respecting (s_j, v)-paths in N'_j and we did not destroy a τ'_j-respecting (s_j, u)-path in N'_j for $u \in U_j \setminus v$ because each arc in Q'_j has τ'_j-value $\tau'_j(Q'_j)$ and each arc in $\rho_{S^1_j}(v)$ has τ'_j-value strictly larger than $\tau'_j(Q'_j)$. On the other hand, if a τ'_j-respecting (s_j, u)-path P contains s_jv (corresponding to $\mu'_v(vw)$ for some $vw \in R^2_j$) in N'_j then $P - s_jv + e_{vw} + f_{vw}$ is a directed (s_j, u)-path in H_j. These arguments imply (c). ■

By Claim 3(a) and Theorem 5, there exists a packing $\mathcal{B}_j$ of s_j-arborescences T^i_j in H_j, each vertex $v \in U_j \cup t_j$ belonging to $\lambda_{H_j}(s_j, v)$ of them. Let us choose such a packing $\mathcal{B}_j$ that minimizes the size of the set $F_{\mathcal{B}_j}$ of the arcs $f_{vw} \in F_j$ such that an arborescence $T^{f_{vw}}_j$ in $\mathcal{B}_j$ contains f_{vw} but not e_{vw}.

Claim 4. *$\mathcal{B}_j$ satisfies the following.*

(a) $d^+_{H_j}(s_j) = |\mathcal{B}_j| = d^-_{H_j}(t_j)$,
(b) $F_{\mathcal{B}_j} = \emptyset$,
(c) $\{T^i_j - s_j - t_j : T^i_j \in \mathcal{B}_j\}$ is a packing of arborescences in Q'_j, each vertex $v \in U_j$ belonging to $\lambda_{H_j}(s_j, v)$ of them.

Proof. (a) By Claim 3(b), t_j belongs to $\lambda_{H_j}(s_j, t_j) = d^-_{H_j}(t_j)$ of the s_j-arborescences in $\mathcal{B}_j$. Thus each arc entering t_j belongs to some s_j-arborescence in $\mathcal{B}_j$ and $d^-_{H_j}(t_j) \leq |\mathcal{B}_j|$. Moreover, by construction and since D'_j is almost Eulerian, we have $d^-_{H_j}(t_j) = d^-_{D'_j}(t_j) - |R^1_j| = d^+_{D'_j}(s_j) - |S^1_j| = d^+_{H_j}(s_j) \geq |\mathcal{B}_j|$, and (a) follows.

(b) Suppose that $F_{\mathcal{B}_j} \neq \emptyset$. Let $E_{\mathcal{B}_j} = \{e_{vw} : f_{vw} \in F_{\mathcal{B}_j}\}$. By (a), every $e_{vw} \in E_{\mathcal{B}_j}$ is contained in an s_j-arborescence $T^{e_{vw}}_j$ in $\mathcal{B}_j$.

First suppose that for some $e_{vw} \in E_{\mathcal{B}_j}$, $T^{e_{vw}}_j$ contains only the arc e_{vw}. Note that $T^{f_{vw}}_j - f_{vw}$ consists of an s_j-arborescence T'_j and a v-arborescence T''_j. Let $\mathcal{B}'_j$ be obtained from $\mathcal{B}_j$ by replacing $T^{f_{vw}}_j$ by T'_j and $T^{e_{vw}}_j$ by $e_{vw} + f_{vw} + T''_j$. Then $\mathcal{B}'_j$ is a packing of s_j-arborescences in H_j such that each vertex $v \in U_j \cup t_j$ belongs to $\lambda_{H_j}(s_j, v)$ of them. Moreover, f_{vw} and e_{vw} belong to the same s_j-arborescence in $\mathcal{B}'_j$, that is $|F_{\mathcal{B}'_j}| < |F_{\mathcal{B}_j}|$ and we have a contradiction.

We may hence suppose that for every $e_{vw} \in E_{\mathcal{B}_j}$, $T^{e_{vw}}_j$ contains another arc, so by (a), contains an arc in $F_{\mathcal{B}_j}$. Let $\mathcal{B}'_j$ be the set of those s_j-arborescences in $\mathcal{B}_j$ that contain an arc of $F_{\mathcal{B}_j}$. Then $|F_{\mathcal{B}_j}| = |E_{\mathcal{B}_j}|$

$\leq |\mathcal{B}'_j| \leq |F_{\mathcal{B}_j}|$. Hence we have equality everywhere. It follows that every s_j-arborescences in $\mathcal{B}'_j$ contains exactly one arc from both $F_{\mathcal{B}_j}$ and $E_{\mathcal{B}_j}$. Then for every $f_{vw} \in F_{\mathcal{B}_j}$, $T_j^{f_{vw}}$ contains an arc $e_{v'w'} \in E_{\mathcal{B}_j}$. Let $\mathcal{B}''_j$ be obtained from $\mathcal{B}_j$ by replacing $e_{v'w'}$ by $e_{vw} \in E_{\mathcal{B}_j}$ in $T_j^{f_{vw}}$ for every $f_{vw} \in F_{\mathcal{B}_j}$. Then $\mathcal{B}''_j$ is a packing of s_j-arborescences in H_j such that each vertex $v \in U_j \cup t_j$ belongs to $\lambda_{H_j}(s_j, v)$ of them. Moreover, $F_{\mathcal{B}''_j} = \emptyset$ and we have a contradiction.

(c) follows from the definition of $\mathcal{B}_j$, (a) and (b). ■

We now finish the definition of μ''_j. Let $vw \in R_j^3$. Then vw corresponds in H_j to an arc $g_{vw} = vt_j$ entering t_j. By Claim 4(a), g_{vw} belongs to an s_j-arborescence $T_j^{g_{vw}}$ in $\mathcal{B}_j$. Let us define $\mu''_j(vw) \in S_j^3$ to be the arc xq''_j of D'' that corresponds to the arc $s_j u$ in H_j of the unique (s_j, t_j)-path of $T_j^{g_{vw}}$. Then $\tau''_j(vw) = \tau'_j(vw) \geq \tau'_j(Q'_j) \geq \tau'_j(xq''_j) = \tau''_j(\mu''_j(vw))$ for all $vw \in R_j^3$.

By the definition of μ''_j and Claim 1, we have a τ''-respecting bijection μ''_j from $\delta_{D''}(q''_j)$ to $\rho_{D''}(q''_j)$ for all $2 \leq j \leq \ell-1$. Recall that D'' is acyclic and almost Eulerian. Then, by Proposition 1 and $d_D^+(s) = d_{D''}^+(s)$, D'' decomposes into τ''-respecting (s,t)-paths $P_1, \ldots, P_{d_D^+(s)}$ such that each vertex $q''_j \neq s$ belongs to $d_{D''}^-(q''_j)$ of them. These paths can be extended, using from Claim 4(c) the arborescences $T_j^i - s_j - t_j$ in Q'_j for $1 \leq i \leq d_{H_j}^+(s_j)$ and $2 \leq j \leq \ell - 1$, to get s-arborescences in D' such that each vertex $v \in V$ belongs to $\lambda_{H_j}(s_j, v) + d_{S_j^1}^-(v) \geq \lambda_{N'_j}(s_j, v) \geq \lambda_{N'}(s, v)$ of them, by Claims 3(b) and 2(c). Since the directed paths $P_1, \ldots, P_{d_D^+(s)}$ are τ''-respecting, that is τ'-respecting and D' is consistent, the arborescences constructed are τ'-respecting. Hence N' has a packing of τ'-respecting s-arborescences $T'_1, \ldots, T'_{d_D^+(s)}$ such that each vertex v of D' distinct from s and t belongs to $\lambda_{N'}(s, v) = \lambda_N(s, v)$ of them, and hence $\{T_1 = T'_1 - t, \ldots, T_{d_D^+(s)} = T'_{d_D^+(s)} - t\}$ is a packing of τ-respecting s-arborescences such that each vertex v of D distinct from s belongs to $\lambda_N(s, v)$ of them. ■

5. Arc-disjoint spanning time-respecting arborescences

Edmonds' arborescence packing theorem [3] states that k-root-connectivity from s implies the existence of a packing of k spanning s-arborescences. The following observation of [1] shows that the natural extension of Edmonds theorem for $k = 1$ is true for temporal networks.

Theorem 6 *([1]). Any τ-respecting root-connected temporal network $N = ((V \cup s, A), \tau)$ contains a spanning τ-respecting s-arborescence.*

The authors of [1] show that high time-respecting root-connectivity of a temporal network does not imply the existence of 2 arc-disjoint spanning time-respecting arborescences.

Theorem 7 *([1]). For all $k \in \mathbb{N}^+$, there exist temporal networks $N = ((V \cup s, A), \tau)$ such that $\lambda_N(s, v) \geq k$ for all $v \in V$ and no packing of 2 spanning τ-respecting s-arborescences exists in N.*

Their construction contains directed cycles but it can be easily modified to get an acyclic example. This acyclic example for $k = 2$ is presented in Figure 2 in [2].

We now relate the spanning time-respecting arborescence packing problem to known problems, namely the Steiner arborescence packing problem and the hypergraph proper 2-coloring problem. To do that we explain how the above mentioned modified construction can be obtained in 3 steps. First, take a k-uniform hypergraph without proper 2-coloring. Then construct a directed graph that is Steiner k-root-connected without 2 arc-disjoint Steiner arborescences. Finally, construct an acyclic temporal network that is time-respecting k-root-connected without 2 arc-disjoint spanning time-respecting arborescences.

There exist many constructions for k-uniform hypergraphs without proper 2-coloring, see [1,5] and Exercise 13.45(b) of [6]. We mention that, by a result of Erdős [7], all examples contain exponentially many hyperedges in k.

Theorem 8 ([7]). *Any k-uniform hypergraph without a proper 2-coloring contains at least 2^{k-1} hyperedges.*

We now show that starting from an arbitrary k-uniform hypergraph $\mathcal{H}_k = (V_k, \mathcal{E}_k)$ without proper 2-coloring how to construct an acyclic directed graph D_k and a vertex set U_k such that $\lambda_{D_k}(s,u) = k$ for all $u \in U_k$ and there exists no packing of two (s, U_k)-arborescences in D_k. Let $G_k := (V_k, U_k; E_k)$ be the bipartite incidence graph of the hypergraph $\mathcal{H}_k$, where the elements of U_k correspond to the hyperedges in $\mathcal{E}_k$. Let $D_k = (V_k \cup U_k \cup s, A_k)$ be obtained from G_k by adding a vertex s and an arc sv for all $v \in V_k$ and directing each edge of E_k from V_k to U_k. By construction D_k is acyclic. Since $\mathcal{H}_k$ is k-uniform, we have $\lambda_{D_k}(s,u) = k$ for all $u \in U_k$.

Theorem 9. *D_k has no packing of two (s, U_k)-arborescences.*

Proof. Suppose that there exists a packing of 2 (s, U_k)-arborescences F_1 and F_2 in D_k. Using this packing, we can define a 2-coloring of V_k: let $v \in V_k$ be colored by 1 if $sv \in A(F_1)$ and by 2 otherwise. Since each vertex in U_k belongs to both F_1 and F_2, no hyperedge of $\mathcal{E}_k$ is monochromatic, that is the above defined 2-coloring of $\mathcal{H}_k$ is proper. This contradicts the fact that $\mathcal{H}_k$ has no proper 2-coloring. ■

As a next step, we show that starting from the acyclic directed graph D_k and the vertex set U_k, how to construct a temporal network N_k such that $\lambda_{N_k}(s,v) = k$ for all vertices v and no packing of 2 spanning time-respecting s-arborescences exists in N. Let us define $N_k := (D^*_k, \tau^*_k)$ as follows: D^*_k is obtained from D_k by adding the set of arcs A^*_k consisting of $k-1$ parallel arcs from s to all $v \in V_k$ and we define $\tau^*_k(a) = 1$ if $a \in A_k$ and 2 if $a \in A^*_k$. Note that since D_k is acyclic, so is D^*_k. Then a spanning s-arborescence F^* of D^*_k is τ^*_k-respecting if and only if $F^* - A^*_k$ is an (s, U_k)-arborescence in D_k. Thus a packing of 2 spanning τ^*_k-respecting s-arborescences in D^*_k would provide a packing of 2 (s, U_k)-arborescences in D_k. Hence, the following result is an immediate consequence of Theorem 9.

Theorem 10. *For all $k \in \mathbb{N}^+$, there exist acyclic temporal networks $N = ((V \cup s, A), \tau)$ such that $\lambda_N(s,v) \geq k$ for all $v \in V$ and no packing of 2 spanning τ-respecting s-arborescences exists in N.*

These examples of acyclic temporal networks that are time-respecting k-root-connected without 2 arc-disjoint spanning time-respecting arborescences contain, by Theorem 8, exponentially many vertices in k. In other words, $k \leq log(n)$ where n is the number of vertices. In the light of this fact, it is natural to ask whether there exist 2 arc-disjoint spanning time-respecting arborescences in a temporal network if k is linear in n. The examples of Fig. 1 show that time-respecting $(n-3)$-root-connectivity does not imply the existence of 2 arc-disjoint spanning time-respecting arborescences. We propose the first steps in this direction. We first remark that n-root-connectivity is enough.

Claim 5. *Let $N = ((V \cup s, A), \tau)$ be a temporal network on $n \geq 1$ vertices such that $\lambda_N(s,v) \geq n$ for all $v \in V$. Then there exists a packing of 2 spanning τ-respecting s-arborescences in N.*

Proof. Since $\lambda_N(s,v) \geq n \geq 1$ for all $v \in V$, there exists, by Theorem 6, a spanning τ-respecting s-arborescence F in N. Further, there exist n arc-disjoint τ-respecting (s,v)-paths $P^v_1, \ldots, P^v_n$ for all $v \in V$. By deleting the arcs of F, we can destroy at most $|A(F)|$ of the (s,v)-paths $P^v_1, \ldots, P^v_n$ for all $v \in V$. Since $|A(F)| = n-1$, this implies that $\lambda_{N-A(F)}(s,v) \geq n-(n-1) = 1$ for all $v \in V$. Then, there exists, by Theorem 6, a spanning τ-respecting s-arborescence F' in $N - A(F)$, and we are done. ■

With some effort we can improve the previous result by 1.

Theorem 11. *Let $N = ((V \cup s, A), \tau)$ be a temporal network on $n \geq 2$ vertices such that $\lambda_N(s, v) \geq n - 1$ for all $v \in V$. Then there exists a packing of 2 spanning τ-respecting s-arborescences in N.*

Proof. Since $\lambda_N(s, v) \geq n - 1 \geq 1$ for all $v \in V$, there exists, by Theorem 6, a spanning τ-respecting s-arborescence F in N. Let $F(v)$ be the unique arc of F entering v for all $v \in V$. Note that $A(F) = \{F(v) : v \in V\}$. If $\lambda_{N-A(F)}(s, v) \geq 1$ for all $v \in V$ then there exists, by Theorem 6, a spanning τ-respecting s-arborescence in $N - A(F)$, and we are done.

Otherwise, $\lambda_{N-A(F)}(s, u) = 0$ for some $u \in V$. By assumption, there exist $n - 1$ arc-disjoint τ-respecting (s, u)-paths $P_1, \ldots, P_{n-1}$. Then, since $|V| = n - 1$, there exists a bijection π from V to $\{1, \ldots, n - 1\}$ such that $F(v)$ is contained in $P_{\pi(v)}$ for all $v \in V$. It follows that no arc leaves u in F. Let $w \in V - u$ be a vertex for which $\tau(F(w))$ is maximum. Let the last arc of $P_{\pi(w)}$ be denoted by xu. Then, since $F(u)$ is the last arc of the path $P_{\pi(u)}$ and the paths are arc-disjoint, $F(u) \neq xu$. By the choice of w and since $P_{\pi(w)}$ is τ-respecting, we have $\tau(F(x)) \leq \tau(F(w)) \leq \tau(xu)$. We obtain that $F' := F - F(u) + xu \neq F$ is also a spanning τ-respecting s-arborescence in N.

By assumption and $|A(F) - F(u)| = n - 2$, we have $\lambda_{N-(A(F)-F(u))}(s, v) \geq (n - 1) - (n - 2) = 1$ for all $v \in V$. Then, by Theorem 6, there exists a spanning τ-respecting s-arborescence F'' in $N - (A(F) - F(u))$. Since F'' contains a unique arc entering u, it does not contain either $F(u)$ or xu. Thus, F'' is arc-disjoint from either F or F', and we are done. ■

We conjecture that the following is true.

Conjecture 1. *Let $N = ((V \cup s, A), \tau)$ be an acyclic temporal network on $n \geq 4$ vertices such that $\lambda_N(s, v) \geq \frac{n}{2}$ for all $v \in V$. Then a packing of 2 spanning τ-respecting s-arborescences exists in N.*

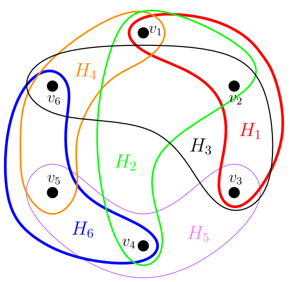
The third example presented in Fig. 1 is of 5 vertices, acyclic, time-respecting 2-root-connected and has no packing of 2 spanning τ-respecting s-arborescences. It follows that time-respecting $\frac{2n}{5}$-root-connectivity is not enough to have a packing of 2 spanning time-respecting s-arborescences in acyclic temporal networks.

6. Complexity results

Lovász [8] proved that the problem of 2-colorings of k-uniform hypergraphs is NP-complete. This implies that the problem of packing 2 Steiner arborescences is also NP-complete. An easier way to see this is to use the NP-complete problem of two arc-disjoint directed paths in a directed graph D, one from r to t and the other from t to r. (See [9].) Construct D' from D by adding a new vertex s and the two arcs sr and st. Then D has an (r, t)-path and a (t, r)-path that are arc-disjoint if and only if D' has a packing of 2 $(s, \{r, t\})$-arborescences. This with the construction presented in the previous section finally imply the following.

Theorem 12. *The problem of packing k spanning time-respecting arborescences is NP-complete even for $k = 2$.*

Let us check what happens if we replace the inequality with equality in the definition of time-respecting directed paths and we consider the values of τ as colors. Then we get monochromatic directed paths. We may hence study the following problem MoChPaSpAr:



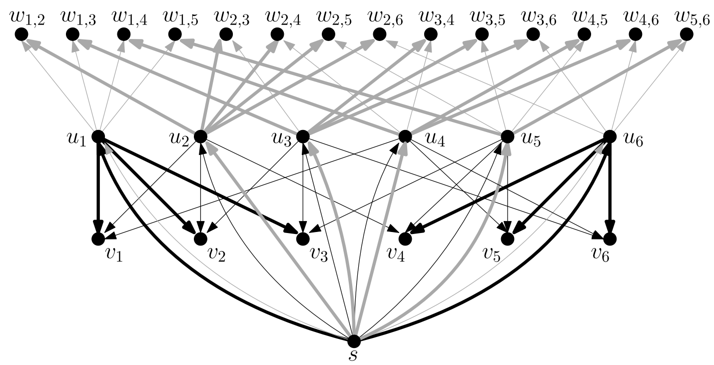


Fig. 2. A 3-regular 3-uniform hypergraph and the constructed colored directed graph for it.

Problem 1. Given a directed graph $D = (Z \cup s, A)$ and a coloring c of the arcs, decide whether there exists a spanning s-arborescence containing only monochromatic directed paths.

We show that this decision problem is difficult. We will reduce the exact cover in 3-regular 3-uniform hypergraphs problem (RXC3) to our problem. In RXC3, we are given a 3-regular 3-uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$, and the problem consists of determining whether there exists a subset $\mathcal{E}'$ of $\mathcal{E}$ such that each vertex in V occurs in exactly one hyperedge in $\mathcal{E}'$. Gonzalez proved in [10] that RXC3 is NP-complete.

Theorem 13. *The problem* MoChPaSpAr *is NP-complete even for acyclic directed graphs and for two colors.*

Proof. It is clear that MoChPaSpAr is in NP. Let us take an instance of RXC3, that is let $\mathcal{H}$ be a 3-regular 3-uniform hypergraph. We construct a polynomial size instance (D, c) of MoChPaSpAr such that $\mathcal{H}$ has an exact cover if and only if (D, c) has a spanning s-arborescence containing only monochromatic directed paths. Since $\mathcal{H}$ is a 3-regular 3-uniform hypergraph, the number of vertices of $\mathcal{H}$ and the number of hyperedges of $\mathcal{H}$ coincide. Let us denote the vertices of $\mathcal{H}$ by $V = \{v_1, \ldots, v_h\}$ and the hyperedges of $\mathcal{H}$ by $\mathcal{E} = \{H_1, \ldots, H_h\}$.

Let $D = (Z \cup s, A)$ be the directed graph where $Z = U \cup V \cup W$ and $A = A_1 \cup A_2 \cup A_3 \cup A_4$ with $U = \{u_1, \ldots, u_h\}$, $W = \{w_{i,j} : H_i \cap H_j \neq \emptyset\}$, $A_1 = \{e_i^1 = su_i : 1 \le i \le h\}$, $A_2 = \{e_i^2 = su_i : 1 \le i \le h\}$, $A_3 = \{u_i v_j : u_i \in U, v_j \in V, v_j \in H_i\}$ and $A_4 = \{u_i w_{i,j}, u_j w_{i,j} : u_i, u_j \in U, w_{i,j} \in W\}$. Let $c(a)$ be equal to black if $a \in A_1 \cup A_3$ and gray if $a \in A_2 \cup A_4$. Note that D is acyclic and c uses only two colors. For an example see Fig. 2.

The size of D is polynomial in h. Indeed, since $\mathcal{H}$ is a 3-regular 3-uniform hypergraph, $|W| \le \frac{1}{2} \cdot 3 \cdot 2 \cdot h$, so $|Z \cup s| = |U| + |V| + |W| + 1 \le h + h + 3h + 1 = 5h + 1$ and $|A| = |A_1| + |A_2| + |A_3| + |A_4| \le h + h + 3h + 2 \cdot 3h = 11h$.

Suppose first that $\mathcal{H}$ has an exact cover $\mathcal{H}'$. Let Z' be the set of vertices of D that can be reached from s by a black directed path starting with an arc su_i with $H_i \in \mathcal{H}'$ and Z'' by a gray directed path starting with an arc su_i with $H_i \notin \mathcal{H}'$. Since $\mathcal{H}'$ is a cover, we have $Z' = V \cup \{u_i : H_i \in \mathcal{H}'\}$. Since the hyperedges in $\mathcal{H}'$ are disjoint, we have $Z'' = \{u_i : H_i \notin \mathcal{H}'\} \cup W$. Since $Z' \cap Z'' = s$, the desired spanning s-arborescence containing only monochromatic directed paths exists. In the example of Fig. 2, $\mathcal{H}' = \{H_1, H_6\}$, $Z' = V \cup \{u_1, u_6\}$, $Z'' = \{u_2, u_3, u_4, u_5\} \cup W$ and the arborescence is represented by bold arcs.

Now suppose that (D, c) has a spanning s-arborescence F containing only monochromatic directed paths. Let $\mathcal{H}' = \{H_j : u_j \in U, v_i \in V, u_j v_i \in F\}$. Since F is a spanning s-arborescence, each vertex v_i has exactly one black arc $u_j v_i$ in F entering. This implies that $\mathcal{H}'$ covers V. Let H_j, H_k $(j < k)$ be hyperedges in $\mathcal{H}'$. If $w_{j,k} \in W$, then, since the directed paths are monochromatic in F, su_j and su_k are black and hence $u_j w_{j,k}$

and $u_k w_{j,k}$ are not contained in F that contradicts the fact that F is a spanning s-arborescence. Thus H_j and H_k are disjoint. It follows that $\mathcal{H}'$ is an exact cover. ■

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