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Packing of maximal independent mixed arborescences

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ABSTRACT

This paper studies the following packing problem: Given a mixed graph F with vertex set V, a matroid M on a set $S = \{s_1, \ldots, s_k\}$ along with a map $\pi : S \to V$, find k mutually edge and arc-disjoint mixed arborescences $T_1, \ldots, T_k$ in F with roots $\pi(s_1), \ldots, \pi(s_k)$, such that, for any $v \in V$, the set $\{s_i : v \in V(T_i)\}$ is independent and its rank reaches the theoretical maximum. This problem was mentioned by Fortier, Király, Léonard, Szigeti and Talon in [Old and new results on packing arborescences in directed hypergraphs, Discrete Appl. Math. 242 (2018), 26-33]; Matsuoka and Tanigawa gave a solution to this in [On reachability mixed arborescence packing, Discrete Optimization 32 (2019) 1-10].

In this paper, we give a new characterization for above packing problem. This new characterization is simplified to the form of finding a supermodular function that should be covered by an orientation of each strong component of a matroid-based rooted mixed graph. Our proofs come along with a polynomial-time algorithm. The technique of using components opens some new ways to explore arborescence packings.

1. Introduction

In this paper, we consider graphs which may have multiple edges or (and) arcs but not loops. Let $D = (V, A)$ be a digraph. For $\emptyset \neq X \subseteq V$, let $d_D^-(X)$ (or $d_A^-(X)$) denote the arcs of D (or A) entering into X. A subdigraph T (it may not be spanning) of D is called an *r-arborescence* if its underlying graph is a tree and for any $u \in V(T)$, there is exactly one directed path in T from r to u. The vertex r is called the *root* of the arborescence T.

Edmonds' arborescence packing theorem [8] characterizes digraphs containing k arc-disjoint spanning arborescences with prescribed roots in terms of a cut condition, this is the starting point of all studies on packing arborescences. This result has extensions in many directions. For the presentation, we introduce some terms and notations.

A mixed graph $F = (V; E, A)$ is a graph consisting of the set E of undirected edges and the set A of directed arcs. By regarding each undirected edge as a directed arc in both directions, each concept in directed graphs can be naturally extended for mixed graphs. Especially, a subdigraph P of F is a *mixed path* if its underlying graph is a path and one end of P can be reached from the other. A subdigraph T (it may not be spanning) of F is called an *r-mixed arborescence* if its underlying graph is a tree and for any $u \in V(T)$, there is exactly one mixed path in T from r to u. Equivalently, a subgraph T of F is an r-mixed arborescence if there exists an orientation of the undirected edges of T such that the obtained subgraph (whose arc set is the union of original arc set and oriented arc set of T) is an r-arborescence.

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Let $X_1, \ldots, X_t$ be disjoint subsets of V; we call $\mathcal{P} = \{X_1, \ldots, X_t\}$ a *subpartition (of V)* and particularly a *partition of V* if $V = \cup_{j=1}^{t} X_j$. For a subpartition $\mathcal{P}$ of V, denote $e_E(\mathcal{P}) = |\{e \in E : e$ connects distinct X_is in $\mathcal{P}$ or connects some X_i and $V \setminus \cup_{j=1}^{t} X_j\}|$.

For nonempty $X, Z \subseteq V$, where X and Z not necessarily disjoint, let $E(X, Z)$ and $A(X, Z)$ denote the set of edges with one endvertex in X and the other in Z and the set of arcs from X to Z respectively. For simplicity, denote $E(X) = E(X, X)$ and $A(X) = A(X, X)$. Let $Z \xrightarrow{F} X$ denote that X and Z are disjoint and X is *reachable* from Z in F, that is, there is a mixed path in F from Z to X. We shall write v for $\{v\}$ for simplicity. Let $W_F(X) := X \cup \{v \in V \setminus X : v \xrightarrow{F} X\}$.

Let $R = \{r_1, \ldots, r_k\} \subseteq V$ be a specified multiset. Let U_i be the set of vertices reachable from r_i. For $u, v \in V$, we say $u \sim v$ if $\{i : u \in U_i\} = \{i : v \in U_i\}$; this $\sim$ is an equivalence relation. Denote equivalence classes for $\sim$ by $\Gamma_1, \ldots, \Gamma_l$, and we call each Γ_j an *atom*. An r_i-mixed arborescence T_i is said to be *maximal* if $V(T_i) = U_i$ (i.e. it spans all the vertices that are reachable from r_i in F). A *packing of maximal mixed arborescences w.r.t.* $R = \{r_1, \ldots, r_k\}$ is a collection $\{T_1, \ldots, T_k\}$ of mutually edge and arc-disjoint mixed arborescences such that T_i has root r_i and $V(T_i) = U_i$. Denote the set $\{1, \ldots, k\}$ by $[k]$.

The remarkable extension of Edmonds' arborescence packing theorem by Kamiyama, Katoh and Takizawa [14] enables us to find a packing of maximal arborescences $\{T_1, \ldots, T_k\}$ w.r.t. R in a digraph (that is $E = \emptyset$).

Let M be a matroid on a set S with rank function r_M, and $\pi : S \to V$ be a (not necessarily injective) map. We may think of π as a placement of the elements of S at vertices of V and different elements of S may be placed at the same vertex. For related definitions and properties of matroids, we refer to [11]. We say that the quadruple (F, M, S, π) is a *matroid-based rooted mixed graph* (or a *matroid-based rooted digraph* if $E = \emptyset$).

The map π is called *M-independent* if $\pi^{-1}(v)$ is independent in M for each $v \in V$. For $X \subseteq V$, denote by S_X the set $\pi^{-1}(X)$. An *M-based packing of mixed arborescences* is a set $\{T_1, \ldots, T_{|S|}\}$ of pairwise edge and arc-disjoint mixed arborescences for which T_i has root $\pi(s_i)$ for $i = 1, \ldots, |S|$ (where $S = \{s_1, \ldots, s_{|S|}\}$), and for each $v \in V$, the set $\{s_j \in S : v \in V(T_j)\}$ is a base of S. An M-based packing of mixed arborescences is an *M-based packing of trees* when each T_i is a tree and an *M-based packing of arborescences* when each T_i is an arborescence.

Katoh and Tanigawa [15] implicitly characterized undirected graphs containing an M-based packing of trees; Durand de Gevigney, Nguyen and Szigeti [7] characterized directed graphs containing an M-based packing of arborescences.

A *maximal M-independent packing of mixed arborescences* is a set $\{T_1, \ldots, T_{|S|}\}$ of pairwise edge and arc-disjoint mixed arborescences for which T_i has root $\pi(s_i)$ for $i = 1, \ldots, |S|$, the set $\{s_i \in S : v \in V(T_i)\}$ is independent in M, and $|\{s_i \in S : v \in V(T_i)\}| = r_M(S_{W_F(v)})$ for each $v \in V$.

Király [16] characterized digraphs containing such a packing.

Theorem 1.1 (*[16]*)**.** *Let $(D = (V, A), M, S, \pi)$ be a matroid-based rooted digraph. There exists a maximal M-independent packing of arborescences in (D, M, S, π) if and only if π is M-independent and*

$$d_A^-(X) \geq r_M(S_{W_D(X)}) - r_M(S_X) \tag{1}$$

holds for each $\emptyset \neq X \subseteq V$.

Fortier, Király, Léonard, Szigeti and Talon [9] characterized mixed graphs containing matroid-based packing of mixed arborescences; this theorem will be used later as the base step in the inductive proof for our main result (Theorem 1.4).

Theorem 1.2 (*[9, Theorem 10], adapted*)**.** *Let $(F = (V; E, A), M, S, \pi)$ be a M-based rooted mixed graph. There exists a matroid-based packing of mixed arborescences in (F, M, S, π) if and only if π is M-independent and*

$$e_E(\mathcal{P}) \geq \sum_{i=1}^{t} (r_M(S) - r_M(X_i) - d_A^-(X_i)) \tag{2}$$

holds for each subpartition $\mathcal{P} = \{X_1, \ldots, X_t\}$ of V.

In [9], Fortier, et al. mentioned the following research problem: how to extend Theorem 1.1 to mixed graphs. Matsuoka and Tanigawa [18] gave a solution to this problem, by using some recent results [17] on the maximal arborescence packing by Király, Szigeti and Tanigawa. Since this research problem is the major topic of this paper, next we present the theorem that is delivered by the solution in [18]. This solution uses concepts of atom and bi-set.

A bi-set $Y = \{Y_O, Y_I\}$ of V is a pair of sets satisfying $Y_I \subseteq Y_O \subseteq V$. For bi-set Y, define $d_A^-(Y) = |\{uv \in A : u \in V \setminus Y_O,$ $v \in Y_I\}|$. The application of bi-sets for arborescence packings was first studied by Bérczi and Frank [1,2], see also [6,17,18].

Let $F = (V; E, A)$ be a mixed graph, and $r_1, \ldots, r_k \in V$ (not necessarily distinct). Let M be a matroid on $[k]$ with rank function r_M, and $\pi : [k] \to V$ be a (not necessarily injective) map such that $\pi(i) = r_i$ for $1 \leq i \leq k$. Let $U_i \subseteq V$ $(i = 1, \ldots, k)$ be the set of vertices reachable from r_i in F, and $\Gamma_1, \ldots, \Gamma_l$ be the atoms of F w.r.t. $\{r_1, \ldots, r_k\}$.

Let $\mathcal{P}_2(V)$ denote all bi-sets of V, i.e. $\mathcal{P}_2(V) := \{(X_O, X_I) : X_I \subseteq X_O \subseteq V\}$. Define

$$\mathcal{F}_j := \{X \in \mathcal{P}_2(V) : \ \emptyset \neq X_I \subseteq \Gamma_j,\ (X_O \setminus X_I) \cap \Gamma_j = \emptyset\} \ \ (1 \leq j \leq l), \quad \mathcal{F} := \cup_{j=1}^{l} \mathcal{F}_j.$$

For $X = (X_O, X_I) \in \mathcal{F}$, define

$$I_X := \{i : \ X_I \subseteq U_i,\ r_i \notin X_I,\ (X_O \setminus X_I) \cap U_i = \emptyset\}, \quad J_X := \{i \in [k] : X_I \subseteq U_i\} \setminus I_X.$$

Theorem 1.3 (*[18]*). *The following statements are equivalent.*

(i) $\exists$ *a maximal* M*-independent packing of mixed arborescences in* $(F, M, [k], \pi)$.
(ii) π *is* M*-independent; and*

$$e_E(\mathcal{P}) + \sum_{q=1}^{t} d_A^-(X^q) \geq \sum_{q=1}^{t} (r_M(I_{X^q} \cup J_{X^q}) - r_M(J_{X^q})) \tag{3}$$

holds for any family of bi-sets $\{X^1, \ldots, X^t\}$ *such that* $\mathcal{P} = \{X_I^1, \ldots, X_I^t\}$ *is a vertex subpartition of some atom* Γ_j *and that* $(X_O^q \setminus X_I^q) \cap \Gamma_j = \emptyset$ *for* $q = 1, \ldots, t$.

In this paper, we give a new characterization for packing of maximal independent mixed arborescences. In comparison with Theorem 1.3, our characterization uses the concepts of strong component and bi-set. Recall that C is a *strong component* of F if it is a maximal subgraph of F for which for any two vertices u, v of C, u and v are reachable from each other in C.

The following theorem is our new characterization, it is simplified to the form of finding an *intersecting supermodular function that should be covered* (to be defined at the beginning of Section 2) by an orientation of each strong component of a matroid-based rooted mixed graph F, the simplified form is Statement (*iii*).

Theorem 1.4. *Let* $(F = (V; E, A), M, S, \pi)$ *be a matroid-based rooted mixed graph. Then the following statements are equivalent.*

(i) $\exists$ *a maximal* M*-independent packing of mixed arborescences in* (F, M, S, π).
(ii) π *is* M*-independent; and*

$$e_E(\mathcal{P}) + \sum_{q=1}^{t} d_A^-(X^q) \geq \sum_{q=1}^{t} (r_M(S_{W_F(V(C))}) - r_M(S_{X_O^q})) \tag{4}$$

holds for any family of bi-sets $\{X^1, \ldots, X^t\}$ *such that* $\mathcal{P} = \{X_I^1, \ldots, X_I^t\}$ *is a vertex subpartition of some strong component* C *and* $X_O^q \setminus X_I^q = W_F(Y)$ *for some* $Y \subseteq W_F(V(C)) \setminus V(C)$, *where* $q = 1, \ldots, t$.
(iii) π *is* M*-independent; and*

$$e_E(\mathcal{P}) \geq \sum_{q=1}^{t} f_C(X_q) \tag{5}$$

holds for any strong component C *of* F *and subpartition* $\mathcal{P} = \{X_1, \ldots, X_t\}$ *of* $V(C)$, *where* $f_C(X_q) = \max\{r_M(S_{W_F(V(C))}) - r_M(S_X) - d_A^-(X) : X_q \subseteq X \text{ and } X \setminus X_q = W_F(Y) \text{ for some } Y \subseteq W_F(V(C)) \setminus V(C)\}$.

Next in Proposition 1.5, we shall give a direct proof that Theorem 1.3 (*ii*) deduces Theorem 1.4 (*ii*), therefore Theorem 1.4 implies Theorem 1.3.

However, we cannot present a direct proof from Theorem 1.4 (*ii*) to Theorem 1.3 (*ii*). Besides the difference that Theorem 1.4 uses 'component'and Theorem 1.3 uses 'atom', there is a slight improvement on the using of bi-set. To see this, we take a look at a special case: Suppose a strong component C is exactly an atom (this can happen if there exist no arcs leaving C and $\{r_1, \ldots, r_k\} \cap V(C) \neq \emptyset$), then it follows from the proof of Proposition 1.5 that (3) is exactly (4) under the condition of $X_O^q \setminus X_I^q = W_F(Y)$ for some $Y \subseteq W_F(V(C)) \setminus V(C)$. Note that Statement (*ii*) in Theorem 1.4 has some extra constraints, this is different than its counterpart.

Indeed, the proof we can present to show that Theorem 1.3 implies Theorem 1.4 is the proof of our main result, which is Section 2 of this paper. Then by Statement (*i*) of Theorems 1.3 and 1.4, these two theorems are equivalent with each other.

Proposition 1.5. *Suppose in Theorem 1.4,* $S = [k]$ *and* $\pi(i) = r_i$ *for* $i = 1, \ldots, k$. *If Theorem 1.3 (ii) holds, then Theorem 1.4 (ii) holds.*

Proof. Suppose (3) holds. Let C be a strong component of F and $X = (X_O, X_I)$ a bi-set such that $X_I \subseteq V(C)$ and $X_O \setminus X_I = W_F(Y)$ for some $Y \subseteq W_F(V(C)) \setminus V(C)$.

Recall that U_i is the set of vertices reachable from r_i in F. Then $X_I \subseteq U_i$ if and only if $r_i \in W_F(V(C))$. Thus $\{i \in [k] : X_I \subseteq U_i\} = S_{W_F(V(C))}$, that is $I_X \cup J_X = S_{W_F(V(C))}$.

Note that $W_F(Y) \cap U_i \neq \emptyset$ if and only if $r_i \in W_F(Y)$. Since $X_I \subseteq V(C)$ and $Y \subseteq W_F(V(C))$, $W_F(Y) \subseteq W_F(V(C))$; thus $r_i \in W_F(Y)$ implies $V(C) \subseteq U_i$ (then $X_I \subseteq U_i$). So

$$\begin{aligned} J_X &= \{i \in [k] : X_I \subseteq U_i\} \setminus I_X \\ &= \{i \in [k] : X_I \subseteq U_i, r_i \in X_I\} \cup \{i \in [k] : X_I \subseteq U_i, W_F(Y) \cap U_i \neq \emptyset\} \\ &= \{i \in [k] : r_i \in X_I\} \cup \{i \in [k] : r_i \in W_F(Y)\} \\ &= S_{X_O}. \end{aligned}$$

Hence, we have

$$r_M(I_X \cup J_X) - r_M(J_X) = r_M(S_{W_F(V(C))}) - r_M(S_{X_O}). \tag{6}$$

For any two $u, v \in V(C)$, $u \sim v$ (by definition); thus $V(C) \subseteq \Gamma_j$ for some atom Γ_j.

If $W_F(Y) \cap \Gamma_j \neq \emptyset$, then $r_i \in W_F(V(C))$ implies $r_i \in W_F(Y)$; thus $S_{W_F(V(C))} \subseteq S_{W_F(Y)} \subseteq S_{X_O}$ and

$$r_M(S_{W_F(V(C))}) - r_M(S_{X_O}) - d_A^-(X) \leq 0. \tag{7}$$

Let $\{X^1, \ldots, X^t\}$ be a family of bi-sets such that $\mathcal{P} = \{X_I^1, \ldots, X_I^t\}$ is a subpartition of $V(C)$ and that $X_O^q \setminus X_I^q = W_F(Y)$ for some $Y \subseteq W_F(V(C)) \setminus V(C)$, where $q = 1, \ldots, t$. Then we have

$$\begin{aligned} e_E(\mathcal{P}) &\geq e_E(\{X_I^q : (X_O^q \setminus X_I^q) \cap \Gamma_j = \emptyset\}) \\ &\geq \sum_{(X_O^q \setminus X_I^q) \cap \Gamma_j = \emptyset} (r_M(I_{X^q} \cup J_{X^q}) - r_M(J_{X^q}) - d_A^-(X^q)) \qquad (\text{by } (3)) \\ &= \sum_{(X_O^q \setminus X_I^q) \cap \Gamma_j = \emptyset} (r_M(S_{W_F(V(C))}) - r_M(S_{X_O^q}) - d_A^-(X^q)) \qquad (\text{by } (6)) \\ &\geq \sum_{q=1}^{t} (r_M(S_{W_F(V(C))}) - r_M(S_{X_O^q}) - d_A^-(X^q)) \qquad (\text{by } (7)). \end{aligned}$$

That is, (4) holds. ■

Note that our Theorem 1.4 appears to be the first evidence to show that the concept of 'atom', which has been widely used in the study of arborescence packings, can be further divided into 'components'under some circumstances. (Indeed, an atom is the union of some strong components of F.) Since component is a much more common concept than the very specialized atom, this technique could make theorems and their proofs simpler.

The very recent paper [13] by Hörsch and Szigeti applied this technique of using 'components', and showed three surprising implications: the result of Edmonds [8] on packing branchings with prescribed root sets $\Rightarrow$ the result of Kamiyama, Katoh, Takizawa [14] on packing maximal arborescences, the result of Durand de Gevigney, Nguyen, Szigeti [7] on matroid-based packing of arborescences $\Rightarrow$ Theorem 1.1, and hypergraphic version of Theorem 1.2 $\Rightarrow$ hypergraphic version of Theorem 1.4. Compared with the existing proofs, these implications of the characterization on packing maximal arborescences [14] and Theorem 1.1 are more direct and easier. Finally, as noted in [13], our proof techniques in this paper can be used to obtain the generalization of Theorem 1.4 to mixed hypergraphs (the remaining open problem in [9]).

2. Main result

Let Ω be a set and $X_1, X_2 \subseteq \Omega$. X_1 and X_2 are *intersecting* if $X_1 \cap X_2 \neq \emptyset$. A function $p : 2^\Omega \to \mathbb{Z}$ is *supermodular (intersecting supermodular)* if the inequality

$$p(X) + p(Y) \leq p(X \cup Y) + p(X \cap Y)$$

holds for all subsets (intersecting subsets, respectively) of Ω. A function b is *submodular* if $-b$ is supermodular. For some recent work related to supermodularity in graph optimization, we refer to [3–5,12].

A family $\mathcal{H}$ of subsets of V is *intersecting* if for any $X, Y \in \mathcal{H}$ such that $X \cap Y \neq \emptyset$, $X \cup Y$ and $X \cap Y \in \mathcal{H}$. For a set function $f : \mathcal{H} \to \mathbb{Z}$, a directed graph $D = (V, A)$ (or just A) is said to *cover* f if $d_A^-(X) \geq f(X)$ holds for all $X \in \mathcal{H}$.

2.1. Preliminaries

Let $(D = (V, A), M, S, \pi)$ be a matroid-based rooted digraph. Suppose (1) holds for each $\emptyset \neq X \subseteq V$, we say $X_0 \subseteq V$ is *tight* if the equality of (1) holds. Note that the in-degree function d_A^- of D and rank function of a matroid is submodular.

Lemma 2.1 (*[16, Lemma 10], adapted*). *Let $(D = (V, A), M, S, \pi)$ be a matroid-based rooted digraph for which (1) holds for each $\emptyset \neq X \subseteq V$. Let $uv \in A$ and X_0 be a minimal tight set such that the arc uv enters X_0. Then $X_0 \subseteq W_D(v)$.*

Lemma 2.2. *Let $(D = (V, A), M, S, \pi)$ be a matroid-based rooted digraph. There exists a maximal M-independent packing of arborescences in (D, M, S, π) if and only if π is M-independent and (1) holds for $X \subseteq V$ such that $v \in X \subseteq W_D(v)$ for some $v \in V$.*

Proof. The necessity comes from Theorem 1.1 directly.

For the sufficiency, suppose to the contrary that D does not have such a packing. By Theorem 1.1, there exists $X_0 \subseteq V$ such that $d_A^-(X_0) < r_M(S_{W_D(X_0)}) - r_M(S_{X_0})$. Let $D' = (V, A')$ be a minimal digraph for which: (*i*) $A \subseteq A'$, (*ii*) $W_D(v) = W_{D'}(v)$ for each $v \in V$, and (*iii*) $d_{A'}^-(X) \geq r_M(S_{W_D(X)}) - r_M(S_X)$ for $X \subseteq V$. Note that such a digraph exists because we can always add arcs uv with $u \in W_D(v)$ till Condition (*iii*) holds. Then $d^-_{A'}(X_0) \geq r_M(S_{W_D(X_0)}) - r_M(S_{X_0}) > d^-_A(X_0)$; and there exists an arc $u_0v_0 \in A' \setminus A$.

By the minimality of D', there exists $X_1 \subseteq V$ such that $d^-_{A'-u_0v_0}(X_1) < r_M(S_{W_D(X_1)}) - r_M(S_{X_1})$. Since $d^-_{A'}(X_1) \geq r_M(S_{W_D(X_1)}) - r_M(S_{X_1})$, we have $d^-_{A'}(X_1) = r_M(S_{W_D(X_1)}) - r_M(S_{X_1})$ (that is X_1 is tight) and u_0v_0 enters X_1. Let X_2 be a minimal tight set of D' such that u_0v_0 enters X_2. Then

$$d^-_A(X_2) \leq d^-_{A'-u_0v_0}(X_2) < d^-_{A'}(X_2) = r_M(S_{W_D(X_2)}) - r_M(S_{X_2}). \tag{8}$$

But by Lemma 2.1, $v_0 \in X_2 \subseteq W_D(v_0)$. Then by the assumption of this lemma, $d^-_A(X_2) \geq r_M(S_{W_D(X_2)}) - r_M(S_{X_2})$, a contradiction to (8). ■

Theorem 2.3 ([10]). *Let $G = (V, E)$ be an undirected graph, $\mathcal{H} \subseteq 2^V$ be an intersecting family with $\emptyset \notin \mathcal{H}$ and $V \in \mathcal{H}$, and $f : \mathcal{H} \to \mathbb{R}$ an intersecting supermodular function with $f(V) = 0$. There exists an orientation of E that covers f (that is $d^-_A(X) \geq f(X)$ for all $X \in \mathcal{H}$, where A is the oriented arc set of E) if and only if*

$$e_E(\mathcal{P}) \geq \sum_{i=1}^{t} f(V_i)$$

holds for every collection $\mathcal{P} = \{V_1, \ldots, V_t\}$ of mutually disjoint members of $\mathcal{H}$.

2.2. Proof of Theorem 1.4

We shall show that (*i*) ⇒ (*ii*), (*ii*) ⇒ (*iii*), and (*iii*) ⇒ (*i*), this will finish the proof. The following claim is needed in the proof.

Claim 2.4. *For mixed graph $F = (V; E, A)$, suppose bi-set $X = (X_O, X_I)$ satisfies that $\emptyset \neq X_I \subseteq V(C)$ for some strong component C and $X_O \setminus X_I = W_F(Y)$ for some $Y \subseteq W_F(V(C)) \setminus V(C)$. Then for any orientation A' of E, we have*

$$d^-_{A'}(X_O) = d^-_{A'}(X_I) = d^-_{A'}(X), \quad d^-_A(X_O) = d^-_A(X). \tag{9}$$

Proof. For bi-set $X = (X_O, X_I)$, by the definition of $d^-_{A'}(X)$, we have

$$d^-_{A'}(X_O) = d^-_{A'}(X) + |A'(V \setminus X_O, X_O \setminus X_I)| = d^-_{A'}(X) + |A'(V \setminus X_O, W_F(Y))|,$$

$$d^-_{A'}(X_I) = d^-_{A'}(X) + |A'(X_O \setminus X_I, X_I)| = d^-_{A'}(X) + |A'(W_F(Y), X_I)|.$$

Similarly, $d^-_A(X_O) = d^-_A(X) + |A(V \setminus X_O, W_F(Y))|$.

By the assumption, $X_O = W_F(Y) \cup X_I$ for some $Y \subseteq W_F(V(C)) \setminus V(C)$, and $\emptyset \neq X_I \subseteq V(C)$. It follows that $W_F(X_O) = W_F(V(C))$, $A'(V \setminus X_O, W_F(Y)) = \emptyset$ and $A(V \setminus X_O, W_F(Y)) = \emptyset$.

By the assumption, $Y \subseteq W_F(V(C)) \setminus V(C)$. Next we deduce that $W_F(Y) \subseteq W_F(V(C)) \setminus V(C)$. Assume to the contrary that $W_F(Y) \cap V(C) \neq \emptyset$, suppose vertex $u \in W_F(Y) \cap V(C)$, then there exist vertex $v \in Y$ and a mixed path P_1 from u to v. Combining that $v \in Y \subseteq W_F(V(C))$ and C is a strong component, there exists a mixed path P_2 from v to u. But then $V(C) \cup P_1 \cup P_2$ is strongly connected, this contradicts that C is a strong component.

Note that since C is a strong component, there is no edge in E between $V(C)$ and $V \setminus V(C)$. Since $X_I \subseteq V(C)$, $W_F(Y) \subseteq W_F(V(C)) \setminus V(C)$, and A' is an orientation of E, we have $A'(W_F(Y), X_I) = \emptyset$. These prove that $d^-_{A'}(X_O) = d^-_{A'}(X_I) = d^-_{A'}(X)$, $\quad d^-_A(X_O) = d^-_A(X)$. ■

(i) ⇒ **(ii)**: The proof follows from Theorem 1.3 and Proposition 1.5.

(ii) ⇒ **(iii)**: For $1 \leq q \leq t$, suppose Y_q satisfies that $f_C(X_q) = r_M(S_{W_F(V(C))}) - r_M(S_{Y_q}) - d^-_A(Y_q)$, and $X_q \subseteq Y_q$ and $Y_q \setminus X_q = W_F(Y)$ for some $Y \subseteq W_F(V(C)) \setminus V(C)$; define bi-set $X^q = (Y_q, X_q)$. Note that $d^-_A(X^q) = d^-_A(X^q_O)$ by Claim 2.4, it is straightforward to check that (5) can be obtained from (4).

(iii) ⇒ **(i)**: Let $\tau(F)$ be the number of strong components of F. We prove that (*iii*) ⇒ (*i*) by induction on $\tau(F)$.

For the base step, suppose $\tau(F) = 1$, i.e., F is strongly connected. Then, for any subpartition $\{X_1, \ldots, X_t\}$ of $V(F)$, by (5), we have

$$e_E(\mathcal{P}) \geq \sum_{q=1}^{t} f_F(X_q) = \sum_{q=1}^{t} (r_M(S) - r_M(S_{X_q}) - d^-_A(X_q)).$$

By Theorem 1.2, there exists a matroid-based packing of mixed arborescences in (F, M, S, π).

For the induction step, suppose $\tau(F) = n \geq 2$, and suppose that (*iii*) ⇒ (*i*) holds for $\tau(F) \leq n - 1$.

Note that there exists a strong component C_0 of F such that no arcs coming out of C_0. Assume otherwise, then each strong component has arcs coming out of it. But then F itself is strongly connected, a contradiction to $\tau(F) \geq 2$.

Suppose C_0 is such a strong component as above, F_1 is the induced mixed graph on vertex set $V(F_1) := V(F) \setminus V(C_0)$. Then $\tau(F_1) = n - 1$.

The following facts are heavily used: $E(V(C_0), V(F_1)) = \emptyset$, $A(V(C_0), V(F_1)) = \emptyset$; therefore for $X_0 \subseteq V(F_1)$, $W_{F_1}(X_0) = W_F(X_0) \subseteq V(F_1)$.

By the induction hypothesis, there exists a maximal $M|S_{V(F_1)}$-independent packing of mixed arborescences in F_1; that is, there exist pairwise edge and arc disjoint $\pi(s_i)$-mixed arborescences T_i' in F_1, where $1 \le i \le |S_{V(F_1)}|$; and for any $v \in V(F_1)$, $\{s_i : v \in V(T_i')\}$ is independent and $|\{s_i : v \in V(T_i')\}| = r_M(S_{W_F(v)})$. Equivalently, $E(F_1)$ can be oriented to A_1 such that there exist pairwise arc disjoint $\pi(s_i)$-arborescences T_i in $D_1 := (V(F_1), A(V(F_1)) \cup A_1)$, where $1 \le i \le |S_{V(F_1)}|$; and for any $v \in V(F_1)$, $\{s_i : v \in V(T_i)\}$ is independent and $|\{s_i : v \in V(T_i)\}| = r_M(S_{W_F(v)})$.

By Theorem 1.1, for any $\emptyset \ne X_0 \subseteq V(F_1)$,

$$d^-_{D_1}(X_0) \ge r_M(S_{W_{D_1}(X_0)}) - r_M(S_{X_0}). \tag{10}$$

Note that if $v \in V(T_i)$, then $\pi(s_i) \in W_{D_1}(v)$. Thus $\{s_i : v \in V(T_i)\} \subseteq S_{W_{D_1}(v)}$, and $|\{s_i : v \in V(T_i)\}| \le r_M(S_{W_{D_1}(v)})$. Since $W_{D_1}(v) \subseteq W_F(v)$, we have $r_M(S_{W_{D_1}(v)}) \le r_M(S_{W_F(v)})$. Since $|\{s_i : v \in V(T_i)\}| = r_M(S_{W_F(v)})$, we have $r_M(S_{W_{D_1}(v)}) = r_M(S_{W_F(v)})$. Thus $r_M(S_{W_{D_1}(X_0)}) = r_M(S_{W_F(X_0)})$. $E(V(C_0), V(F_1)) = A(V(C_0), V(F_1)) = \emptyset$ gives that $d^-_{D_1}(X_0) = d^-_{A\cup A_1}(X_0)$. So (10) can be transformed to:

$$d^-_{A\cup A_1}(X_0) \ge r_M(S_{W_F(X_0)}) - r_M(S_{X_0}). \tag{11}$$

Define $f_{C_0} : 2^{V(C_0)} \setminus \{\emptyset\} \to \mathbb{Z}$, $f_{C_0}(X) = \max\{r_M(S_{W_F(V(C_0))}) - r_M(S_{X_0}) - d^-_A(X_0) : X \subseteq X_0$ *and* $X_0 \setminus X = W_F(Y)$ *for some* $Y \subseteq W_F(V(C_0)) \setminus V(C_0)\}$. Then we have the following claim.

Claim 2.5. *f_{C_0} is intersecting supermodular.*

Proof. Suppose $X_1, X_2 \subseteq V(C_0)$ are intersecting sets, $Y_1, Y_2 \subseteq W_F(V(C_0)) \setminus V(C_0)$ satisfy that $f_{C_0}(X_i) = r_M(S_{W_F(V(C_0))}) - r_M(S_{X_i\cup W_F(Y_i)}) - d^-_A(X_i \cup W_F(Y_i))$, where $i = 1, 2$.

Note that $W_F(Y_1) \cup W_F(Y_2) = W_F(Y_1 \cup Y_2)$. Let $Y_3 = W_F(Y_1) \cap W_F(Y_2)$, note that $W_F(Y_3) = Y_3$; thus $W_F(Y_1) \cap W_F(Y_2) = W_F(Y_3)$. Since r_M and d^-_A are submodular,

$$\begin{aligned} f_{C_0}(X_1) + f_{C_0}(X_2) &= r_M(S_{W_F(V(C_0))}) - r_M(S_{X_1\cup W_F(Y_1)}) - d^-_A(X_1 \cup W_F(Y_1)) \\ &\quad + r_M(S_{W_F(V(C_0))}) - r_M(S_{X_2\cup W_F(Y_2)}) - d^-_A(X_2 \cup W_F(Y_2)) \\ &\le r_M(S_{W_F(V(C_0))}) - r_M(S_{X_1\cup X_2\cup W_F(Y_1\cup Y_2)}) - d^-_A(X_1 \cup X_2 \cup W_F(Y_1 \cup Y_2)) \quad \square \\ &\quad + r_M(S_{W_F(V(C_0))}) - r_M(S_{(X_1\cap X_2)\cup W_F(Y_3)}) - d^-_A((X_1 \cap X_2) \cup W_F(Y_3)) \\ &\le f_{C_0}(X_1 \cup X_2) + f_{C_0}(X_1 \cap X_2). \quad \blacksquare \end{aligned}$$

Using Claim 2.5 and (5), by Theorem 2.3, we know that there exists an orientation A_0 of $E(C_0)$ such that A_0 covers f_{C_0}, i.e., for any $\emptyset \ne Z \subseteq V(C_0)$ and Z_0 such that $Z \subseteq Z_0$ and $Z_0 \setminus Z = W_F(Y)$ for some $Y \subseteq W_F(V(C_0)) \setminus V(C_0)$,

$$d^-_{A_0}(Z) \ge r_M(S_{W_F(V(C_0))}) - r_M(S_{Z_0}) - d^-_A(Z_0). \tag{12}$$

Using orientation A_1 of $E(F_1)$ and A_0 of $E(C_0)$, we have a directed graph D (of F) with arc set $A \cup A_0 \cup A_1$.

Apply (9) to bi-set (Z_0, Z), we have $d^-_{A_0\cup A_1}(Z_0) = d^-_{A_0\cup A_1}(Z)$; by the definition of Z and Z_0, no arc of A_1 enters Z or Z_0; therefore $d^-_{A_0}(Z_0) = d^-_{A_0}(Z)$.

Then using (12), for each $Z_0 \subseteq W_F(V(C_0))$ such that $Z := Z_0 \cap V(C_0) \ne \emptyset$ and $W_F(Z_0 \setminus V(C_0)) = Z_0 \setminus V(C_0)$, we have

$$d^-_{A\cup A_0}(Z_0) = d^-_A(Z_0) + d^-_{A_0}(Z_0) = d^-_A(Z_0) + d^-_{A_0}(Z) \ge r_M(S_{W_F(V(C_0))}) - r_M(S_{Z_0}). \tag{13}$$

Lemma 2.6. *Suppose $v \in V(C_0)$ and $v \in X_0 \subseteq W_F(v)$. Then we have*

$$d^-_{A\cup A_0\cup A_1}(X_0) \ge r_M(S_{W_F(X_0)}) - r_M(S_{X_0}). \tag{14}$$

Proof. Since $v \in V(C_0)$, then $X_0 \subseteq W_F(v) = W_F(V(C_0))$, and $v \in X_0 \cap V(C_0) \ne \emptyset$. By (13), it suffices to consider the case where $Y := X_0 \setminus V(C_0)$ and $Y \subsetneqq W_F(Y)$.

Since $Y \subseteq V(F_1)$, as noted before, $W_F(Y) \subseteq V(F_1)$. Then $X_0 \cap W_F(Y) \subseteq X_0 \cap \overline{V(C_0)} \subseteq Y \subseteq X_0 \cap W_F(Y)$, this gives $X_0 \cap W_F(Y) = Y$. Let $X := X_0 \cap V(C_0)$, then $X_0 \cup W_F(Y) = X \cup W_F(Y)$. Combining that r_M and $d^-_{A\cup A_0\cup A_1}$ are submodular, we have

$$\begin{aligned} &(r_M(S_{W_F(X_0)}) - r_M(S_{X_0}) - d^-_{A\cup A_0\cup A_1}(X_0)) \\ &+ (r_M(S_{W_F(Y)}) - r_M(S_{W_F(Y)}) - d^-_{A\cup A_0\cup A_1}(W_F(Y))) \\ \le\ &(r_M(S_{W_F(X_0)}) - r_M(S_{X\cup W_F(Y)}) - d^-_{A\cup A_0\cup A_1}(X \cup W_F(Y))) \\ &+ (r_M(S_{W_F(Y)}) - r_M(S_Y) - d^-_{A\cup A_0\cup A_1}(Y)). \end{aligned} \tag{15}$$

Note that $W_F(X_0) = W_F(V(C_0))$, apply (13) (with $Z_0 = X \cup W_F(Y)$), we have

$$r_M(S_{W_F(X_0)}) - r_M(S_{X\cup W_F(Y)}) - d^-_{A\cup A_0\cup A_1}(X \cup W_F(Y)) \le 0.$$

Apply (11) to Y, we have

$$r_M(S_{W_F(Y)}) - r_M(S_Y) - d^-_{A\cup A_0\cup A_1}(Y) \leq 0.$$

Notice that $d^-_{A\cup A_0\cup A_1}(W_F(Y)) = 0$. Thus (15) gives

$$r_M(S_{W_F(X_0)}) - r_M(S_{X_0}) - d^-_{A\cup A_0\cup A_1}(X_0) \leq 0.$$

This proves the lemma. ∎

We are ready to show $(iii) \Rightarrow (i)$ by applying Lemma 2.2:

Suppose $X_0 \subseteq V(F)$, and for some $v \in V(F)$, $v \in X_0 \subseteq W_F(v)$. If $X_0 \subseteq V(F_1)$, then by (11), (1) holds. Since $V(F) = V(F_1) \cup V(C_0)$, the only left case is $X_0 \cap V(C_0) \neq \emptyset$. Notice that, in this case, the above v satisfies $v \in V(C_0)$. (Otherwise, $v \in V(F_1)$; but then $W_F(v) \subseteq V(F_1)$; it follows that $X_0 \subseteq W_F(v) \subseteq V(F_1)$; this contradicts $X_0 \cap V(C_0) \neq \emptyset$.) Then by Lemma 2.6, (1) holds. Apply Lemma 2.2, there exists a maximal M-independent packing of mixed arborescences in $(F = (V; E, A), M, S, \pi)$. This finishes the proof. ∎

Remarks on the complexity: Frank [10] showed that the problem of covering an intersecting supermodular function by orienting edges can be solved in polynomial time. Hence, we can orient all strong components C of F such that the obtained digraph D covers f_C in polynomial time. Then, according to the polynomial-time algorithm given in [16], a maximal M-independent packing of mixed arborescences in F can be found in polynomial time.

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