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On packing arborescences in temporal networks

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ABSTRACT

A temporal network is a directed graph in which each arc has a time label specifying the time at which its end vertices communicate. An arborescence in a temporal network is said to be time-respecting, if the time labels on every directed path from the root in this arborescence are monotonically non-decreasing. In this paper, we consider a characterization of the existence of arc-disjoint time-respecting arborescences in temporal networks.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers. For each directed graph D, we denote by $V(D)$ and $A(D)$ the sets of vertices and arcs of D, respectively. Furthermore, for each directed graph D and each vertex v of D, let $\delta_D(v)$ and $\varrho_D(v)$ be the sets of arcs of D leaving and entering v, respectively. We denote by $a=(u,v)$ an arc a from u to v.

A *temporal network* N is a pair (D,τ) of a directed graph D and a time label function $\tau\colon A(D)\to\mathbb{N}$. For each arc a of D, the time label $\tau(a)$ specifies the time at which its end vertices communicate. This model is used for modeling communication in distributed networks and scheduled transportation networks. See [1] for applications of temporal networks. If we communicate along a directed path P in a temporal network, then the time labels of the arcs of P must be monotonically non-decreasing. Formally speaking, a directed path P in a temporal network $N=(D,\tau)$ is said to be *time-respecting*, if

$$\tau(a_1)\le\tau(a_2)\le\cdots\le\tau(a_k),$$

where we assume that P passes through arcs $a_1,a_2,\dots,a_k$ in this order.

Besides a directed path, an arborescences is another important concept in a directed graph from not only a theoretical point of view but also a practical point of view. Formally speaking, a subgraph T of a directed graph D with a specified vertex r is called an *r-arborescence* or an *arborescence rooted at r*, if

1. $V(T)=V(D)$,
2. there exists a directed path in T from r to every vertex v of D, and
3. for each vertex v of D,

$$|\varrho_T(v)|=\begin{cases}0 & \text{if } v=r\\ 1 & \text{otherwise.}\end{cases}$$

It is not difficult to see that an r-arborescence is a spanning tree in D whose arcs are directed away from r.

Assume that we are given a temporal network $N=(D,\tau)$ with a specified vertex r. For each r-arborescence

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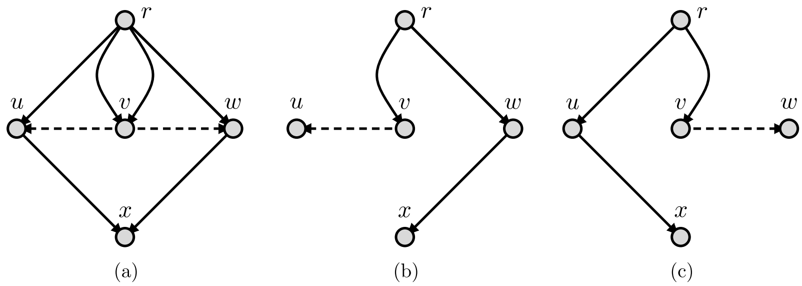

Fig. 1. (a) An example of a temporal network. A time label of each arc illustrated by a real line is equal to 1. A time label of each arc illustrated by a broken line is equal to 2. (b, c) Two arc-disjoint time-respecting r-arborescences in the temporal network illustrated in (a).

T in N and each vertex v of D with $v \neq r$, T is said to be *time-respecting on* v, if

$$\forall a \in \delta_T(v) \colon \tau\big(\mathrm{in}(v)\big) \leq \tau(a), \tag{1}$$

where $\mathrm{in}(v)$ represents the unique arc in $\varrho_T(v)$. Furthermore, an r-arborescence T in N is said to be *time-respecting*, if T is time-respecting on every vertex v of D with $v \neq r$. It is not difficult to see that an r-arborescence T in N is time-respecting if and only if for every vertex v of T, the unique directed path from r to v in T is time-respecting.

In this paper, we consider a characterization of the existence of arc-disjoint time-respecting arborescences rooted at a specified vertex in a temporal network (see Fig. 1). Packing problems are very important problems in graph theory and combinatorial optimization. Furthermore, it is practically natural to think that a network in which there exist many arc-disjoint arborescences has high robustness against troubles.

2. Problem formulation

For defining our problem, we first consider the case where the time label of every arc is the same, i.e., we consider a characterization of the existence of arc-disjoint arborescences rooted at a specified vertex in a directed graph. For this case, the following important theorem was proved by Edmonds [2].

Theorem 1. *(See Edmonds [2].) For each directed graph D with a specified vertex r, there exist k arc-disjoint r-arborescences if and only if for every vertex v of D, there exist k arc-disjoint directed paths from r to v.*

Theorem 1 is one of the most important theorems in graph theory and combinatorial optimization. Furthermore, it gives us the following algorithmic implication. For checking the existence of k arc-disjoint r-arborescences, it is suffice to decide whether there exist k arc-disjoint directed paths from r to every vertex. Since we can decide in polynomial time whether there exist k arc-disjoint directed paths from r to every vertex (see, e.g., [3]), Theorem 1 implies that we can decide in polynomial time whether there exist k arc-disjoint r-arborescences. It should be noted that Theorem 1 was extended to various settings (see, e.g., [4–6]).

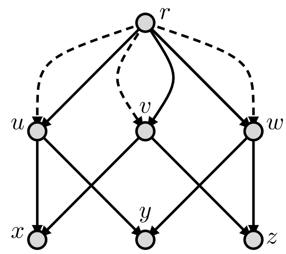

Fig. 2. A counterexample. A time label of each arc illustrated by a real line is equal to 1. A time label of each arc illustrated by a broken line is equal to 2.

In this paper, we consider the following extension of Theorem 1. "For each temporal network $N = (D, \tau)$ with a specified vertex r, there exist k arc-disjoint time-respecting r-arborescences if and only if for every vertex v of D, there exist k arc-disjoint time-respecting directed paths from r to v." Unfortunately, it is known [1] that this statement is not always true. Although the counterexample given in [1] has a directed cycle, we can construct an acyclic temporal network in which this statement is not true by slightly modifying the counterexample in [1] (see Fig. 2), where a temporal network $N = (D, \tau)$ is said to be *acyclic*, if D is acyclic.

In this paper, we consider the above statement in some special case. Precisely speaking, we consider the case where an input temporal network satisfies the pre-flow condition. See the next section for the definition of the pre-flow condition. The pre-flow condition in a directed graph was introduced in [7]. Our definition is a natural extension of this. It should be noted that the temporal network illustrated in Fig. 2 is not pre-flow.

3. Our contributions

Assume that we are given a temporal network $N = (D, \tau)$ with a specified vertex r. A subgraph T of D is called a *partial r-arborescence*, if $r \in V(T)$ and T is an r-arborescence in the subgraph of D induced by $V(T)$. Notice that $V(T)$ is not necessarily equal to $V(D)$. For each vertex v of D with $v \neq r$, we define $\lambda_N(v)$ as the maximum number of arc-disjoint time-respecting directed paths from r to v in N. In addition, define $\lambda_N(r) := \infty$. A partial r-arborescence in N is said to be *time-respecting*, if (1) holds for every vertex v of T with $v \neq r$. For each vertex v of D and each positive integer i, we define $\sigma_N(v,i)$ and $\gamma_N(v,i)$ by

$$\sigma_N(v,i) := \left|\left\{a \in \varrho_D(v) \;\middle|\; \tau(a) \le i\right\}\right|$$
$$\gamma_N(v,i) := \left|\left\{a \in \delta_D(v) \;\middle|\; \tau(a) \le i\right\}\right|,$$

respectively. A temporal network $N = (D, \tau)$ with a specified vertex r is said to be *pre-flow*, if

$$\sigma_N(v,i) \ge \gamma_N(v,i) \tag{2}$$

for every vertex v of D with $v \ne r$ and every positive integer i.

The main result of this paper can be described as follows. Theorem 2 can be regarded as a variation of Corollary 2.1 of Bang-Jensen, Frank and Jackson [7] in an acyclic temporal network.

Theorem 2. *For each acyclic and pre-flow temporal network $N = (D, \tau)$ with a specified vertex r and each positive integer k, there exist k arc-disjoint time-respecting partial r-arborescences $T_1, T_2, \ldots, T_k$ such that each vertex v of D is contained in exactly $\min\{k, \lambda_N(v)\}$ partial arborescences of $T_1, T_2, \ldots, T_k$.*

As a corollary of Theorem 2, we can obtain the following corollary. This can be regarded as a variation of Theorem 1 in an acyclic temporal network.

Corollary 3. *For each acyclic and pre-flow temporal network $N = (D, \tau)$ with a specified vertex r, there exist k arc-disjoint time-respecting r-arborescences if and only if for each vertex v of D, there exist k arc-disjoint time-respecting directed paths from r to v.*

Unfortunately, there exists a pre-flow temporal network with directed cycles in which the extension of Theorem 1 does not hold. See Section 5 for such a counterexample.

4. Proof of Theorem 2

Here we give the proof of Theorem 2. Assume that we are given an acyclic and pre-flow temporal network $N = (D, \tau)$ with a specified vertex r and a positive integer k. Since removing vertices u of D with $\lambda_N(u) = 0$ does not affect $\lambda_N(w)$ of vertices w of D with $\lambda_N(w) > 0$, we assume that $\lambda_N(v) > 0$ for every vertex v of D.

For each vertex v of D with $v \ne r$, we define a bipartite graph $G_v = (P_v \cup Q_v, E_v)$ as follows. The vertex set P_v contains a vertex $p(a)$ for each arc a in $\varrho_D(v)$, and the vertex set Q_v contains a vertex $q(a)$ for each arc a in $\delta_D(v)$. Furthermore, the edge set E_v contains an edge between a vertex $p(a)$ in P_v and a vertex $q(b)$ in Q_v, if $\tau(a) \le \tau(b)$. These are all arcs of E_v.

Lemma 4. *For each vertex v of D with $v \ne r$, there exists a matching M_v in G_v such that it covers all vertices in Q_v, i.e., for every vertex q in Q_v, there exists an edge in M_v that is incident to q.*

Proof. Let v be a vertex v of D with $v \ne r$. It is known [8] that there exists a matching M_v in G_v covering all vertices of Q_v if and only if

$$\forall X \subseteq Q_v \colon \left|\Gamma(X)\right| \ge |X|, \tag{3}$$

where $\Gamma(X)$ is the set of vertices in P_v that are adjacent to a vertex in X. Let us fix a subset X of Q_v. Define

$$t := \max\left\{\tau(a) \;\middle|\; a \in \delta_D(v) \text{ with } q(a) \in X\right\}.$$

Let a^* be an arc in $\delta_D(v)$ with $\tau(a^*) = t$. Since $\tau(a) \le t$ for every arc a in $\delta_D(v)$ such that $q(a)$ is contained in X, we have

$$\gamma_N(v,t) \ge |X|. \tag{4}$$

It follows from the definition of E_v that there exists an edge between $p(a)$ and $q(a^*)$ for every arc a in $\varrho_D(a)$ with $\tau(a) \le t$. So, we have

$$\left|\Gamma(X)\right| = \sigma_N(v,t). \tag{5}$$

It follows from (2), (4) and (5) that (3) holds. □

In the sequel, for each vertex v of D with $v \ne r$, we fix the matching M_v in Lemma 4. For each vertex v of D with $v \ne r$ and each arc $a = (v, w)$ in $\delta_D(v)$, we denote by $\mu(a)$ the arc b in $\varrho_D(v)$ such that there exists an edge between $p(b)$ and $q(a)$ in M_v.

The following lemma plays an important role in the proof of Theorem 2.

Lemma 5. *For each vertex v of D with $v \ne r$, we have*

$$\left|\varrho_D(v)\right| = \lambda_N(v).$$

Proof. Let v be a vertex of D with $v \ne r$. Define $d := \lambda_N(v)$. It follows from the definition of $\lambda_N(v)$ that

$$\left|\varrho_D(v)\right| \ge d. \tag{6}$$

Assume that (6) strictly holds. Since D is acyclic, for each arc a in $\varrho_D(v)$ we have a directed path P_a from r to v that passes through arcs

$$a,\ \mu(a),\ \mu\big(\mu(a)\big),\ \mu\Big(\mu\big(\mu(a)\big)\Big),\ \ldots,$$

in the reverse order. It follows from the definition of $\mu(\cdot)$ that the directed paths P_a and P_b are clearly arc-disjoint for every distinct arcs a and b in $\varrho_D(v)$, which implies that there exist more than d arc-disjoint time-respecting directed paths from r to v in N. This contradicts the fact that $\lambda_N(v) = d$. □

Now we are ready to prove Theorem 2. We propose an algorithm for finding desired arc-disjoint time-respecting partial r-arborescences. Let n be the number of vertices of D. It is well known (see, e.g., [3]) that since D is acyclic, there exists a bijective function $\pi \colon V(D) \to \{1, 2, \ldots, n\}$ such that $\pi(u) > \pi(v)$ if there exists an arc $a = (u, v)$ of D. Since D is acyclic and $\lambda_N(v) > 0$ for every vertex v of D, we have that $\varrho_D(r) = \emptyset$ and $\pi(r) = n$. Since we can naturally assume that k is not more than $|A(D)|$, Algorithm 1 is a polynomial-time algorithm.

Algorithm 1.

Step 1: For each $i = 1, 2, \ldots, k$, set $A_i^0 := \emptyset$. Furthermore, set $t := 1$.

Step 2: If $t = n$, then halt and output $A_1^{n-1}, A_2^{n-1}, \ldots, A_k^{n-1}$.

Step 3: Set v be the vertex of D with $\pi(v) = t$, and do the following steps.

(3-a) Define

$$I^+ := \left\{ i = 1, 2, \ldots, k \;\middle|\; \delta_D(v) \cap A_i^{t-1} \neq \emptyset \right\},$$

and let I^- be an arbitrary subset of positive integers in $\{1, 2, \ldots, k\} \setminus I^+$ such that

$$\left|I^-\right| = \min\{k, \lambda_N(v)\} - \left|I^+\right|.$$

(3-b) For each positive integer i in I^+, find an arc a^* in $\delta_D(v) \cap A_i^{t-1}$ such that

$$\tau\left(a^*\right) = \min\left\{ \tau(a) \;\middle|\; a \in \delta_D(v) \cap A_i^{t-1} \right\}, \quad (7)$$

and then set $a_i^t := \mu(a^*)$.

(3-c) For each positive integer i in I^-, choose an arbitrary arc a_i^t in $\varrho_D(v)$ such that

$$\forall j, j' \in I^- \quad \text{s.t.} \quad j \neq j' : a_j^t \neq a_{j'}^t,$$
$$\forall j \in I^-, \ \forall j' \in I^+ : a_j^t \neq a_{j'}^t.$$

(3-d) For each $i = 1, 2, \ldots, k$, set $A_i^t := A_i^{t-1} \cup \{a_i^t\}$.

(3-e) Update $t := t + 1$, and then go to Step 2. □

We first prove that Algorithm 1 is well-defined. For this, we first prove that $|I^+| \leq \lambda_N(v)$ in Step (3-a). Since $A_1^{t-1}, A_2^{t-1}, \ldots, A_k^{t-1}$ are arc-disjoint, it follows from (2) and Lemma 5 that

$$\left|I^+\right| \leq \left|\delta_D(v)\right| \leq \left|\varrho_D(v)\right| = \lambda_N(v).$$

Furthermore, since $|\varrho_D(v)| = \lambda_N(v)$ follows from Lemma 5, we have $|\varrho_D(v)| \geq \min\{k, \lambda_N(v)\}$. Thus, Step (3-c) is well-defined.

Assume that Algorithm 1 outputs subsets $A_1^{n-1}, A_2^{n-1}, \ldots, A_k^{n-1}$ of A. For each $i = 1, 2, \ldots, k$, let T_i be a subgraph of D satisfying

$$V(T_i) := \{r\} \cup \left\{ v \in V(D) \;\middle|\; \varrho_D(v) \cap A_i^{n-1} \neq \emptyset \right\},$$
$$A(T_i) := A_i^{n-1}.$$

Notice that we can prove that for each $i = 1, 2, \ldots, k$, every end vertices of an arc of T_i are contained in $V(T_i)$ as follows. It follows from the definition of Step (3-a) that for every vertex v of D with $v \neq r$, if $\delta_D(v) \cap A_i^{n-1}$ is not empty, then $\varrho_D(v) \cap A_i^{n-1}$ is not empty. This implies that the tail of the unique arc in $\varrho_D(v) \cap A_i^{n-1}$ is contained in $V(T_i)$ for every vertex v of D such that $\varrho_D(v) \cap A_i^{n-1}$ is not empty. Moreover, from the definition of I^+ and I^-, it follows that each vertex v of D is contained in exactly $\min\{k, \lambda_N(v)\}$ graphs of $T_1, T_2, \ldots, T_k$. Thus, Theorem 2 immediately follows from the following lemma.

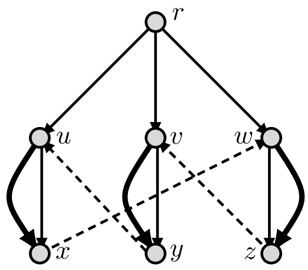

Fig. 3. A counterexample. A time label of each arc illustrated by a thin line is equal to 1. A time label of each arc illustrated by a broken line is equal to 2. A time label of each arc illustrated by a thick line is equal to 3.

Lemma 6. *For every $i = 1, 2, \ldots, k$, T_i is a time-respecting partial r-arborescence.*

Proof. For each $i = 1, 2, \ldots, k$ and each vertex v of T_i with $v \neq r$, we have $|\varrho_{T_i}(v)| = 1$. Since D is acyclic, this implies that T_i is a partial r-arborescence for every $i = 1, 2, \ldots, k$.

Now we prove that T_i is time-respecting. Fix a vertex v of T_i with $v \neq r$, and assume that $\pi(v) = t$. Since D is acyclic,

$$\forall j = t, t+1, \ldots, n-1 : \delta_D(v) \cap A_i^j = \delta_D(v) \cap A_i^{t-1}. \quad (8)$$

If $\delta_D(v) \cap A_i^{t-1}$ is empty, then it follows from (8) that T_i is time-respecting on v. Hence, we consider the case where $\delta_D(v) \cap A_i^{t-1}$ is not empty. Let $\mathrm{in}(v)$ be the unique arc in $\varrho_{T_i}(v)$, and let a^* be the unique arc in $\delta_{T_i}(v)$ with $\mathrm{in}(v) = \mu(a^*)$. It follows from (7) and (8) that

$$\begin{aligned} \tau\left(a^*\right) &= \min\left\{ \tau(a) \;\middle|\; a \in \delta_D(v) \cap A_i^{t-1} \right\} \\ &= \min\left\{ \tau(a) \;\middle|\; a \in \delta_D(v) \cap A_i^{n-1} \left(= \delta_{T_i}(v)\right) \right\}. \end{aligned}$$

This and the definition of $\mu(\cdot)$ imply that

$$\begin{aligned} \tau\left(\mathrm{in}(v)\right) &= \tau\left(\mu\left(a^*\right)\right) \leq \tau\left(a^*\right) \\ &= \min\left\{ \tau(a) \;\middle|\; a \in \delta_{T_i}(v) \right\} \leq \tau(b) \end{aligned}$$

for every arc b in $\delta_{T_i}(v)$. This completes the proof. □

5. Counterexample

In this section, we give a pre-flow temporal network with directed cycles in which the extension of Theorem 1 does not hold. The pre-flow temporal network described in Fig. 3 is such a counterexample.

It is not difficult to see that there exist two arc-disjoint time-respecting directed paths from r to every vertex. However, we can prove that there does not exist two arc-disjoint time-respecting r-arborescences as follows. Assume that there exist two arc-disjoint time-respecting r-arborescences T_1 and T_2. Since three arcs leave r, one of T_1 and T_2 contains only one arc leaving r. Assume that T_1 contains only one arc leaving r and it is (r, u). Note that there exists no time-respecting directed path from u to v. Thus, T_1 is not a time-respecting r-arborescence. This completes the proof.

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