

Optimisation Combinatoire 2A

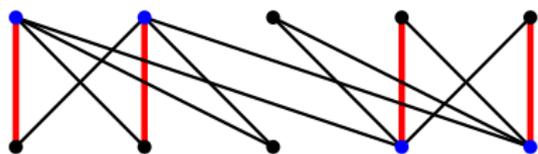
Couplages dans les graphes bipartis

Zoltán Szigeti

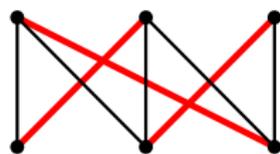
Ensimag
Grenoble INP

Définitions : $G = (V, E)$

- 1 **Couplage** : $M \subseteq E$ tel que $d_M(v) \leq 1 \forall v \in V$.
- 2 **Couplage parfait** : $M \subseteq E$ tel que $d_M(v) = 1 \forall v \in V$.
- 3 **Transversal** : $T \subseteq V$ tel que $T \cap \{u, v\} \neq \emptyset \forall uv \in E$.
- 4 $\nu(G)$:= $\max\{|M| : M \text{ couplage de } G\}$.
- 5 $\tau(G)$:= $\min\{|T| : T \text{ transversal de } G\}$.



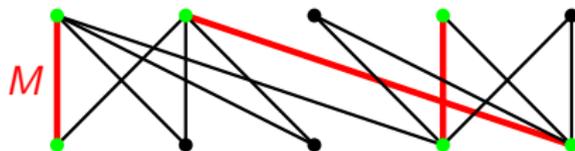
$$\nu(G_1) = 4 = \tau(G_1)$$



$$\nu(G_2) = 3 = \tau(G_2)$$

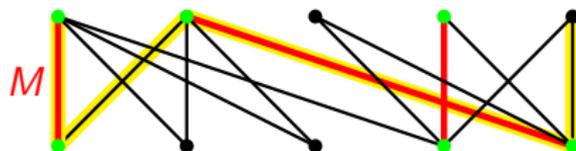
Définitions : $G = (V, E)$

- 1 Sommet **M -saturé** : $v \in V$ tel que $d_M(v) = 1$.
- 2 Sommet **M -insaturé** : $v \in V$ tel que $d_M(v) = 0$.
- 3 Chaîne **M -alternée** : si ses arêtes sont alternées en M et en $E \setminus M$.
- 4 Chaîne **M -augmentante** : si M -alternée d'extrémités M -insaturées.
- 5 **Graphe biparti** : $V = A \cup B, E(A) = \emptyset = E(B)$.



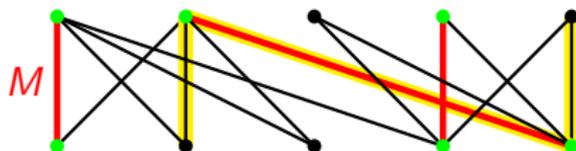
Définitions : $G = (V, E)$

- 1 Sommet **M -saturé** : $v \in V$ tel que $d_M(v) = 1$.
- 2 Sommet **M -insaturé** : $v \in V$ tel que $d_M(v) = 0$.
- 3 Chaîne **M -alternée** : si ses arêtes sont alternées en M et en $E \setminus M$.
- 4 Chaîne **M -augmentante** : si M -alternée d'extrémités M -insaturées.
- 5 **Graphe biparti** : $V = A \cup B, E(A) = \emptyset = E(B)$.



Définitions : $G = (V, E)$

- 1 Sommet **M -saturé** : $v \in V$ tel que $d_M(v) = 1$.
- 2 Sommet **M -insaturé** : $v \in V$ tel que $d_M(v) = 0$.
- 3 Chaîne **M -alternée** : si ses arêtes sont alternées en M et en $E \setminus M$.
- 4 Chaîne **M -augmentante** : si M -alternée d'extrémités M -insaturées.
- 5 **Graphe biparti** : $V = A \cup B, E(A) = \emptyset = E(B)$.



Définitions : $G = (V, E)$

- 1 Sommet **M -saturé** : $v \in V$ tel que $d_M(v) = 1$.
- 2 Sommet **M -insaturé** : $v \in V$ tel que $d_M(v) = 0$.
- 3 Chaîne **M -alternée** : si ses arêtes sont alternées en M et en $E \setminus M$.
- 4 Chaîne **M -augmentante** : si M -alternée d'extrémités M -insaturées.
- 5 **Graphe biparti** : $V = A \cup B, E(A) = \emptyset = E(B)$.

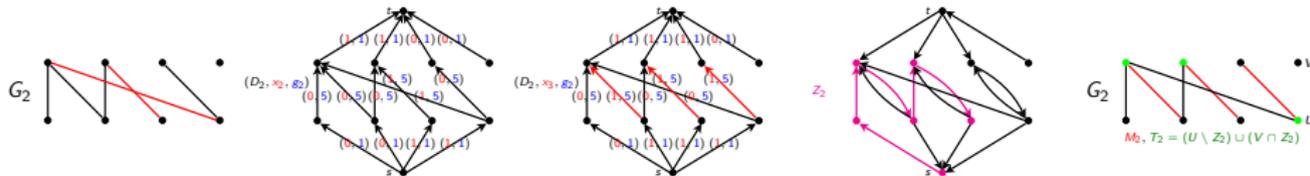


Théorèmes

- 1 **Berge** : Un couplage M est de cardinal maximum \iff il n'existe pas de chaîne M -augmentante.
- 2 **König** : Dans un graphe biparti G , $\nu(G) = \tau(G)$.

Remarques

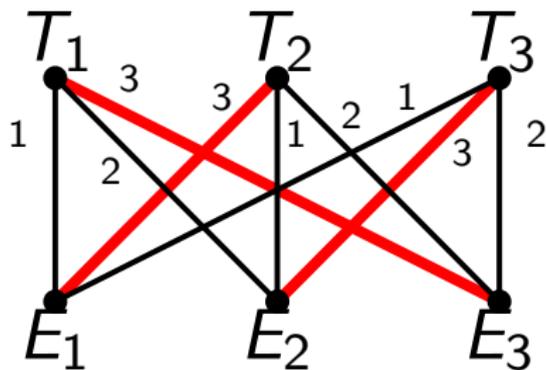
- 1 Théorème de Ford-Fulkerson implique Théorème de König.
- 2 Par l'algorithme d'Edmonds-Karp on trouve dans un graphe biparti
 - 1 un couplage de cardinal maximum et
 - 2 un ensemble transversal de cardinal minimum.



Problème de couplage de coût maximum

Problème

P_1 : Etant donné un graphe biparti $G = (U, V; E)$ et un coût c sur les arêtes, trouver un couplage M de G de coût $(\sum_{e \in M} c(e))$ maximum.



- P_1 couplage de coût maximum dans un graphe biparti.
- P_2 le coût est **non-négatif** : on enlève les arêtes de coût négatif car le couplage du coût maximum ne contient aucune arête de coût négatif.
- P_3 couplage **parfait** de coût maximum dans un graphe biparti **complet** : on ajoute des nouveaux sommets et des nouvelles arêtes de coût 0.
- P_4 couplage parfait de coût **minimum** dans un graphe biparti complet : on multiplie les coûts par -1 .
- P_5 couplage parfait de coût minimum dans un graphe biparti complet avec coût **non-négatif** : on ajoute à chaque coût $L :=$ valeur absolue maximale du coût d'une arête. Le nouveau coût est non-négatif et le coût de chaque couplage parfait a augmenté par constante $(\frac{n}{2}L)$.
- P_6 couplage parfait de coût minimum dans un graphe biparti **ayant un couplage parfait** avec coût non-négatif : plus général que P_5 .

Programmation Linéaire : $G = (U, V; E)$ biparti

Primal

$$\sum_{e \in \delta(w)} x(e) = 1 \quad \forall w \in U \cup V,$$

$$x(e) \geq 0 \quad \forall e \in E,$$

$$\sum_{e \in E} c(e)x(e) = w(\min).$$

Dual

$$y(u) + y(v) \leq c(uv) \quad \forall uv \in E,$$

$$\sum_{w \in U \cup V} y(w) = z(\max).$$

Théorème des écarts complémentaires

- 1 Si x et y sont des solutions réalisables de (P) et (D) et
- 2 les conditions des écarts complémentaires sont satisfaites :
$$x(uv) > 0 \implies y(u) + y(v) = c(uv) \quad (uv \text{ est } y\text{-serrée}).$$
- 3 alors x et y sont des solutions optimales de (P) et (D).

Algorithme pour trouver un couplage parfait de coût minimum dans un graphe biparti

Algorithme méthode hongroise (Kuhn)

Entrée : $G = (U, V; E)$ un graphe biparti qui admet un couplage parfait et c un coût non-négatif sur E .

Sortie : Un couplage parfait de G de c -coût minimum.

Idée : On aura à chaque étape :

- 1 un vecteur x (le vecteur caractéristique d'un couplage M),
- 2 une solution réalisable y du (D),
- 3 tels que les conditions des écarts complémentaires soient satisfaites :
 - $x(e) > 0 \implies e$ est y -serrée, c'est-à-dire
 - M est un couplage du graphe partiel induit par les arêtes y -serrées.

Algorithme méthode hongroise (Kuhn)

Etape 0. *Initialisation.*

$$M_0 := \emptyset, i := 1.$$

$$y_1(w) := \begin{cases} \min\{c(wv) : wv \in E\} & \text{si } w \in U, \\ 0 & \text{si } w \in V, \end{cases}$$

Etape 1. *Construction du graphe G_i des arêtes serrées.*

$$G_i := (U, V; E_i) \text{ où } E_i = \{uv \in E : y_i(u) + y_i(v) = c(uv)\}.$$

Etape 2. *Construction d'un couplage maximum et d'un transversal minimum dans G_i .*

En partant de M_{i-1} et en utilisant les flots, trouver un couplage M_i de G_i de cardinal maximum et un ensemble transversal T_i de G_i de cardinal minimum.

Etape 3. *Condition d'arrêt.*

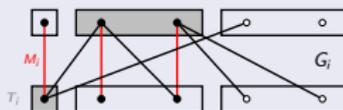
Si M_i est un couplage parfait de G_i , alors arrêter avec M_i .

Etape 4. *Changement de solution du dual.*

$$\varepsilon_i := \min\{c(uv) - y_i(u) - y_i(v) : uv \in E(G - T_i)\}$$

$$y_{i+1}(w) := \begin{cases} y_i(w) + \varepsilon_i & \text{si } w \in U \setminus T_i, \\ y_i(w) - \varepsilon_i & \text{si } w \in V \cap T_i, \\ y_i(w) & \text{sinon.} \end{cases}$$

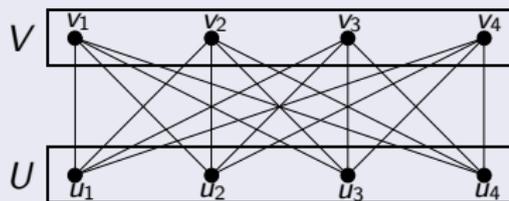
$i := i + 1$ et Aller à l'Etape 1.



Exercice 6.8

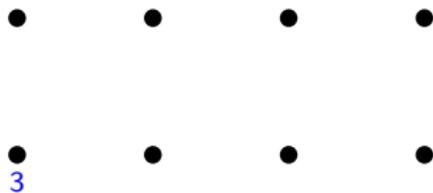
Énoncé

Exécuter l'Algorithme méthode hongroise pour trouver un couplage parfait de coût minimum dans le graphe biparti suivant.

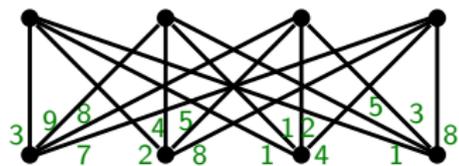


	u_1	u_2	u_3	u_4
v_1	3	2	1	1
v_2	9	4	1	5
v_3	8	5	2	3
v_4	7	8	4	8

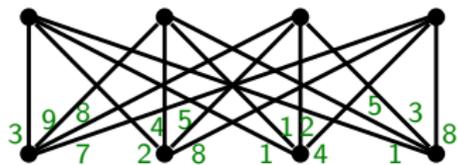
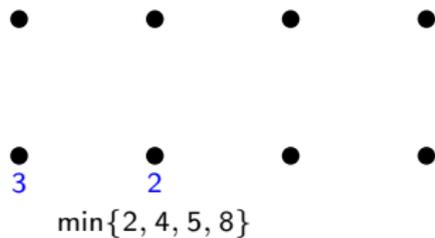
Exercise 6.8



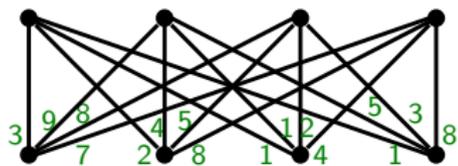
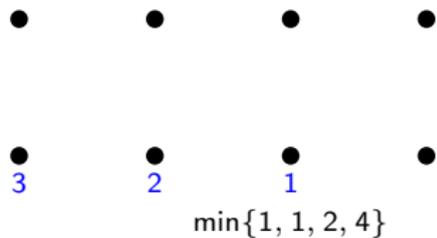
$$\min\{3, 9, 8, 7\}$$



Exercise 6.8



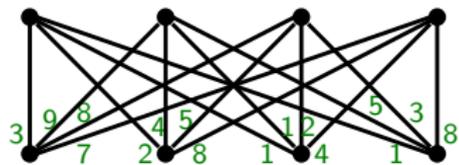
Exercise 6.8



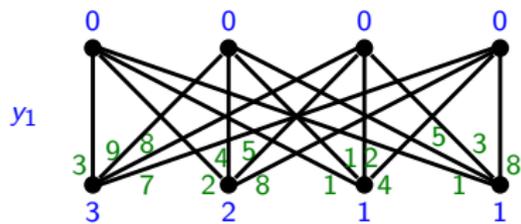
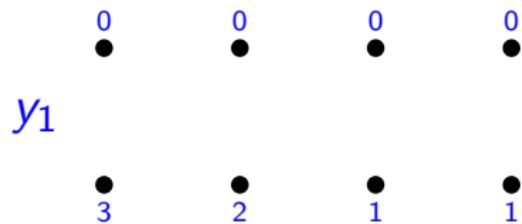
Exercise 6.8



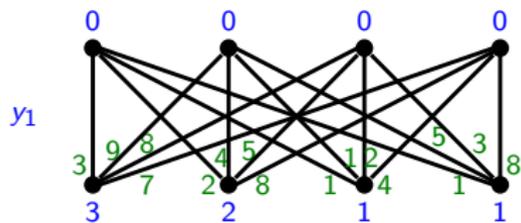
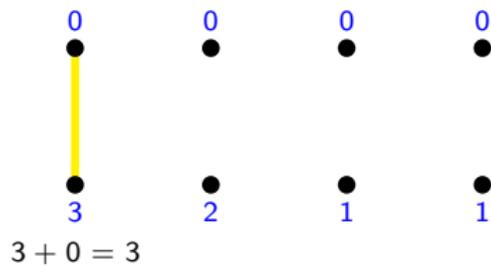
$\min\{1, 5, 3, 8\}$



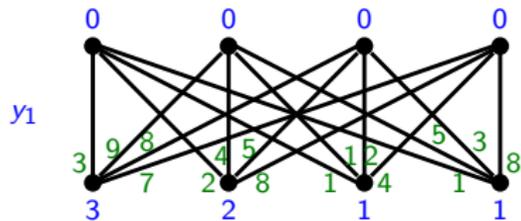
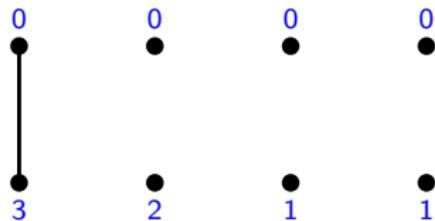
Exercise 6.8



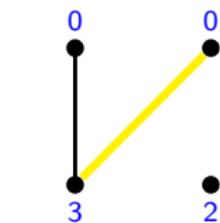
Exercise 6.8



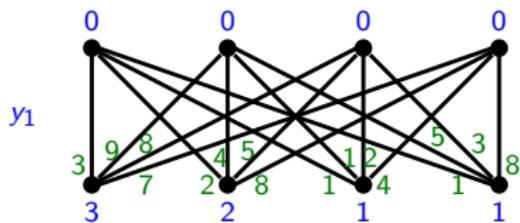
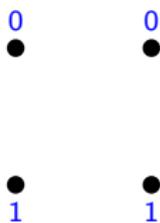
Exercise 6.8



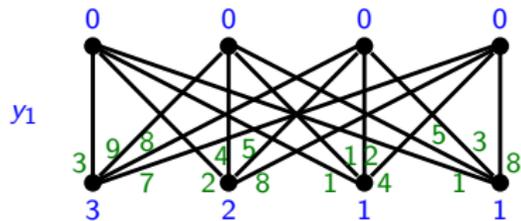
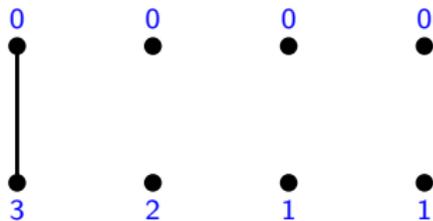
Exercise 6.8



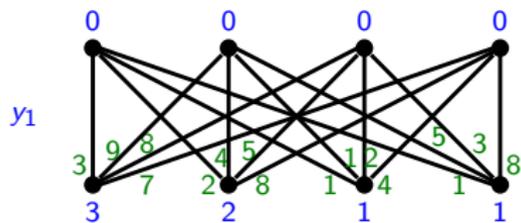
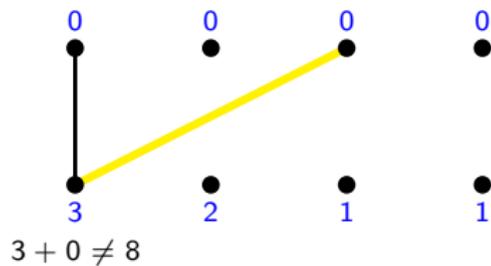
$$3 + 0 \neq 9$$



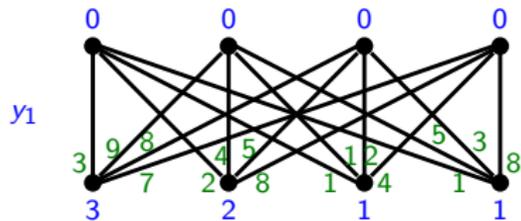
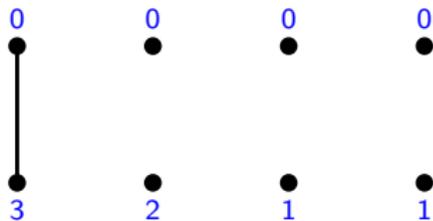
Exercise 6.8



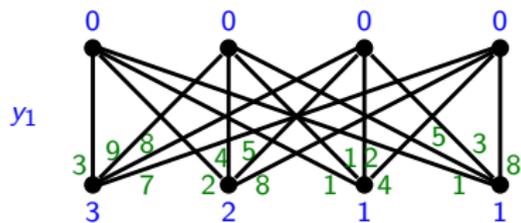
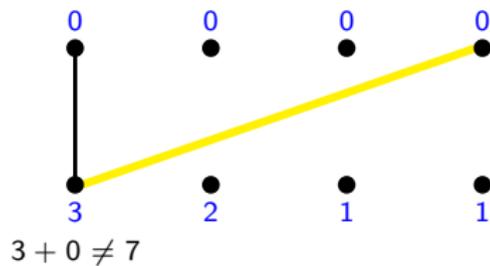
Exercise 6.8



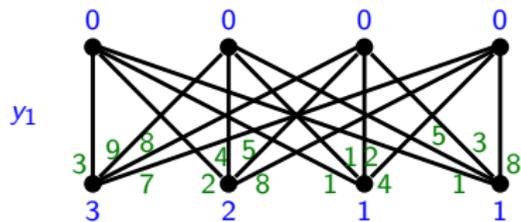
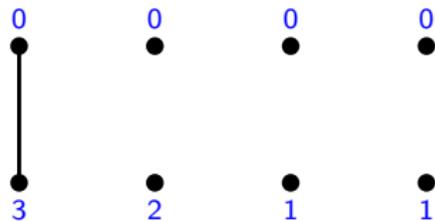
Exercise 6.8



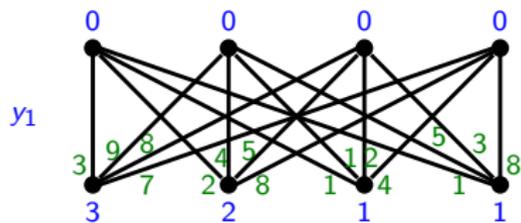
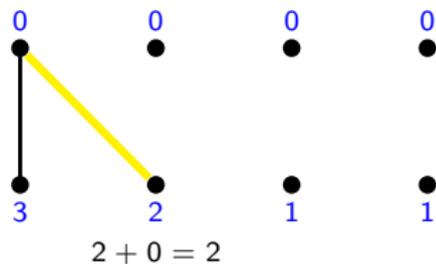
Exercise 6.8



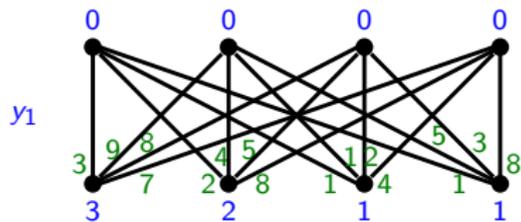
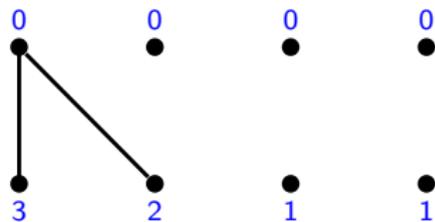
Exercise 6.8



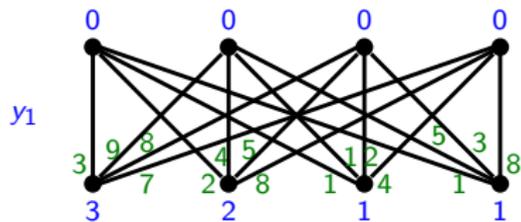
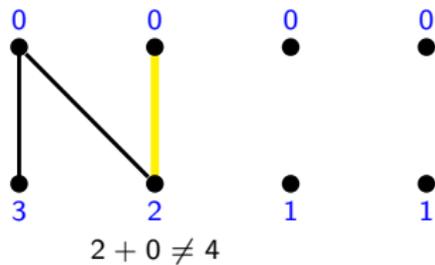
Exercise 6.8



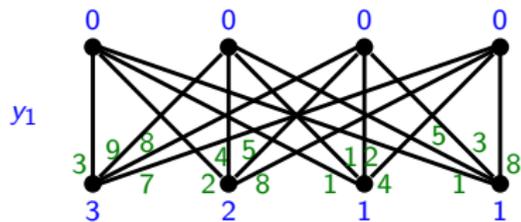
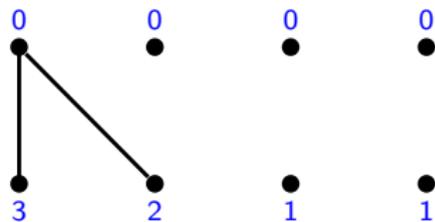
Exercise 6.8



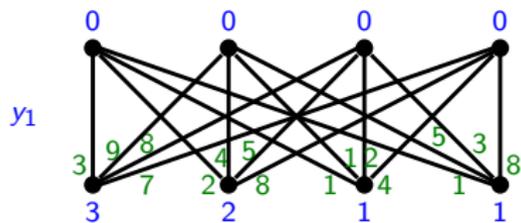
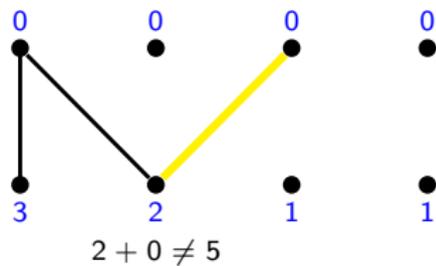
Exercise 6.8



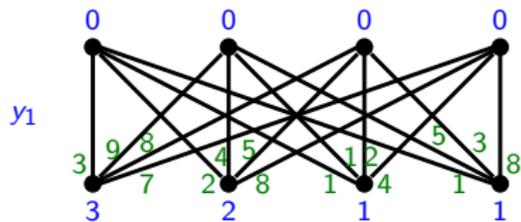
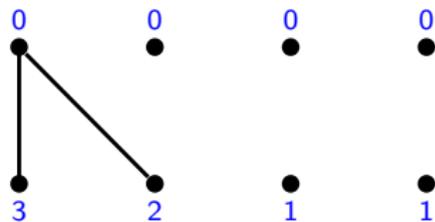
Exercise 6.8



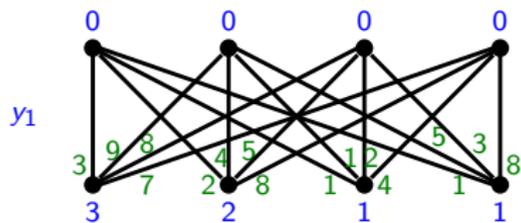
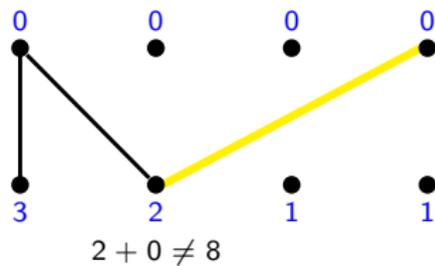
Exercise 6.8



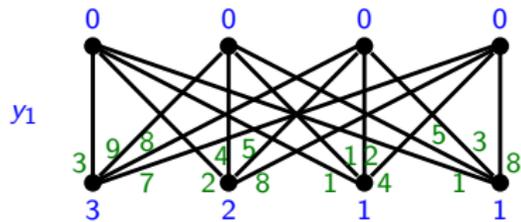
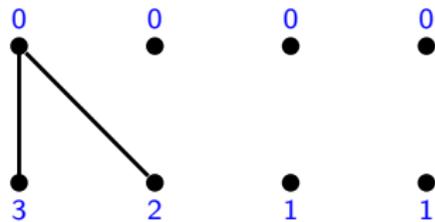
Exercise 6.8



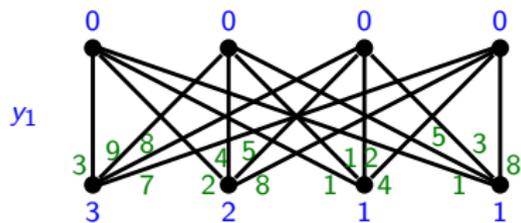
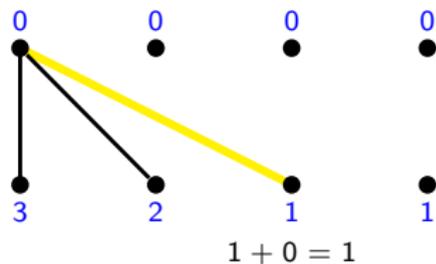
Exercise 6.8



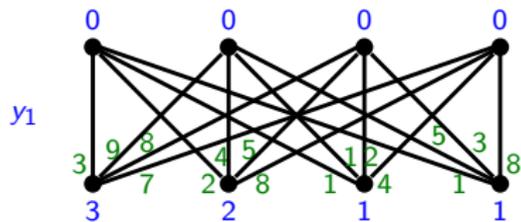
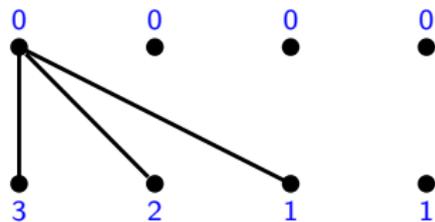
Exercise 6.8



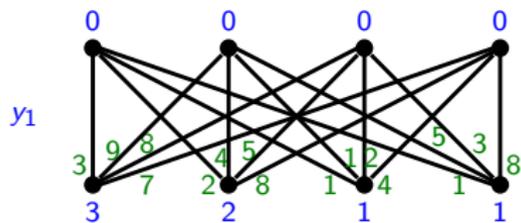
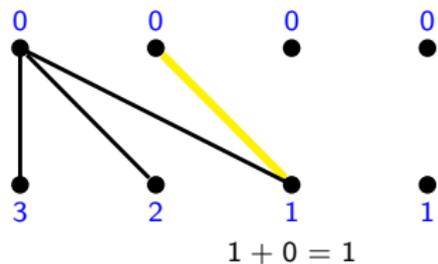
Exercise 6.8



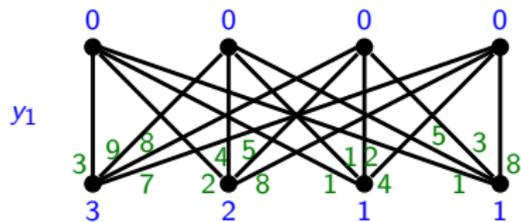
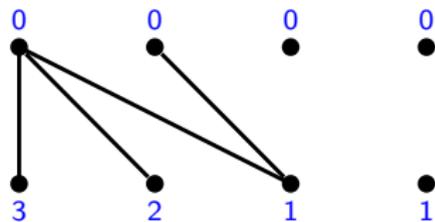
Exercise 6.8



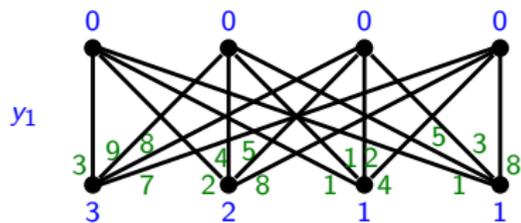
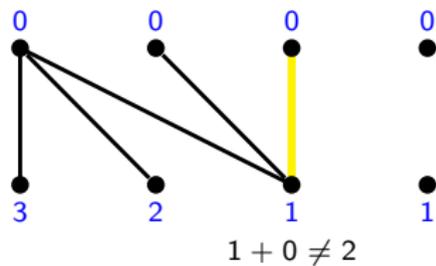
Exercise 6.8



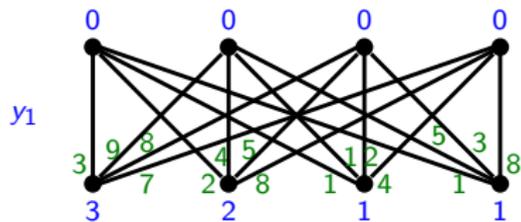
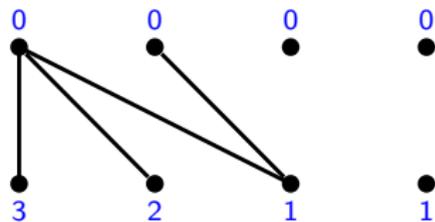
Exercise 6.8



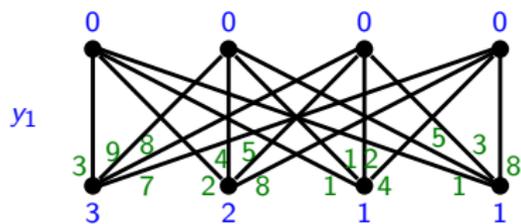
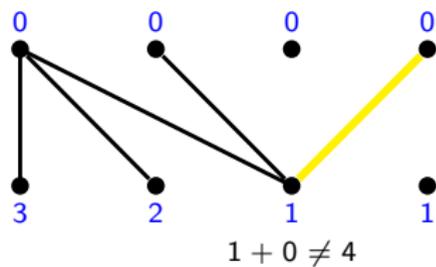
Exercise 6.8



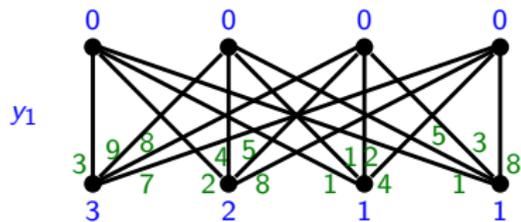
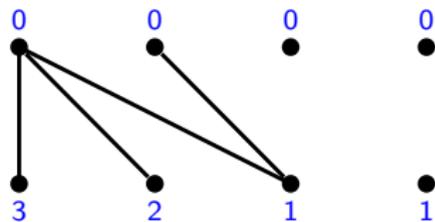
Exercise 6.8



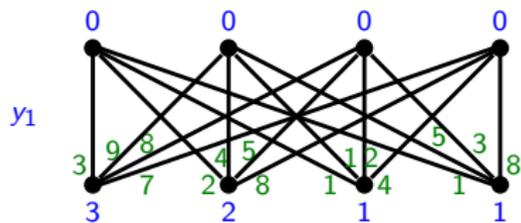
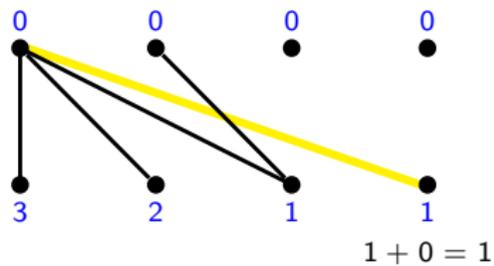
Exercise 6.8



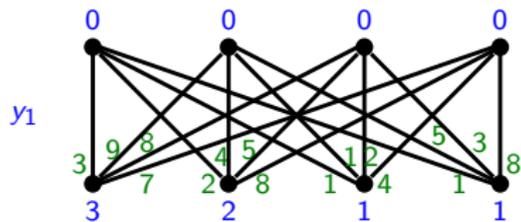
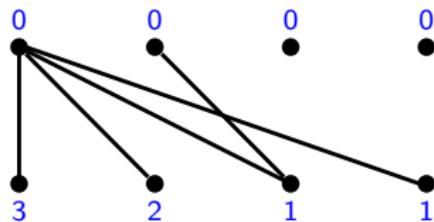
Exercise 6.8



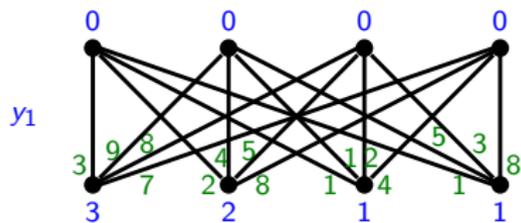
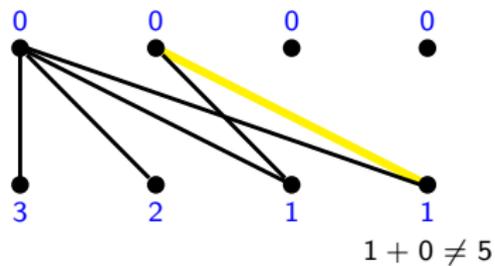
Exercise 6.8



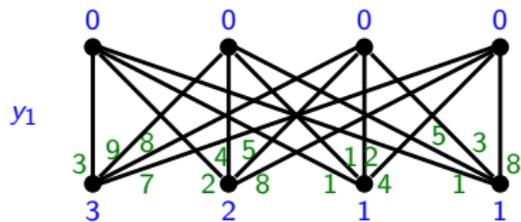
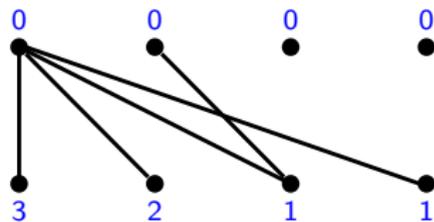
Exercise 6.8



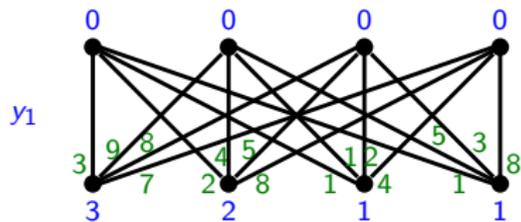
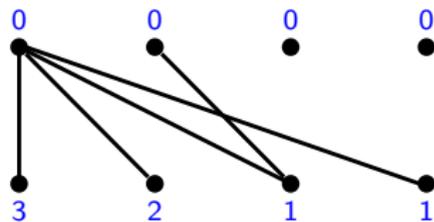
Exercise 6.8



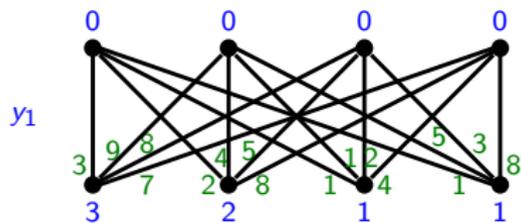
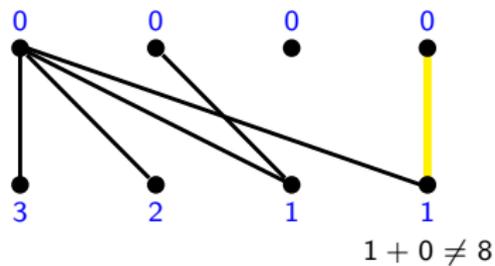
Exercise 6.8



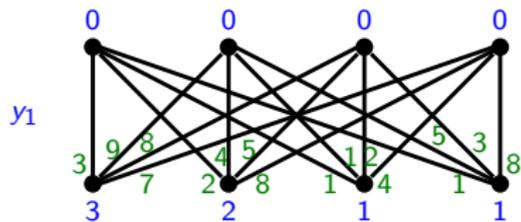
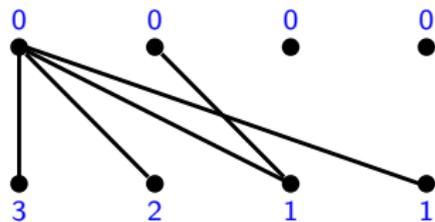
Exercise 6.8



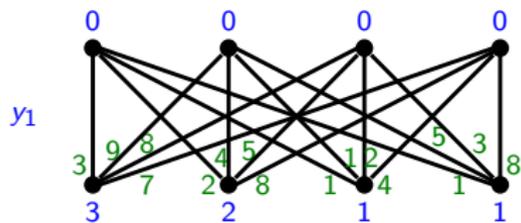
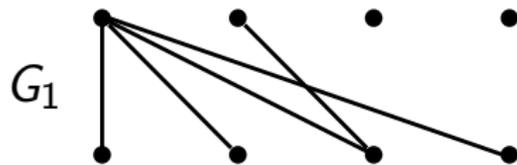
Exercise 6.8



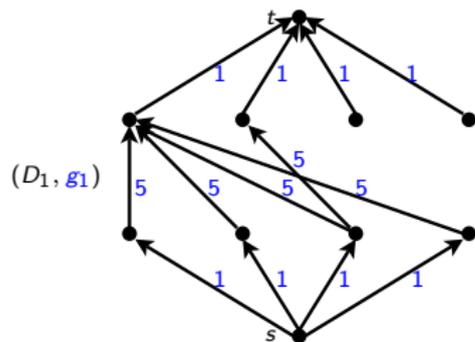
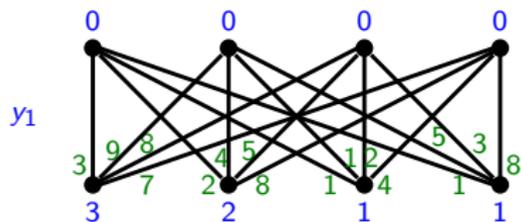
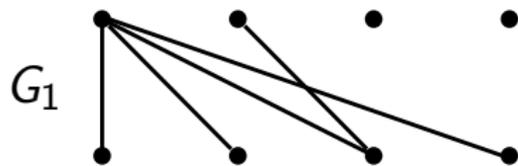
Exercise 6.8



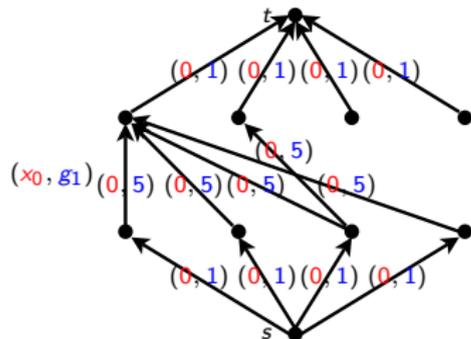
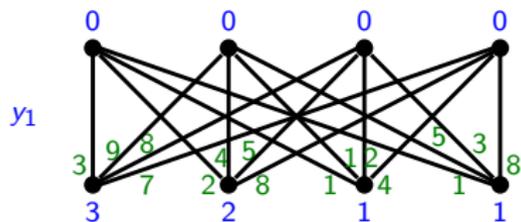
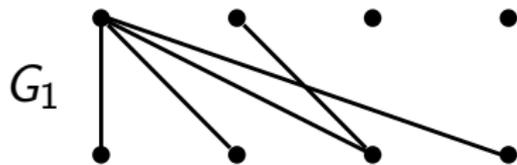
Exercise 6.8



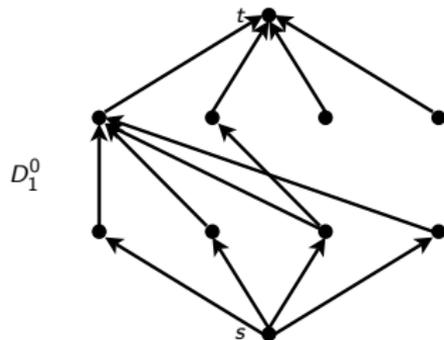
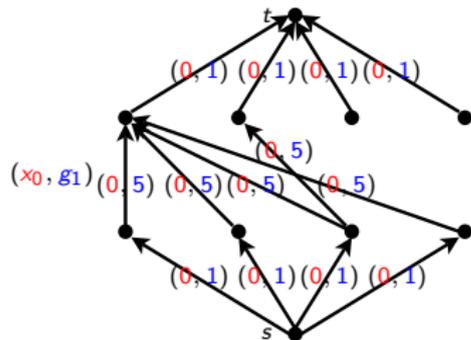
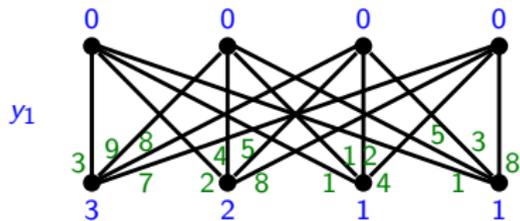
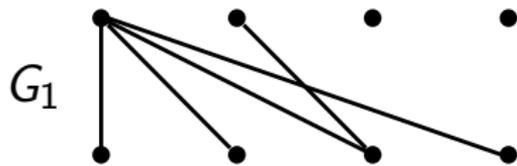
Exercise 6.8



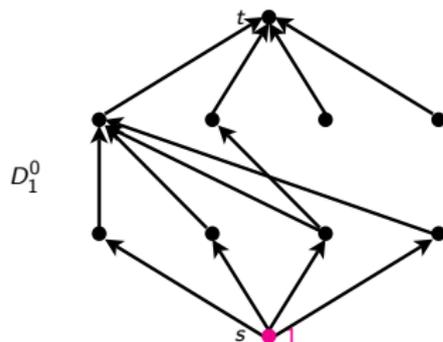
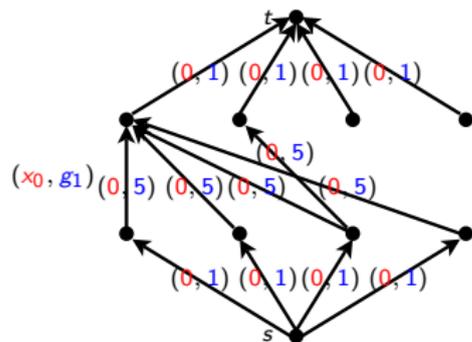
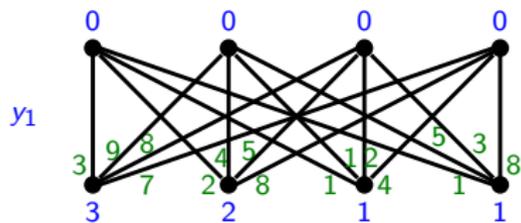
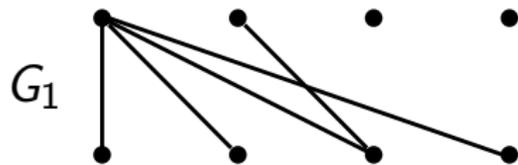
Exercise 6.8



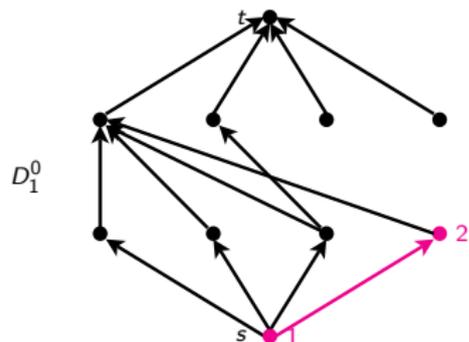
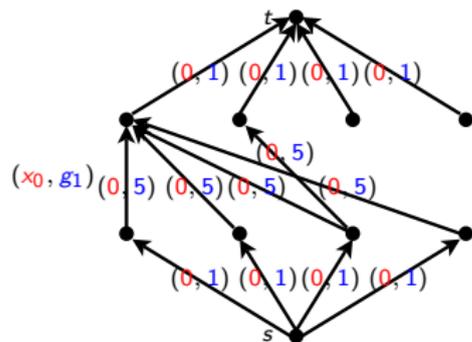
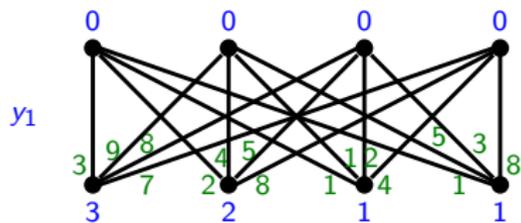
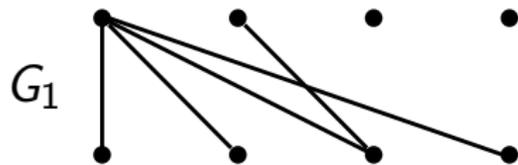
Exercise 6.8



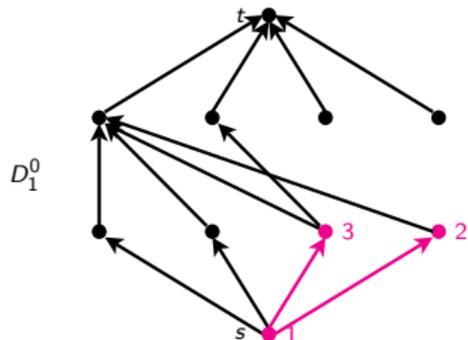
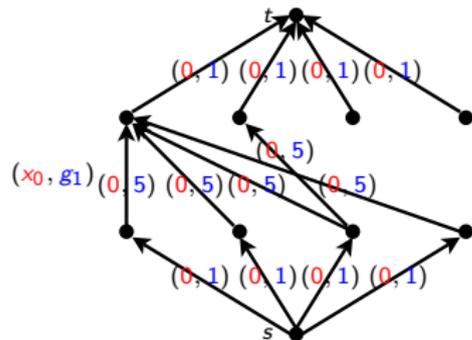
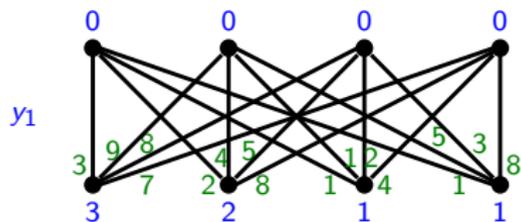
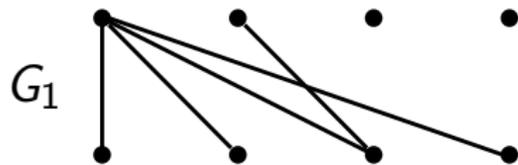
Exercise 6.8



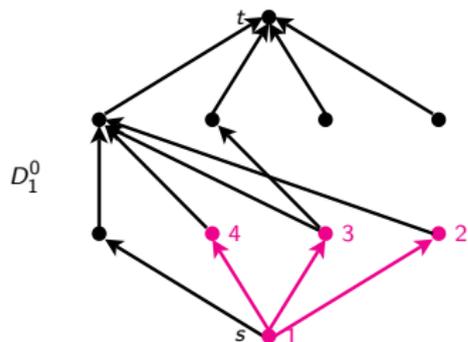
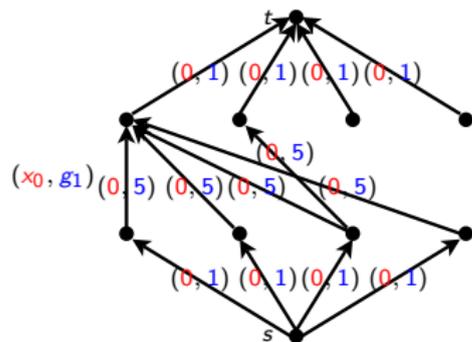
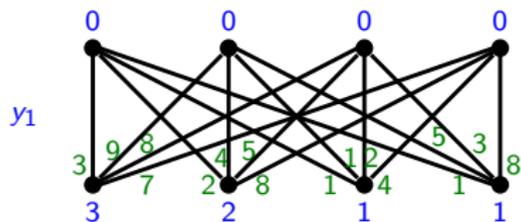
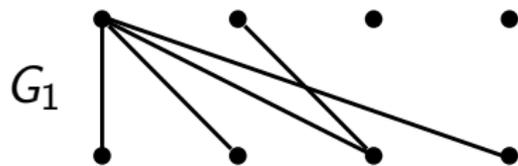
Exercise 6.8



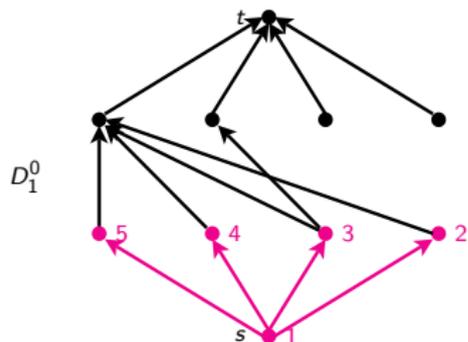
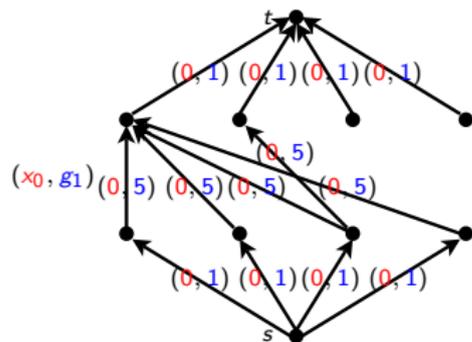
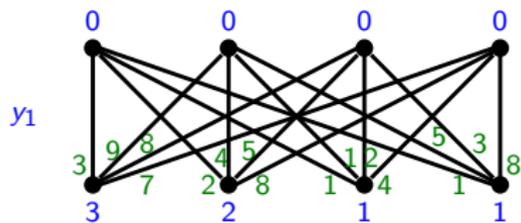
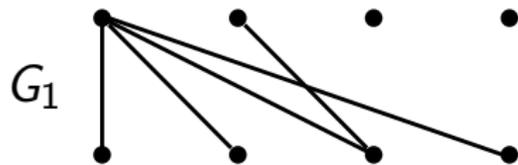
Exercise 6.8



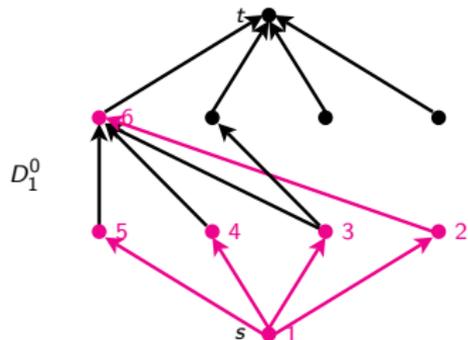
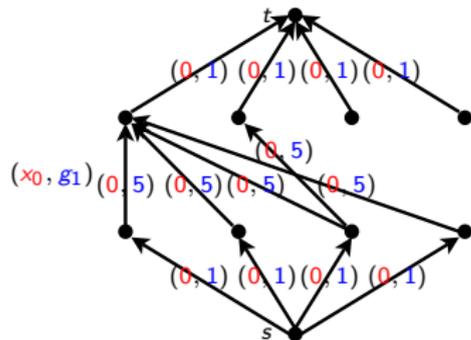
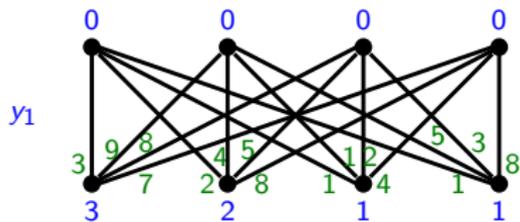
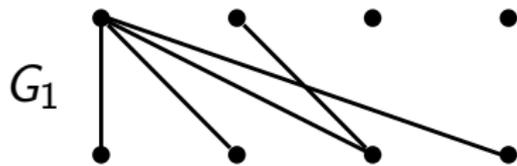
Exercise 6.8



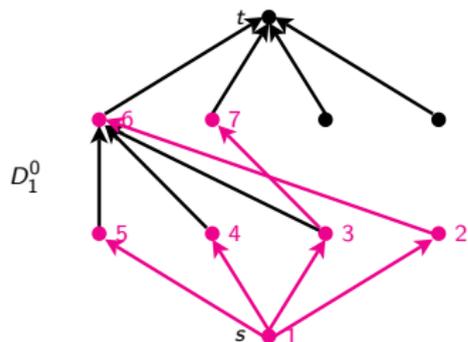
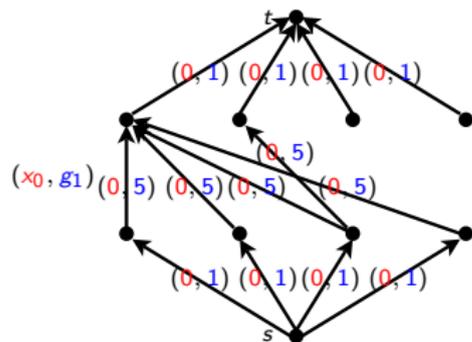
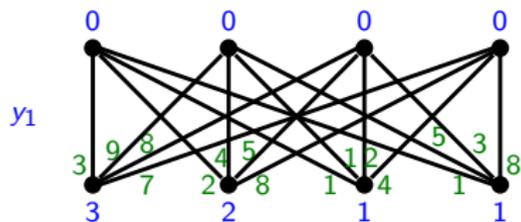
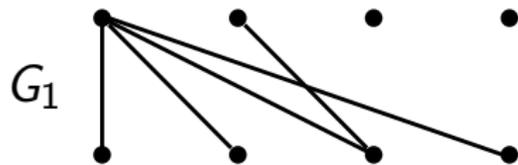
Exercise 6.8



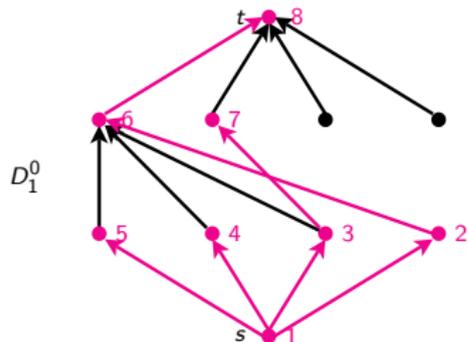
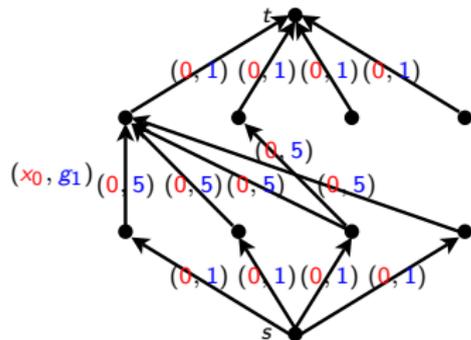
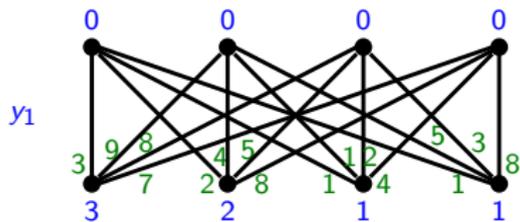
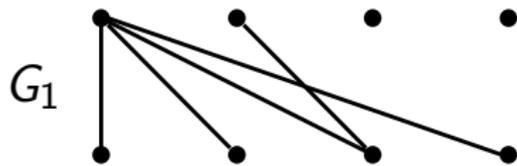
Exercise 6.8



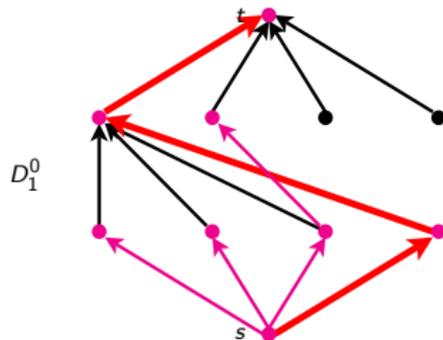
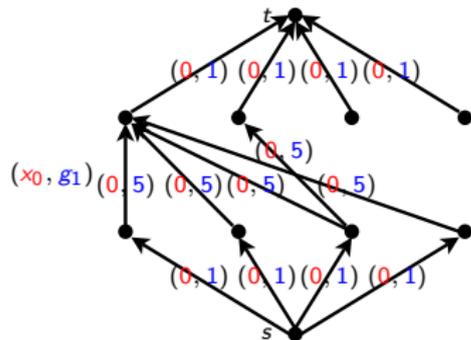
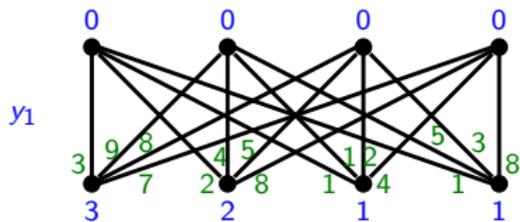
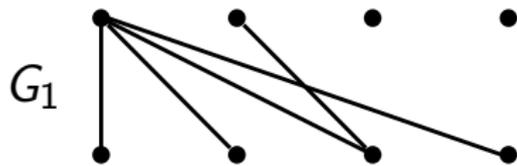
Exercise 6.8



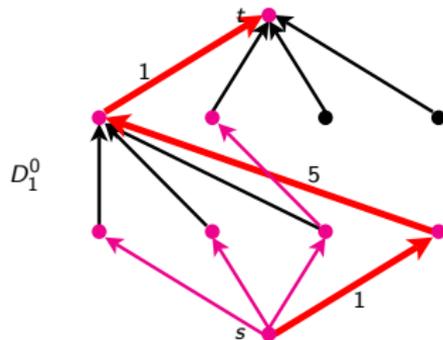
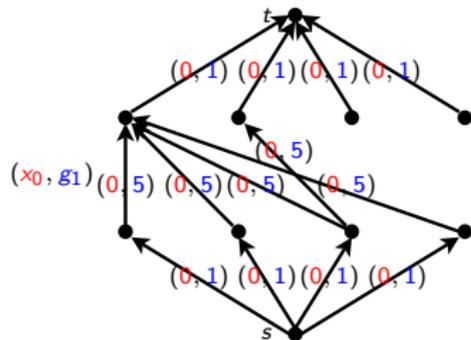
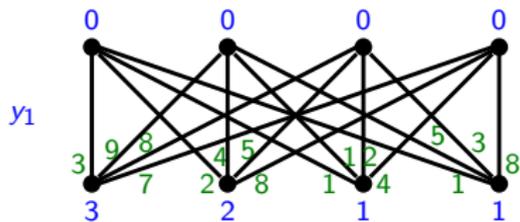
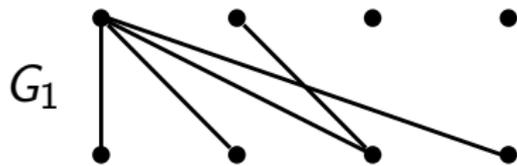
Exercise 6.8



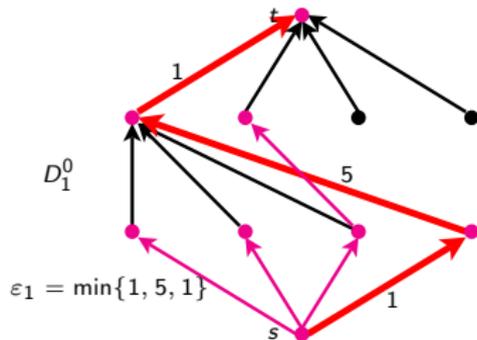
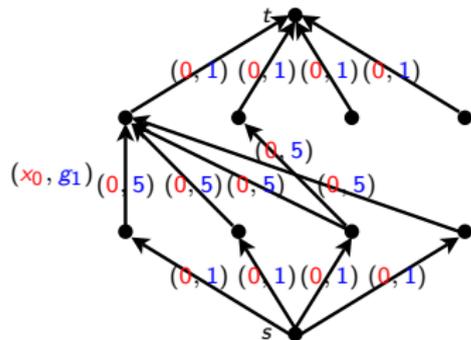
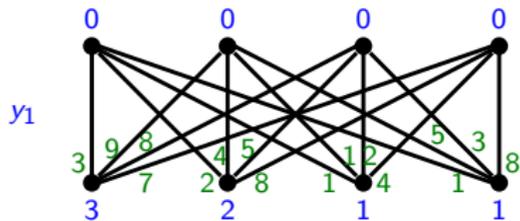
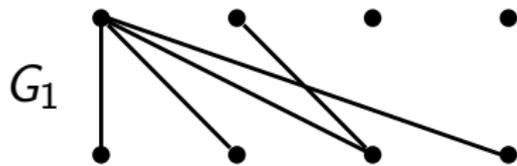
Exercise 6.8



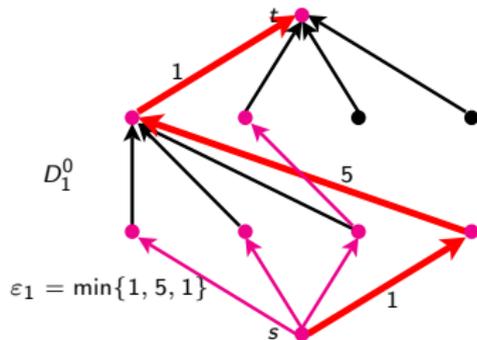
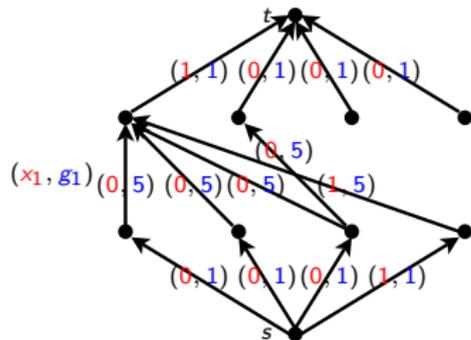
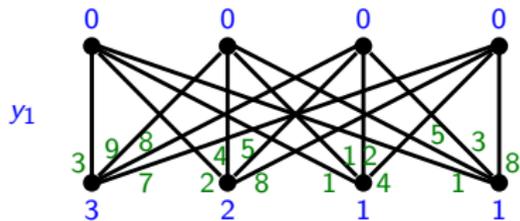
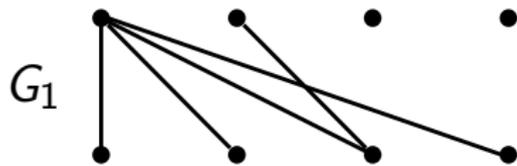
Exercise 6.8



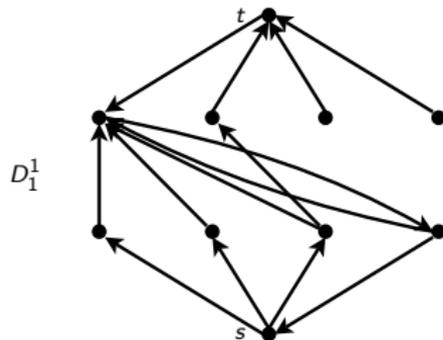
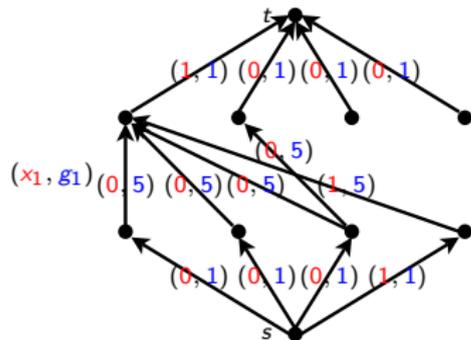
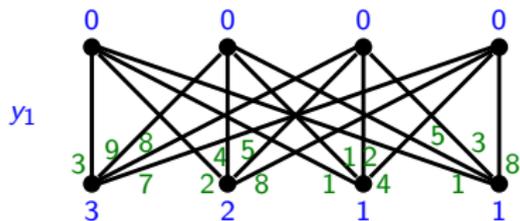
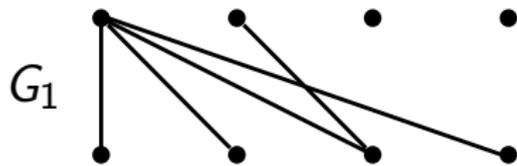
Exercise 6.8



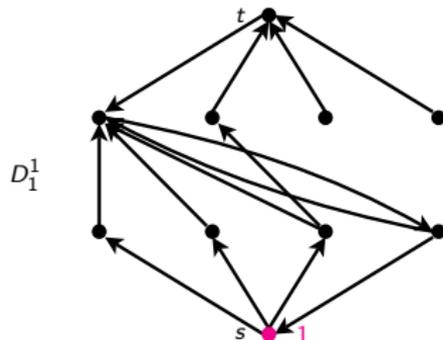
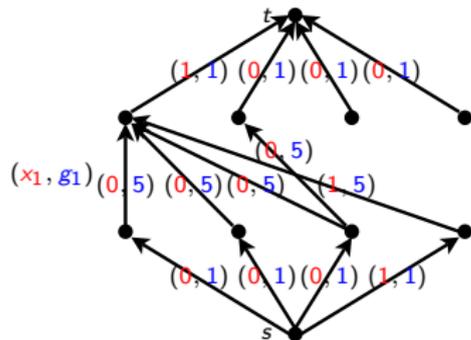
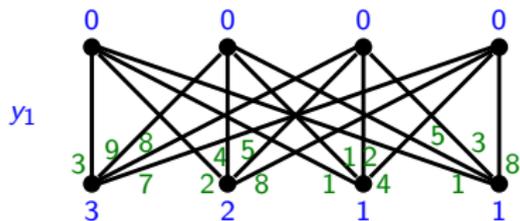
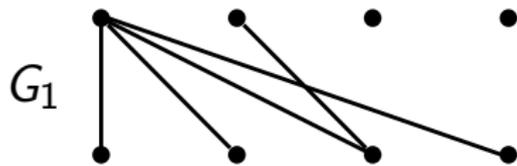
Exercise 6.8



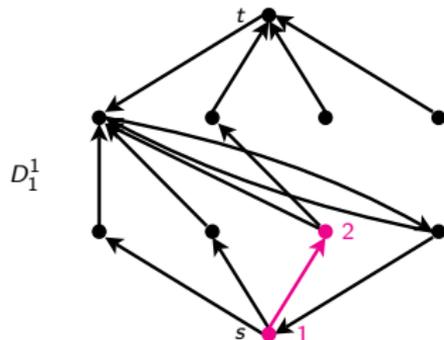
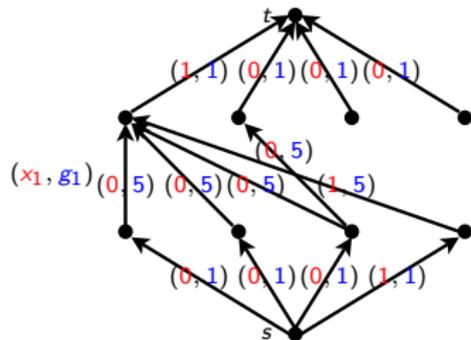
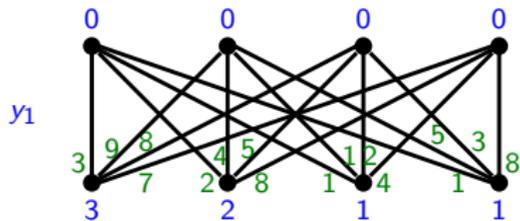
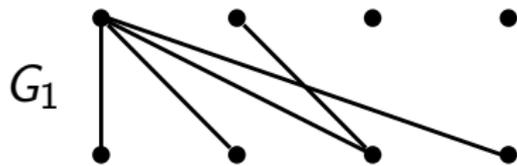
Exercise 6.8



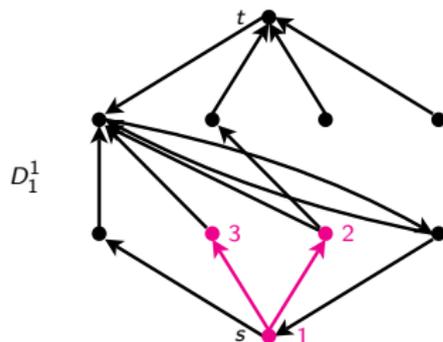
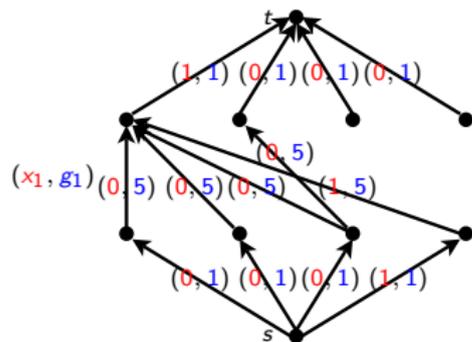
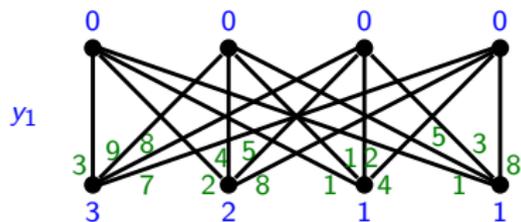
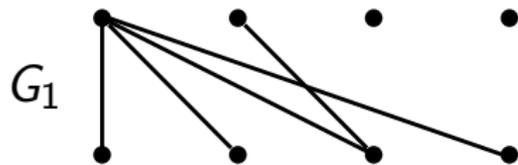
Exercise 6.8



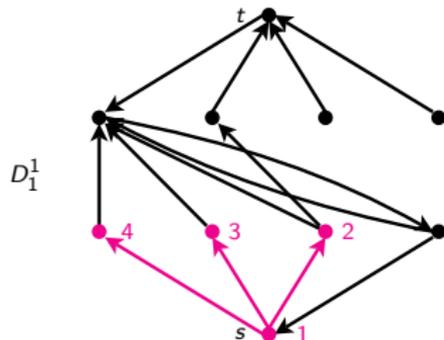
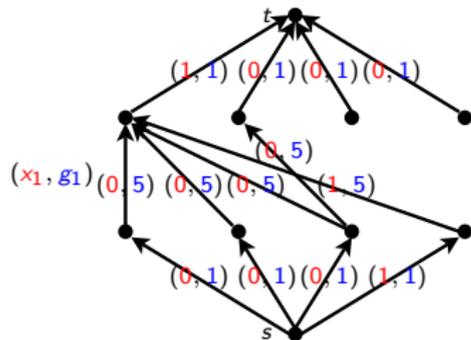
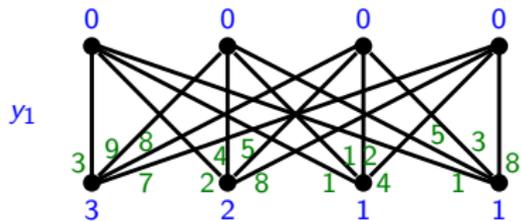
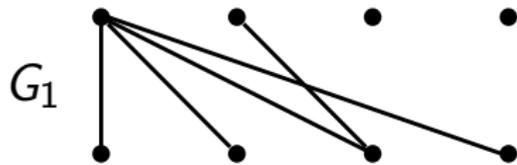
Exercise 6.8



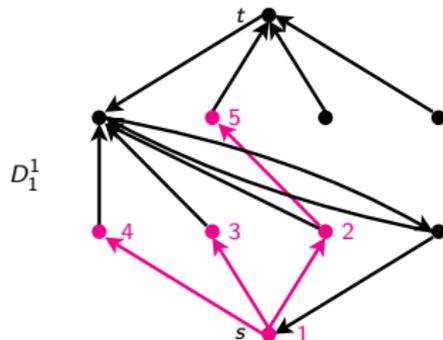
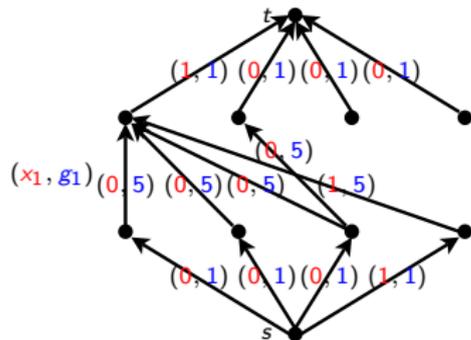
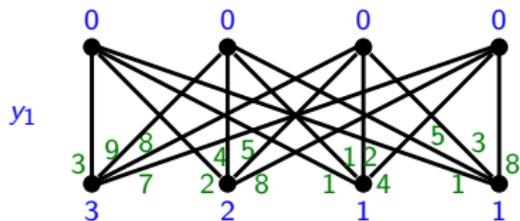
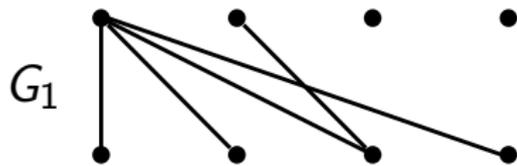
Exercise 6.8



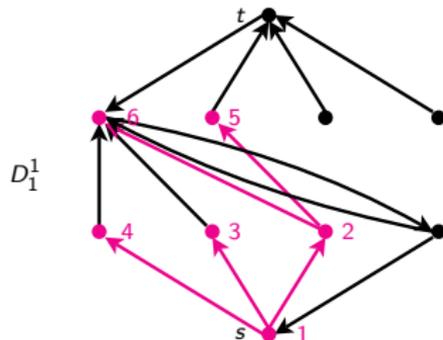
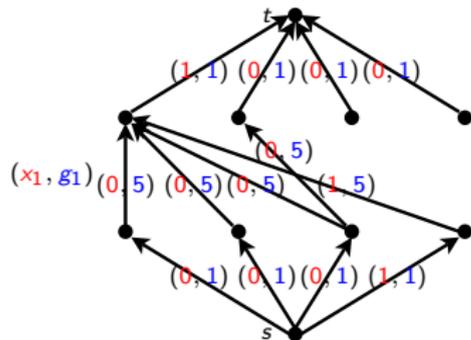
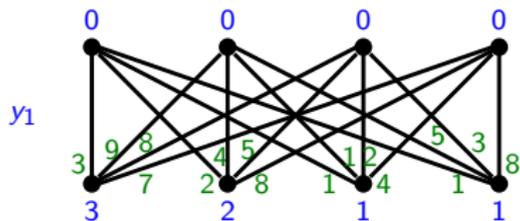
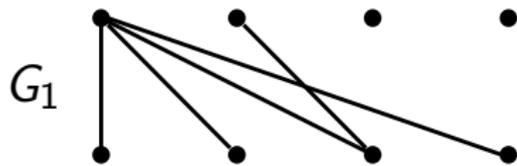
Exercise 6.8



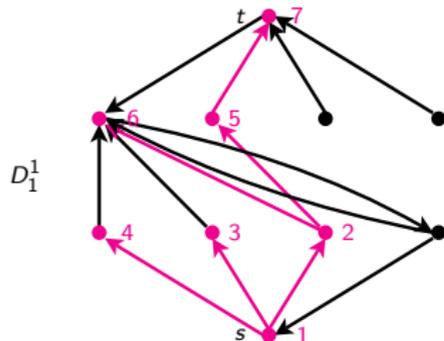
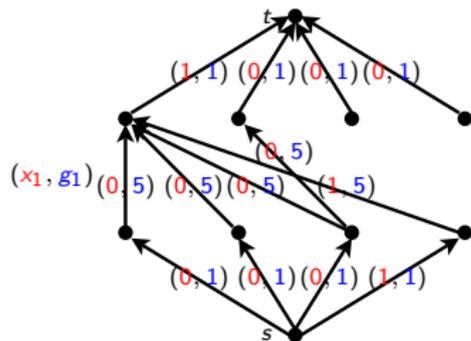
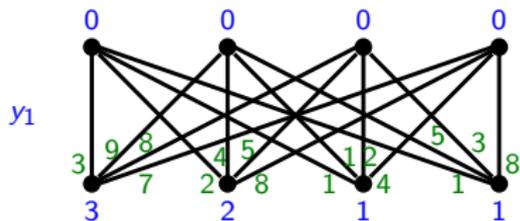
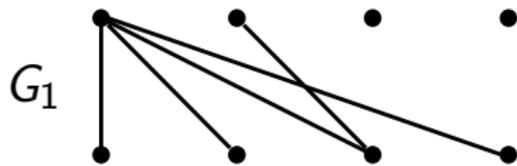
Exercise 6.8



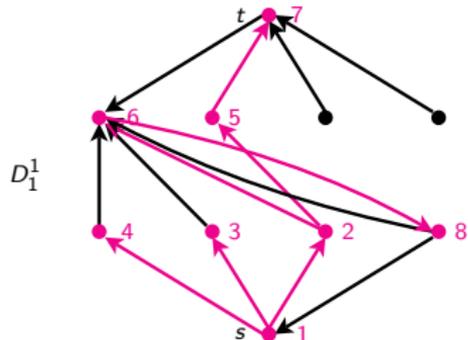
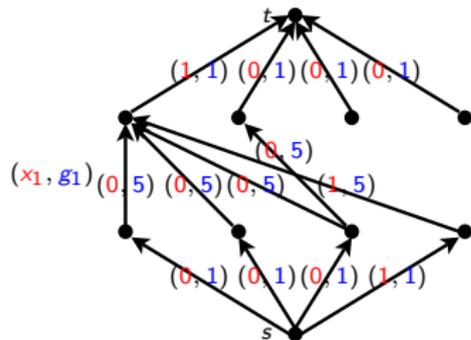
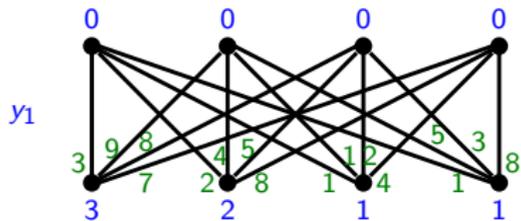
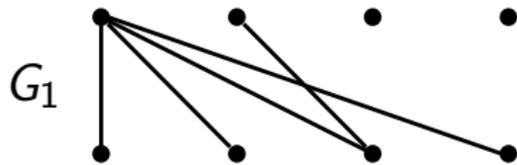
Exercise 6.8



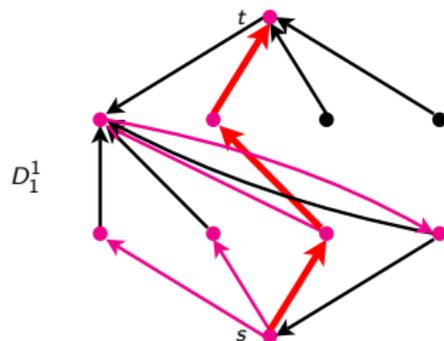
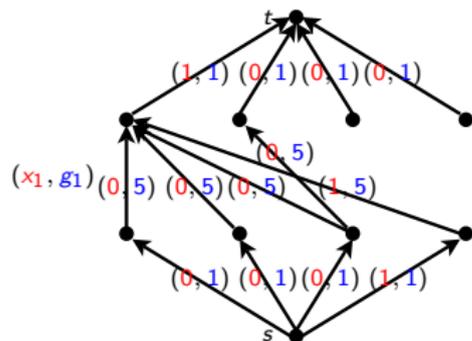
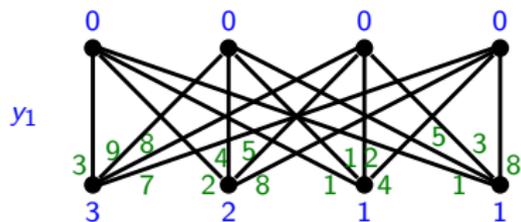
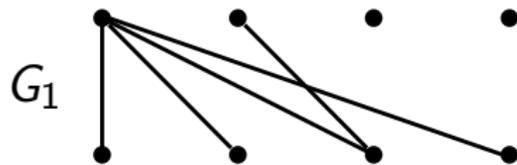
Exercise 6.8



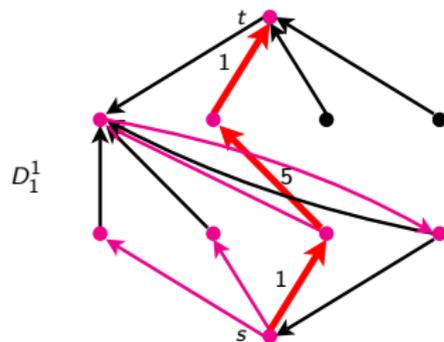
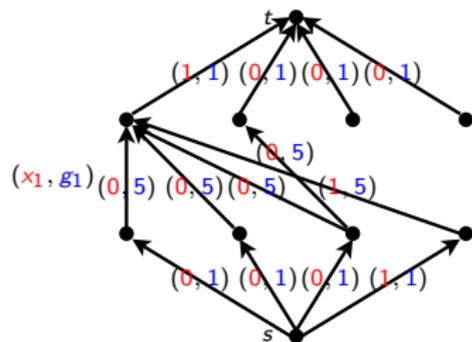
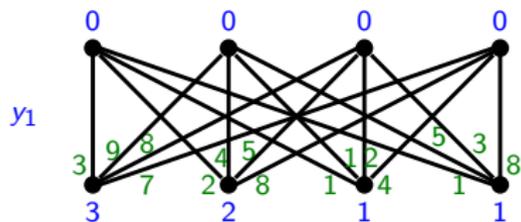
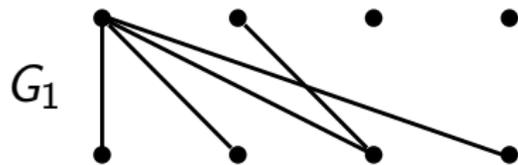
Exercise 6.8



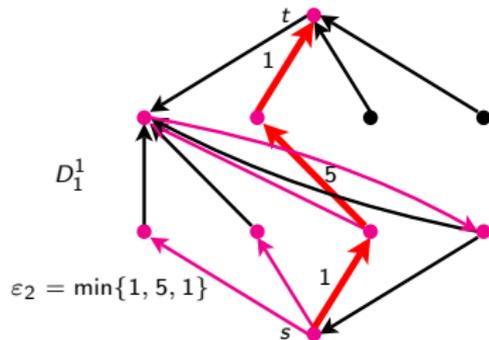
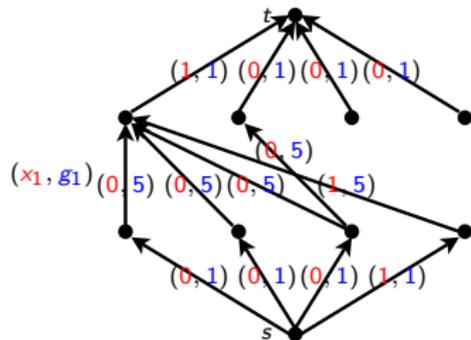
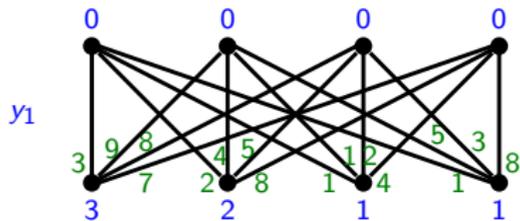
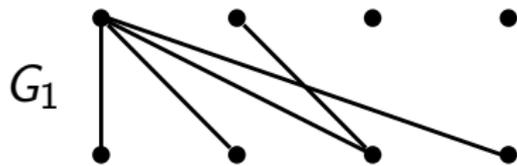
Exercise 6.8



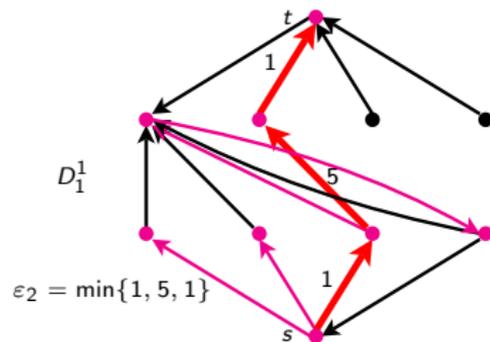
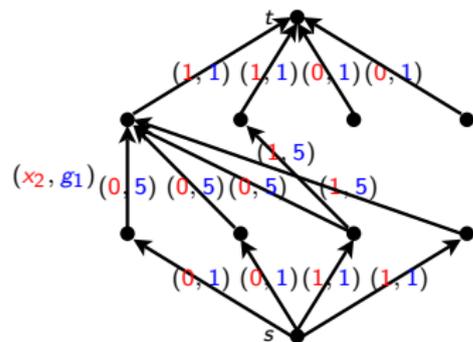
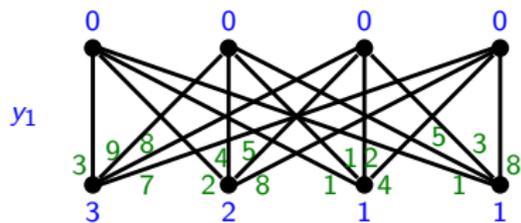
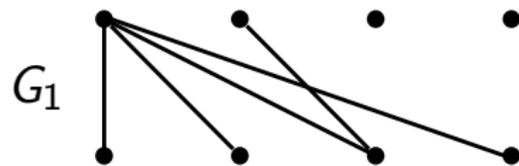
Exercise 6.8



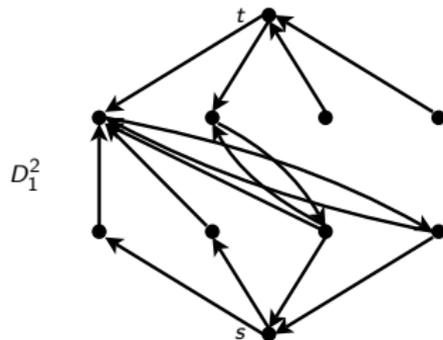
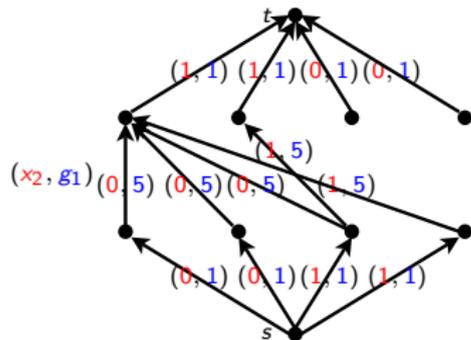
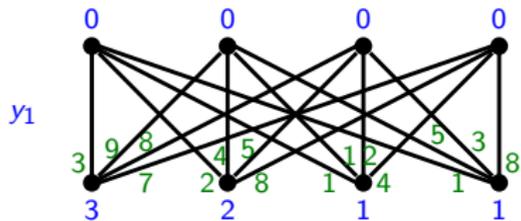
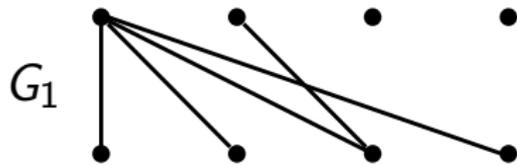
Exercise 6.8



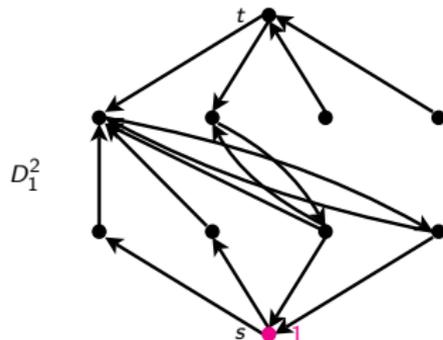
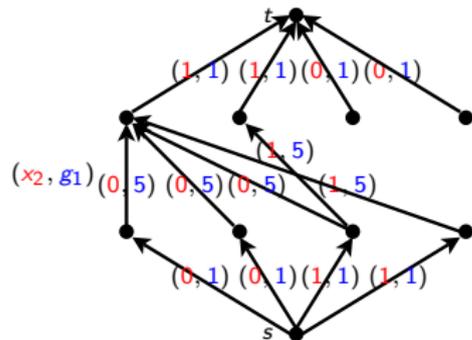
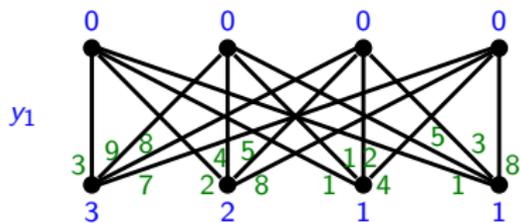
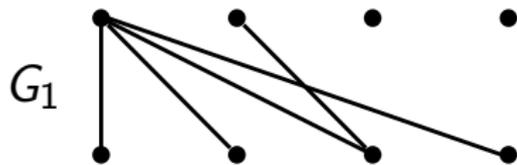
Exercise 6.8



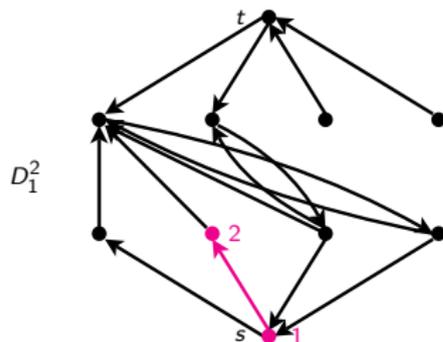
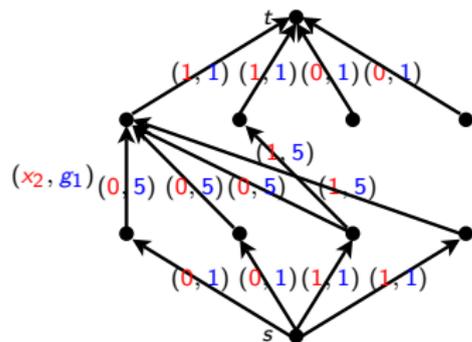
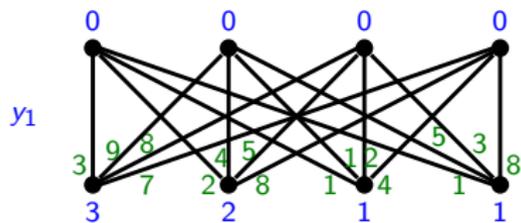
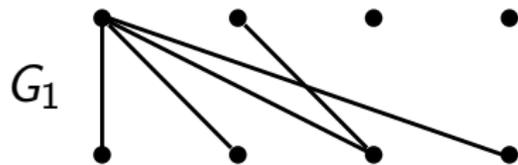
Exercise 6.8



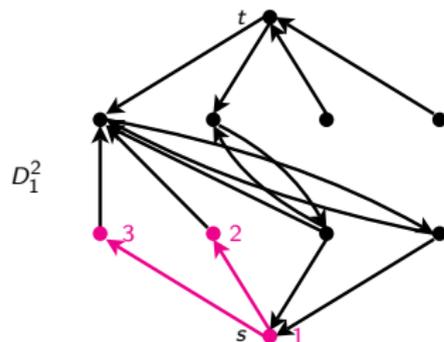
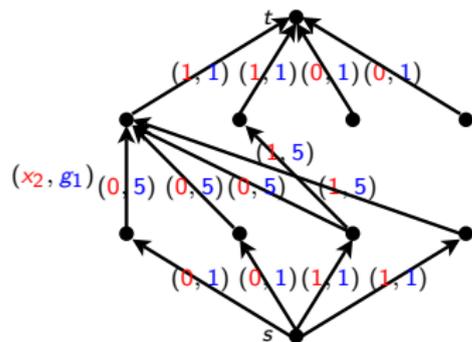
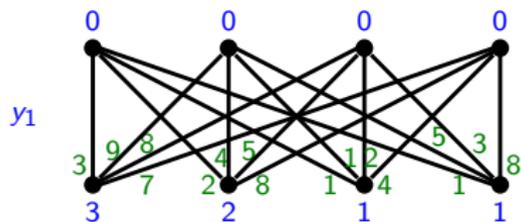
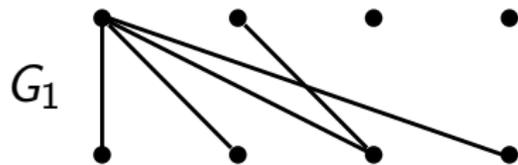
Exercise 6.8



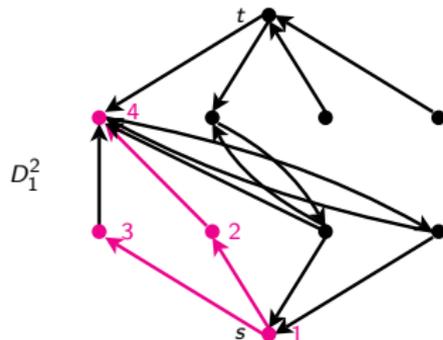
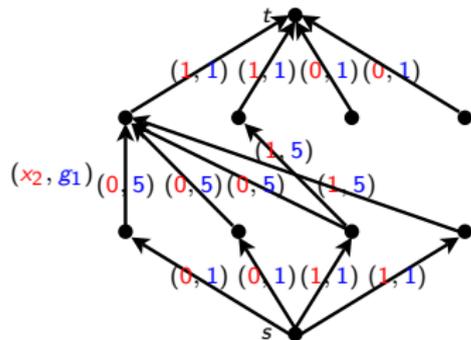
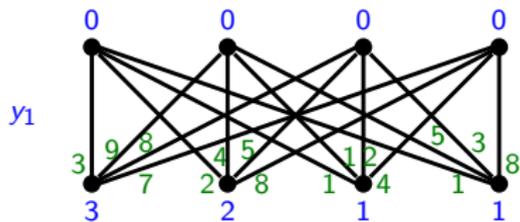
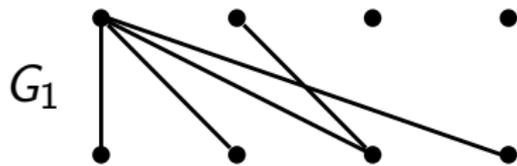
Exercise 6.8



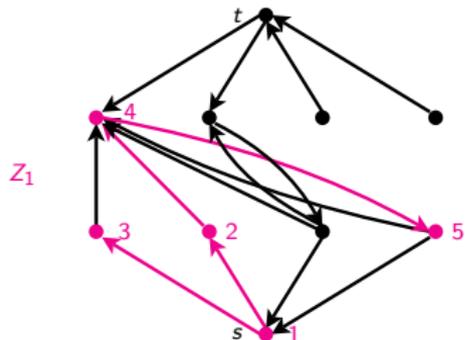
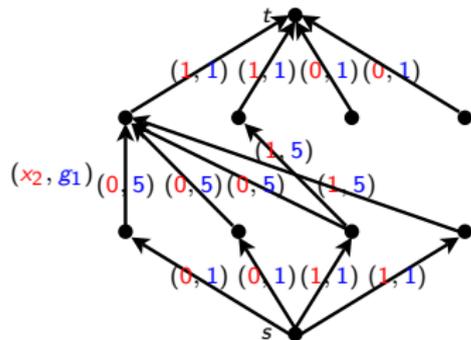
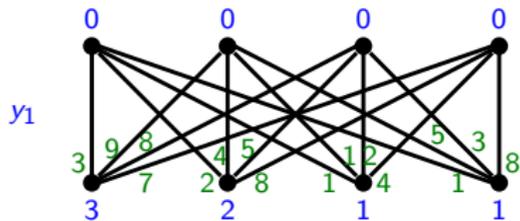
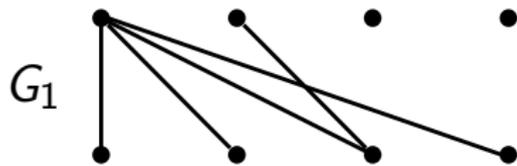
Exercise 6.8



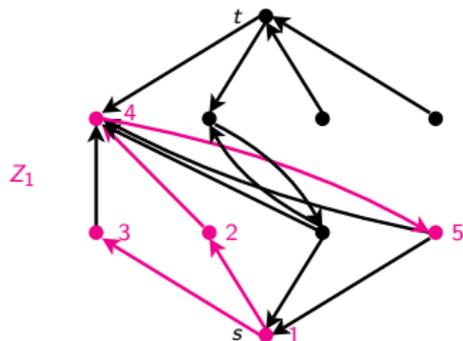
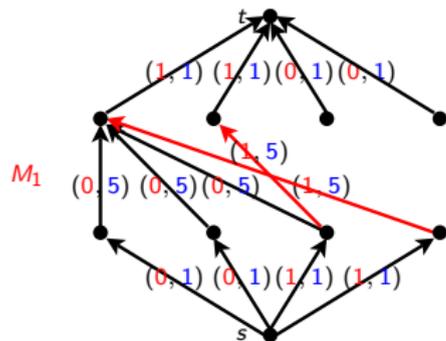
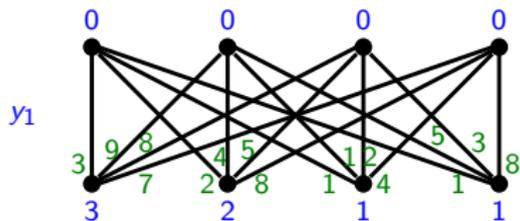
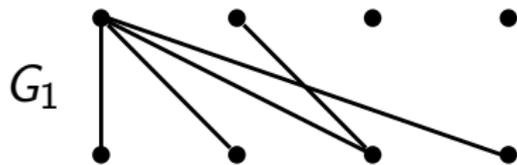
Exercise 6.8



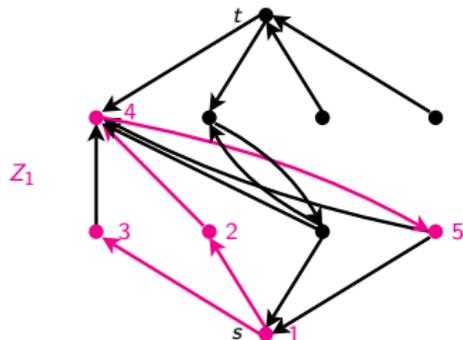
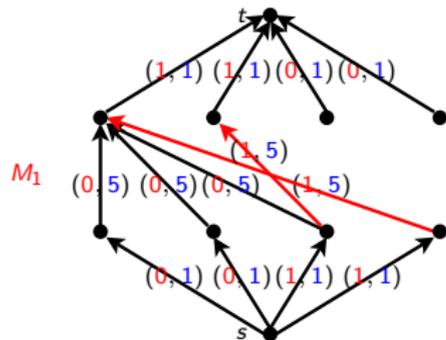
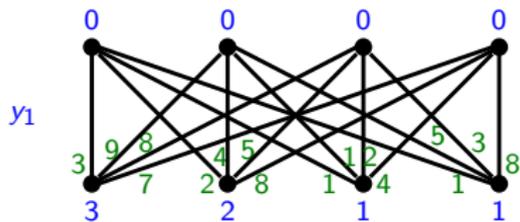
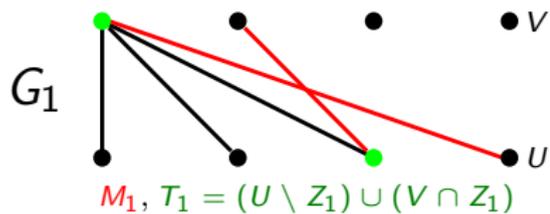
Exercise 6.8



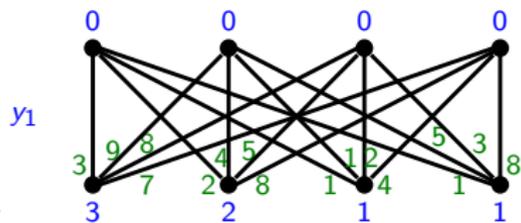
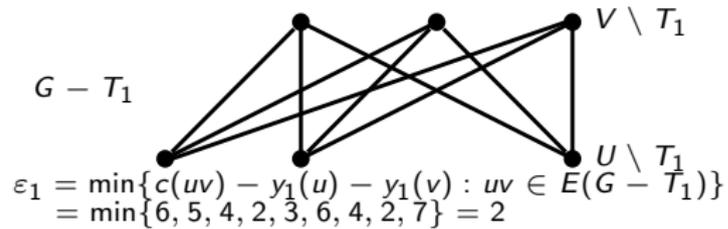
Exercice 6.8



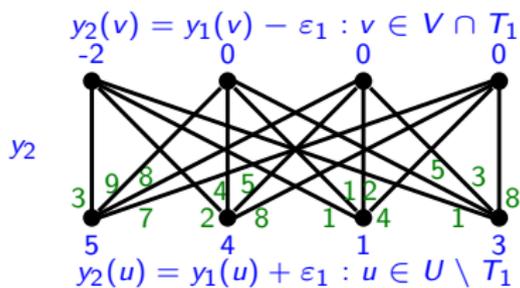
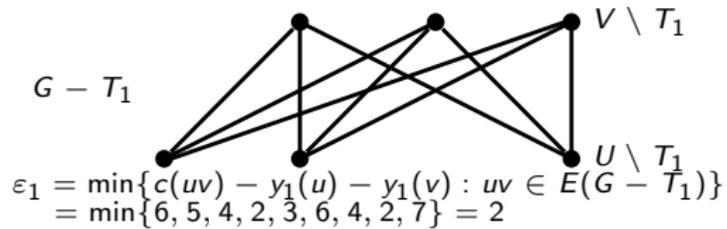
Exercise 6.8



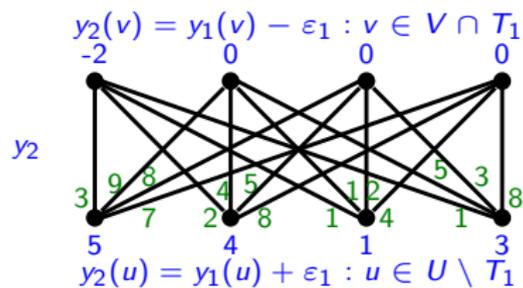
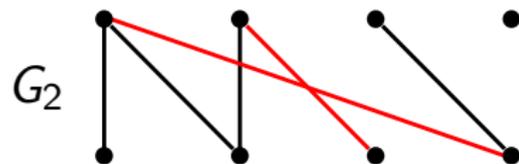
Exercise 6.8



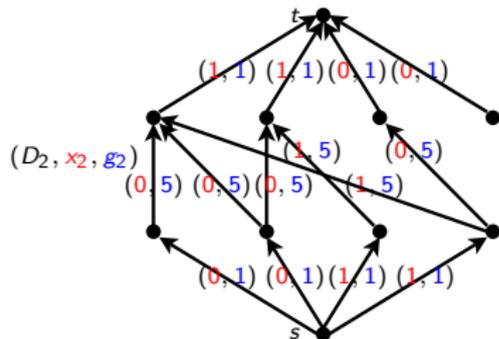
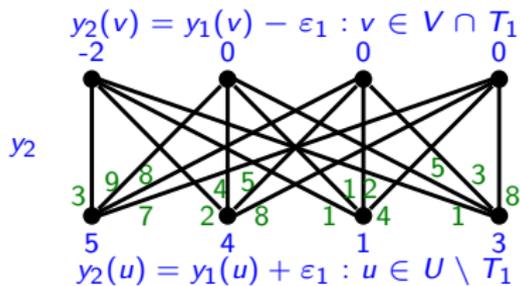
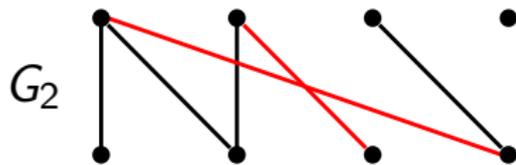
Exercise 6.8



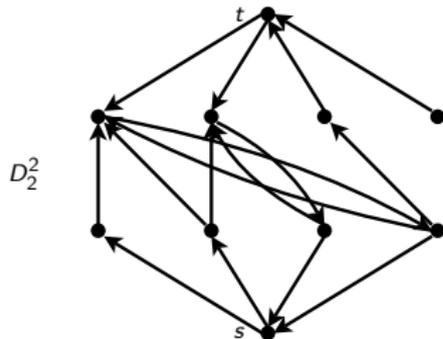
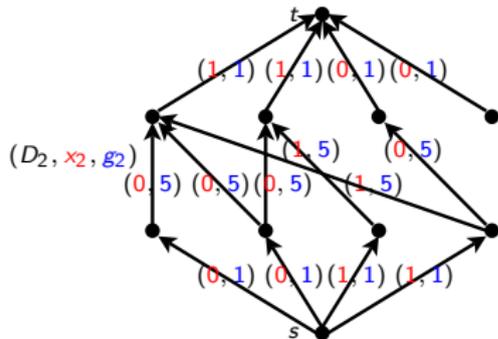
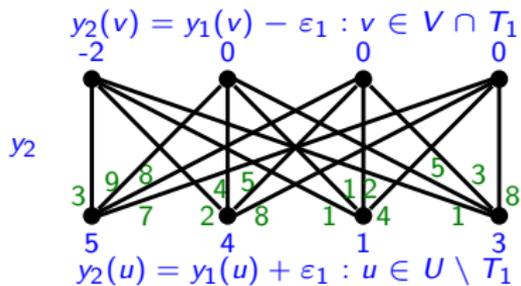
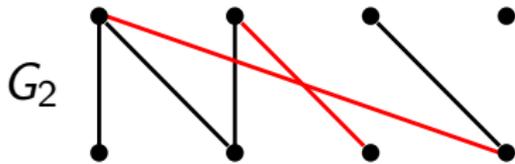
Exercise 6.8



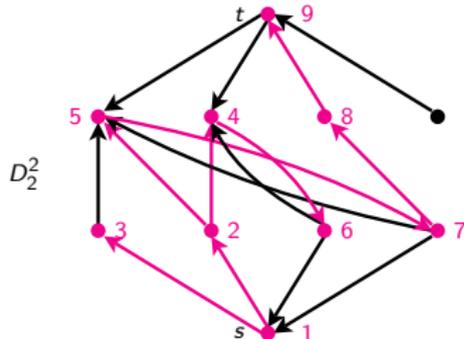
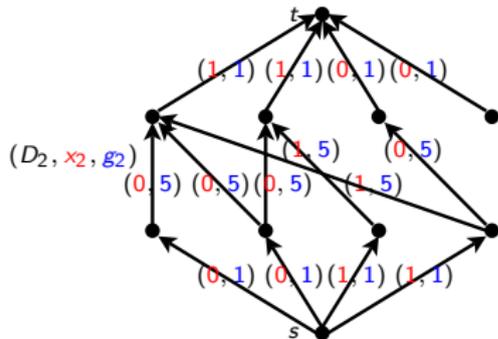
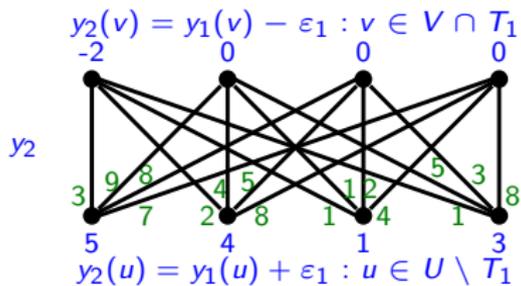
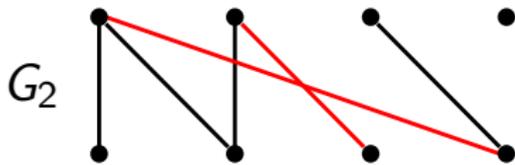
Exercise 6.8



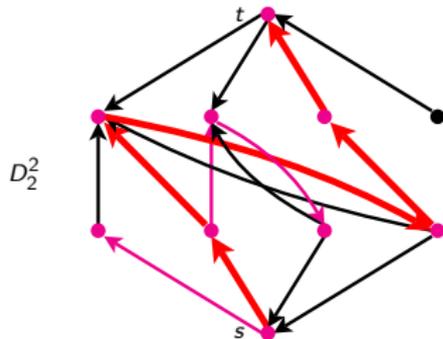
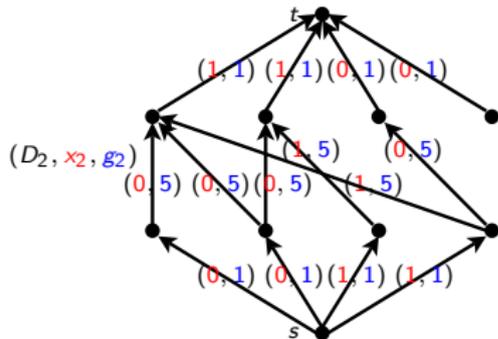
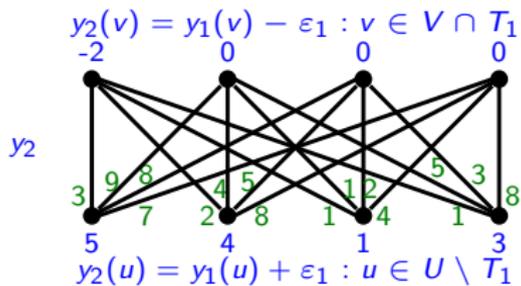
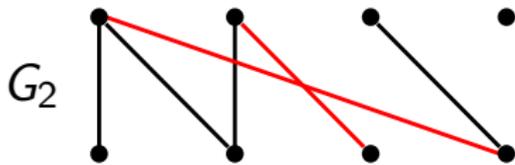
Exercise 6.8



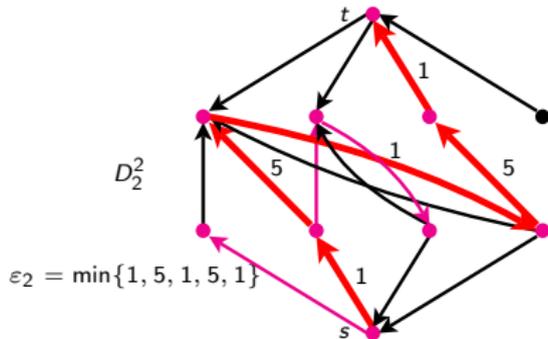
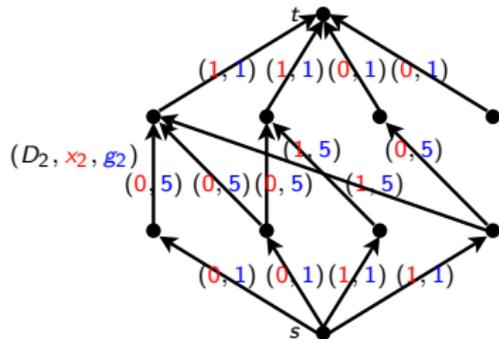
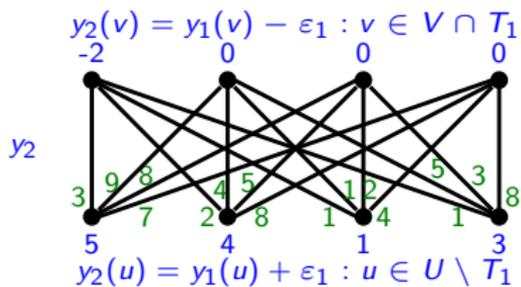
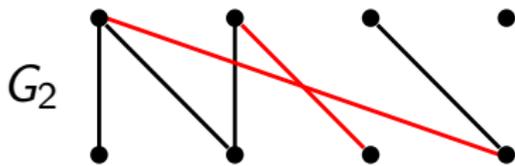
Exercise 6.8



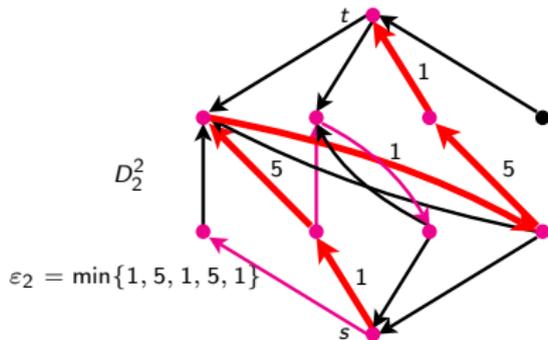
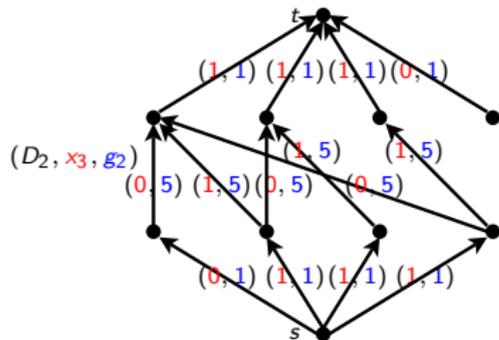
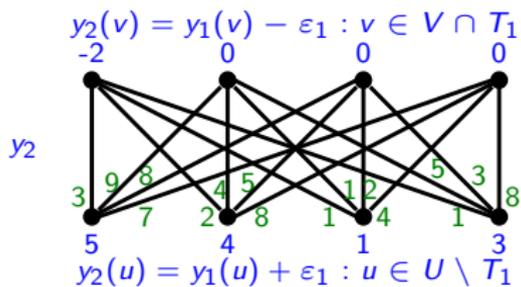
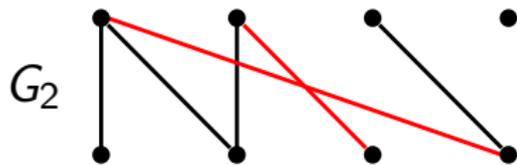
Exercise 6.8



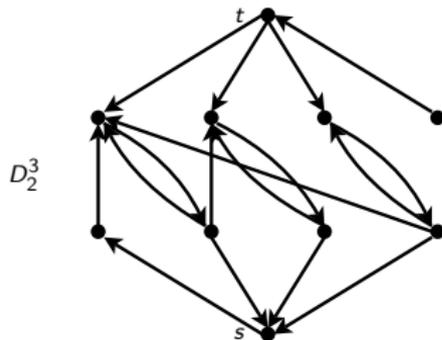
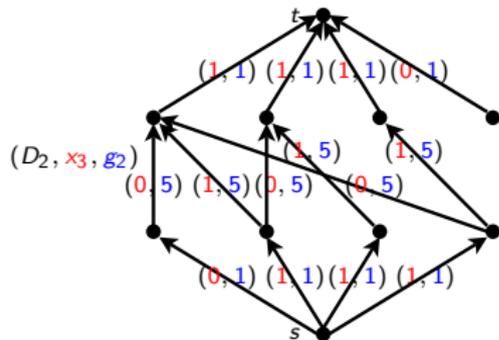
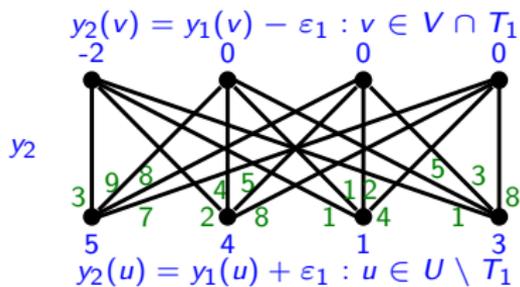
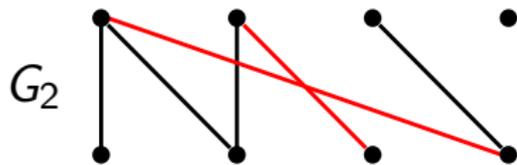
Exercise 6.8



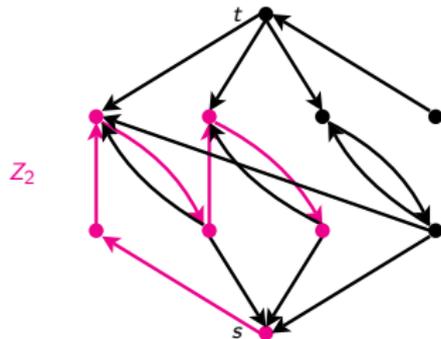
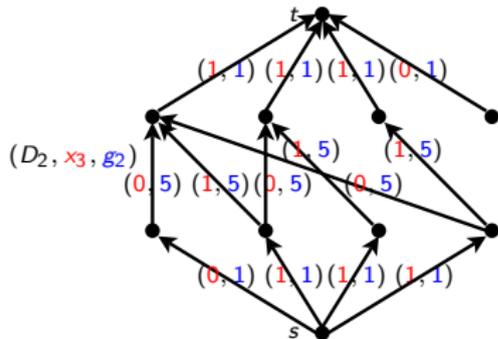
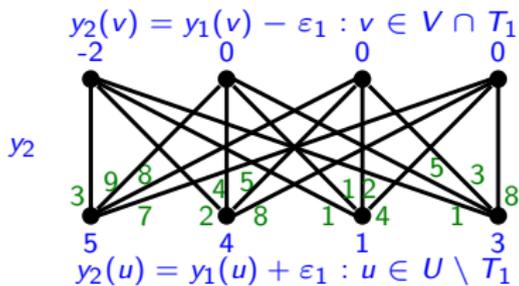
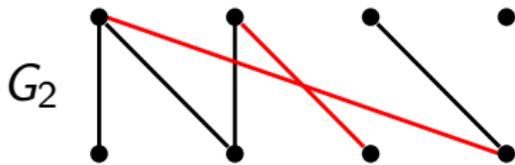
Exercise 6.8



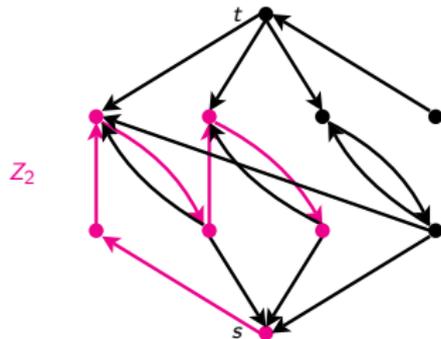
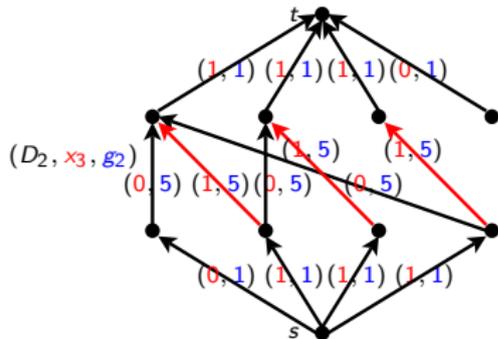
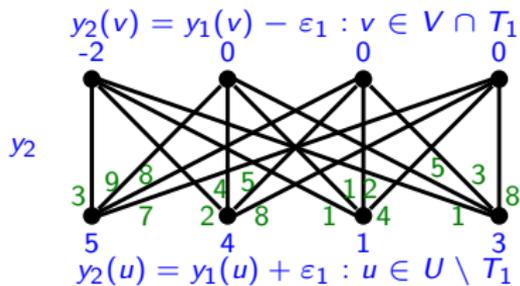
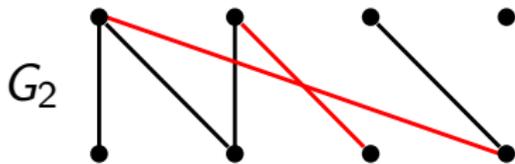
Exercise 6.8



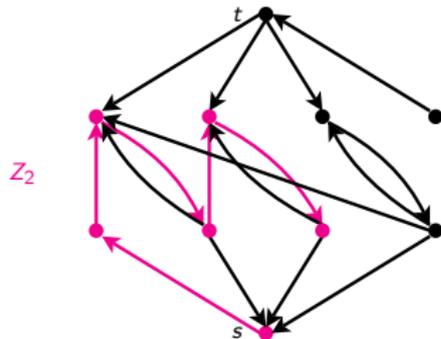
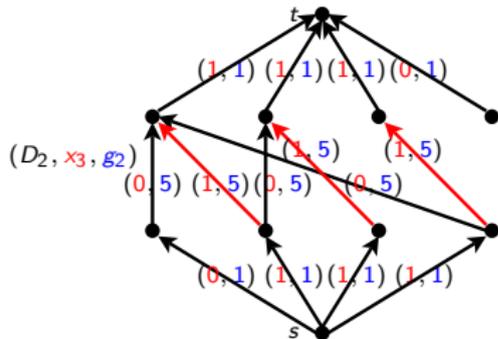
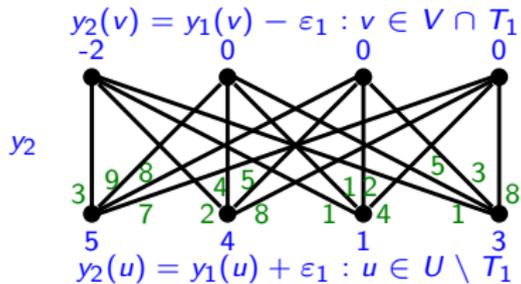
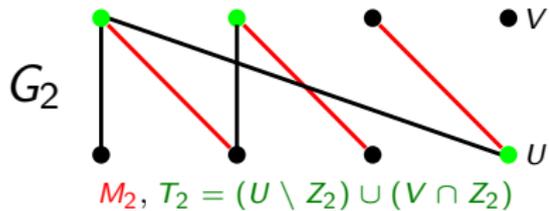
Exercise 6.8



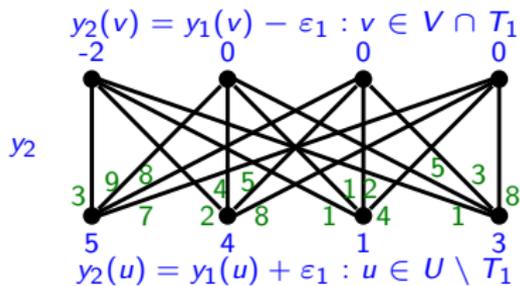
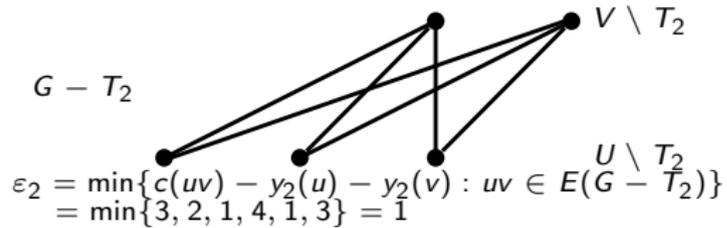
Exercise 6.8



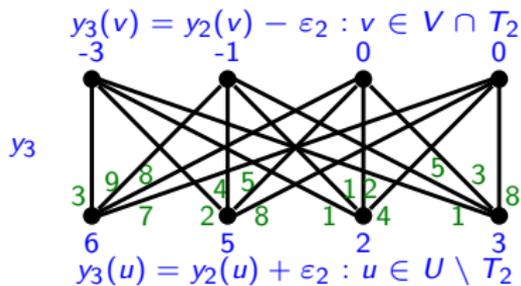
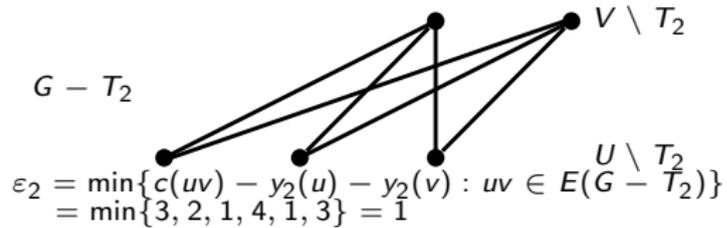
Exercise 6.8



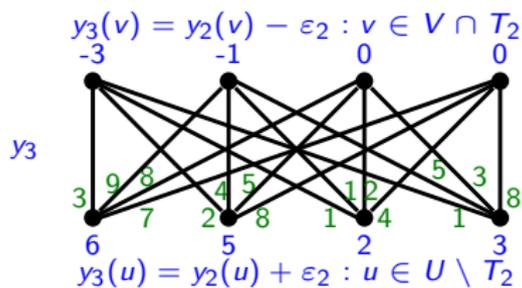
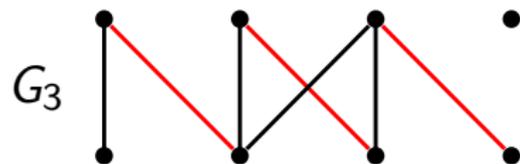
Exercise 6.8



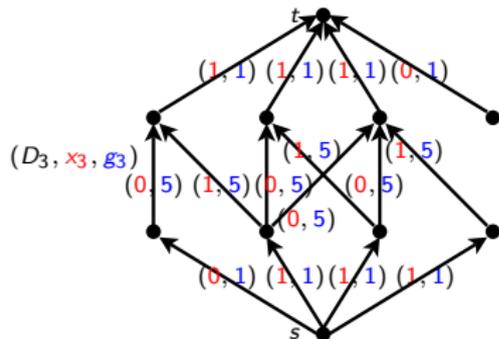
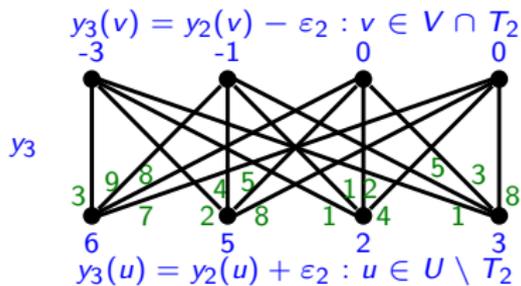
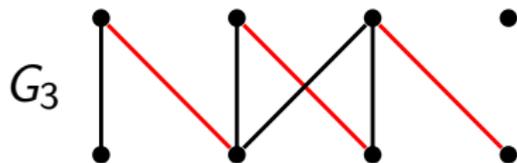
Exercise 6.8



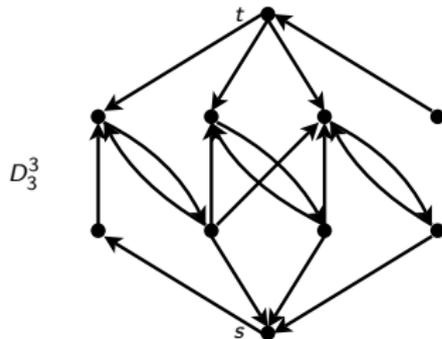
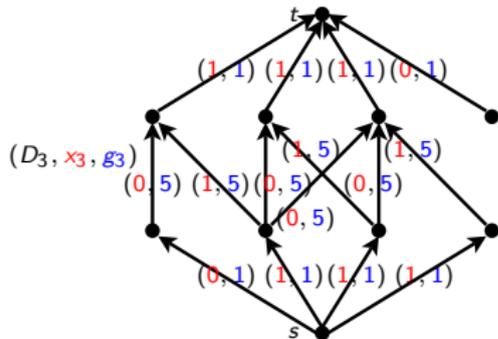
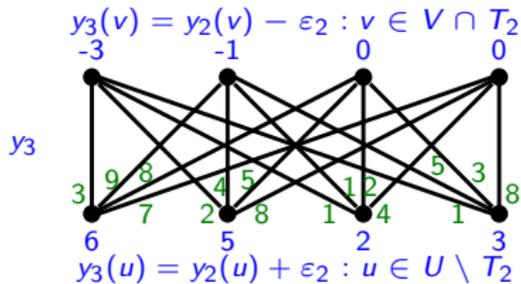
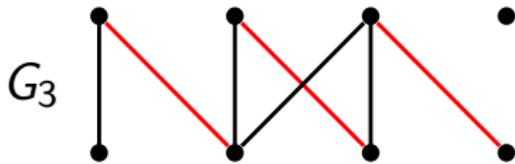
Exercise 6.8



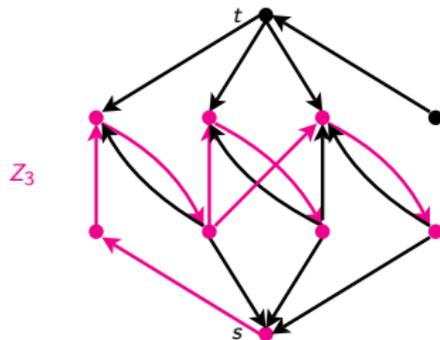
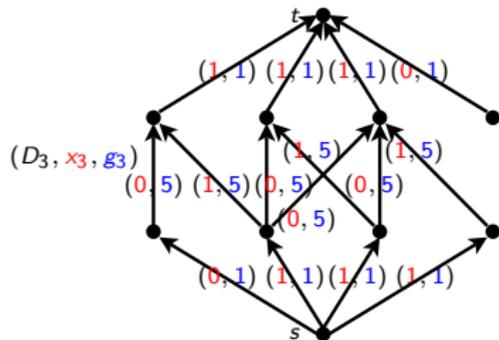
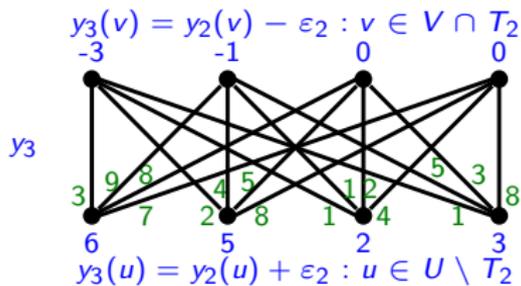
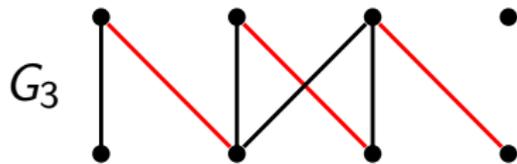
Exercise 6.8



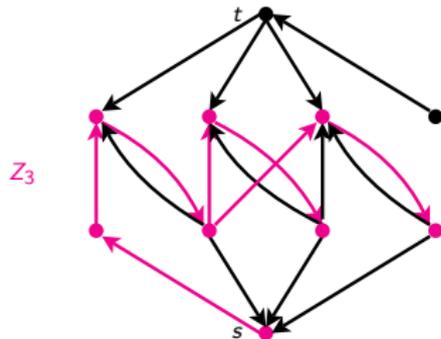
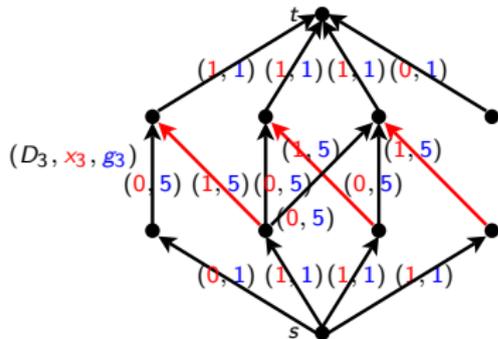
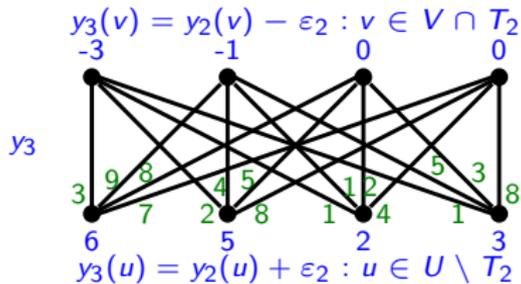
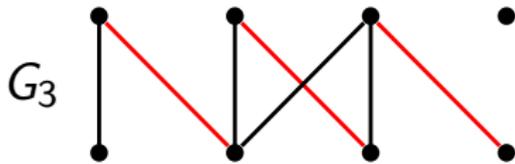
Exercise 6.8



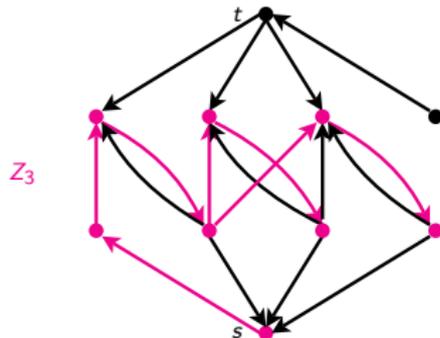
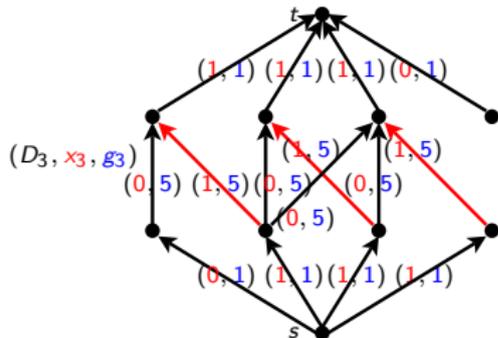
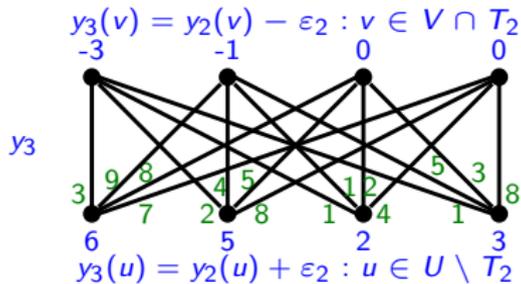
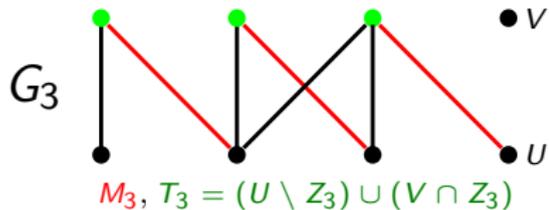
Exercise 6.8



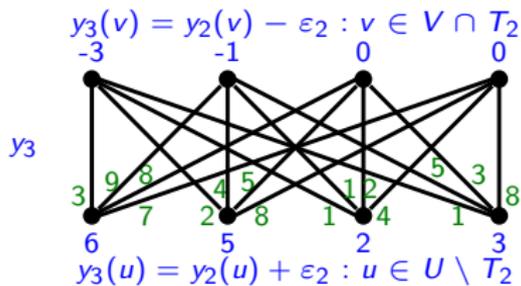
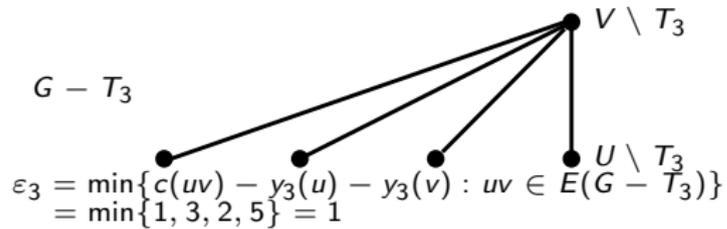
Exercise 6.8



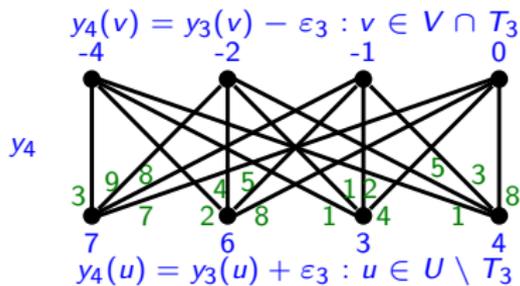
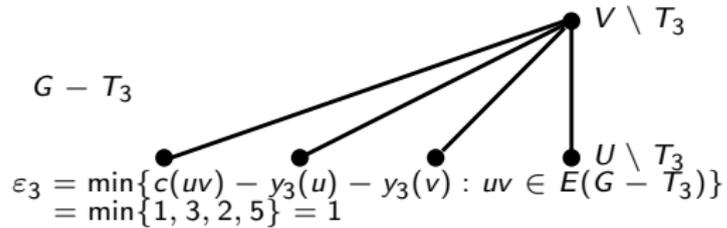
Exercise 6.8



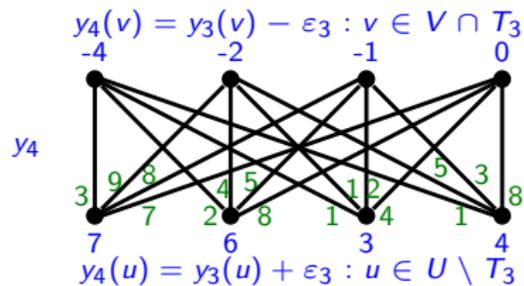
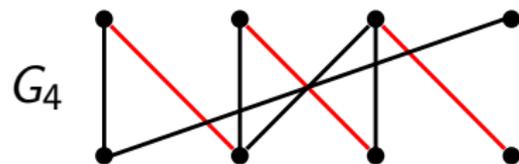
Exercise 6.8



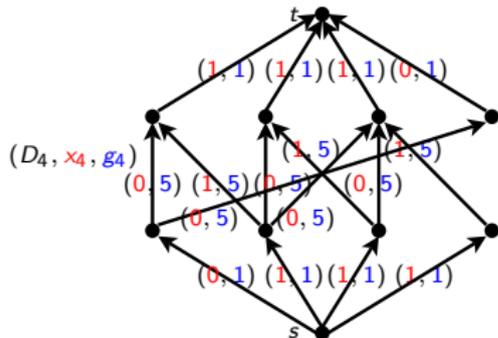
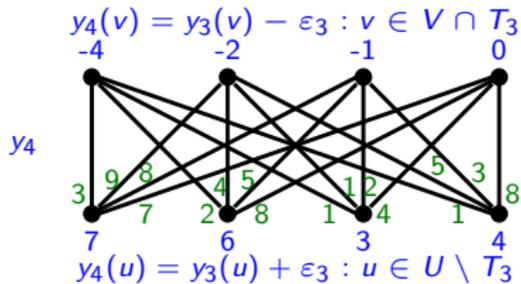
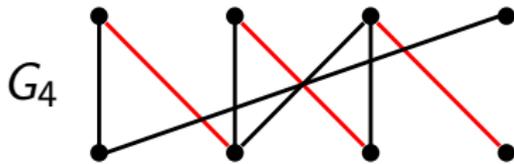
Exercise 6.8



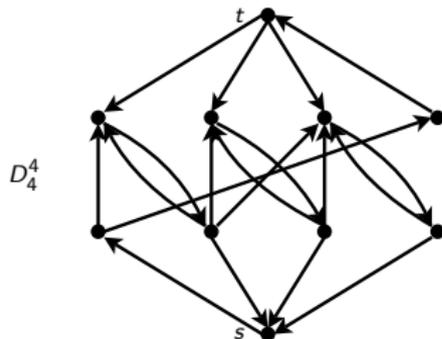
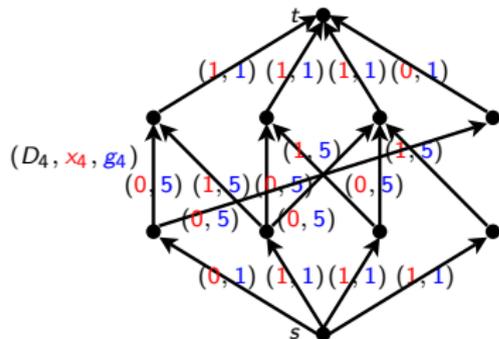
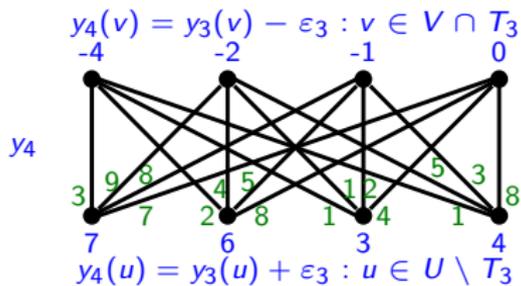
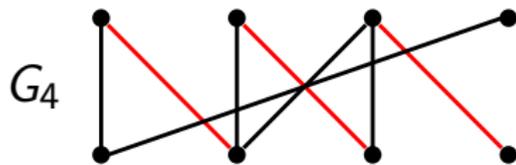
Exercise 6.8



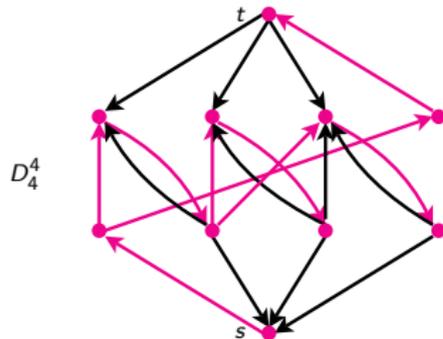
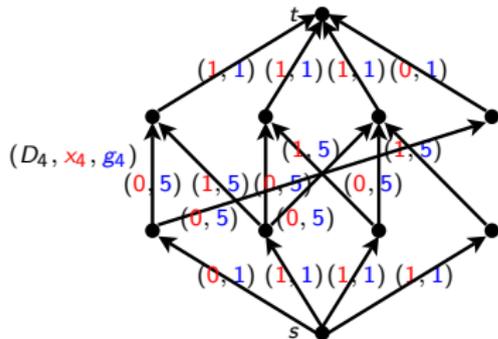
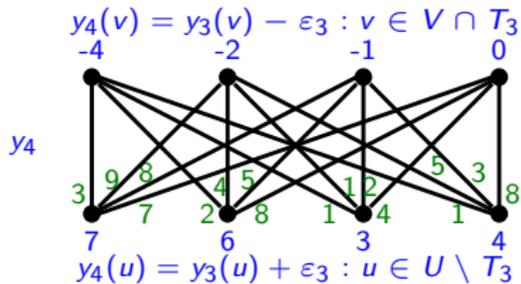
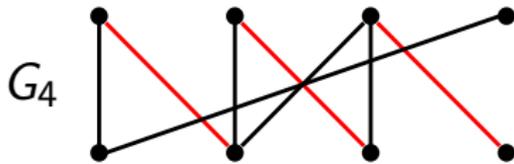
Exercise 6.8



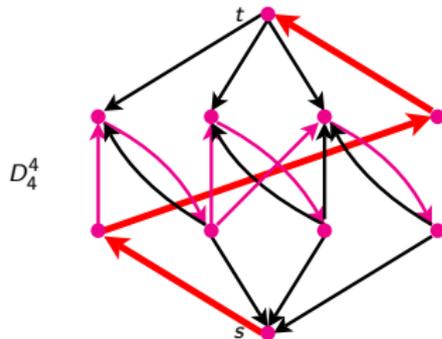
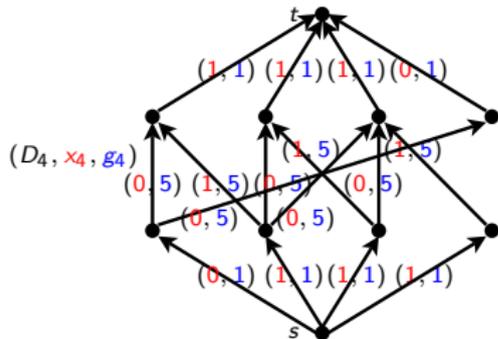
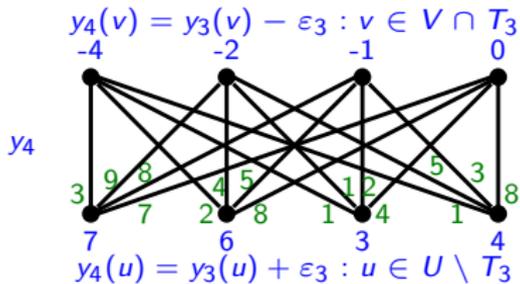
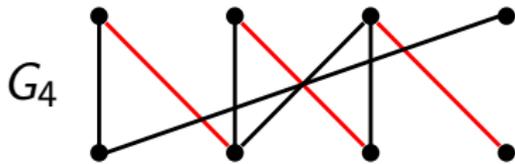
Exercise 6.8



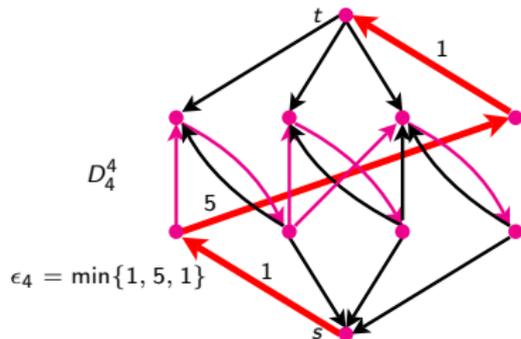
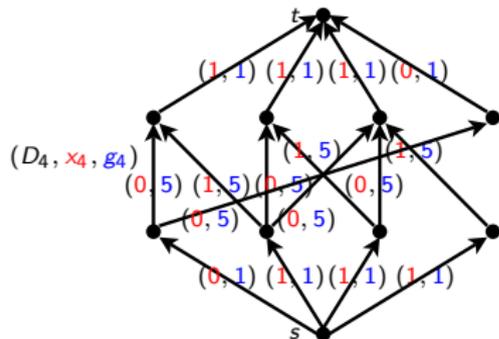
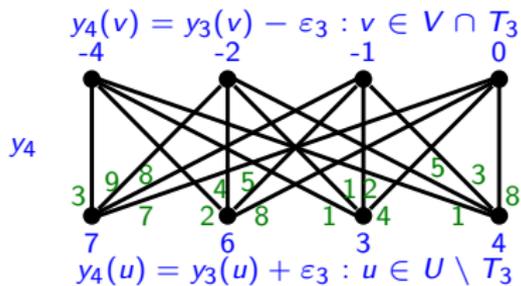
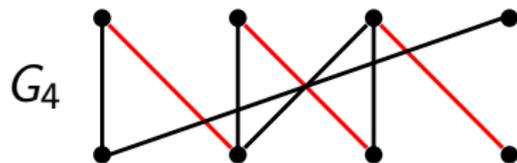
Exercise 6.8



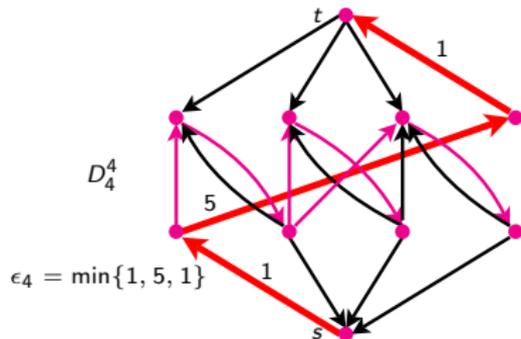
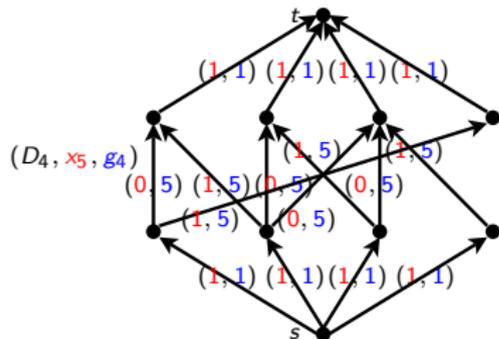
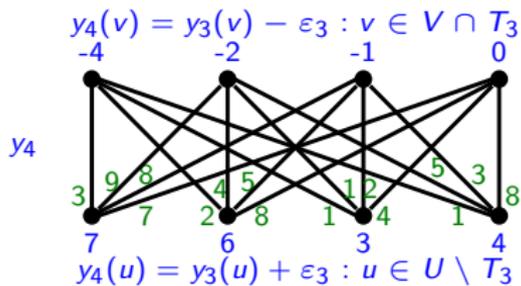
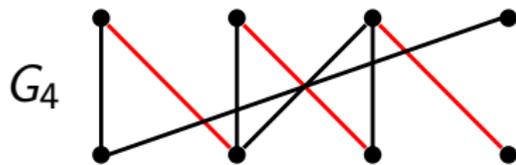
Exercise 6.8



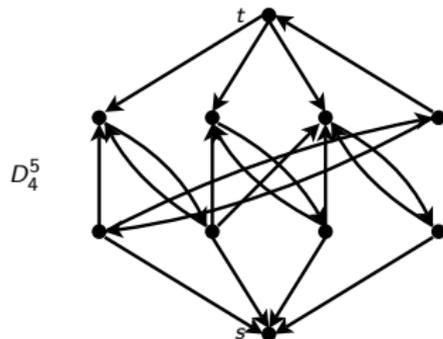
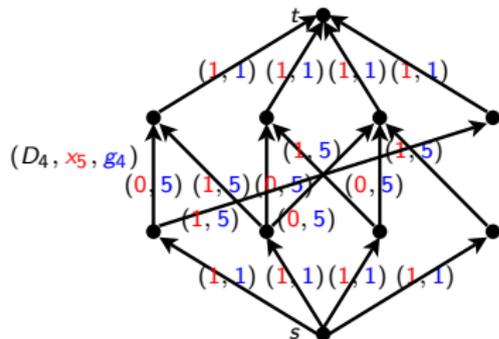
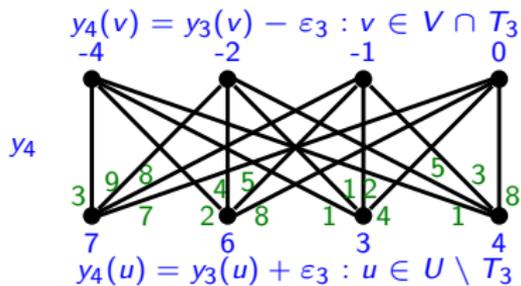
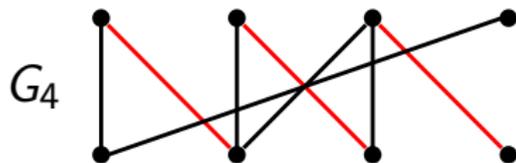
Exercise 6.8



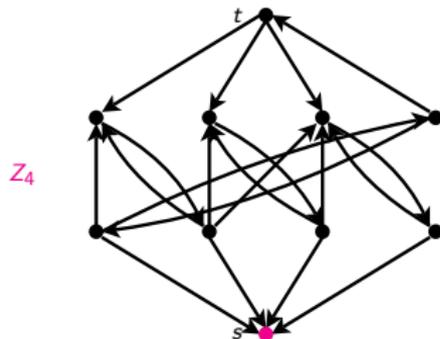
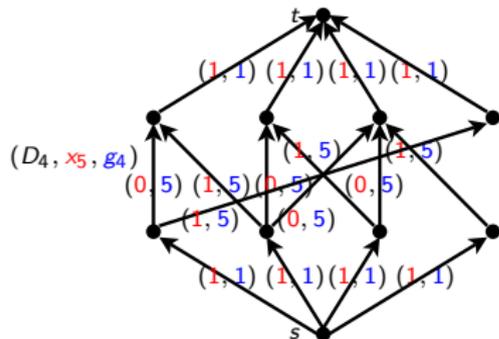
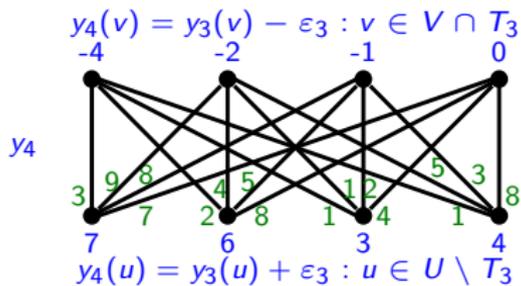
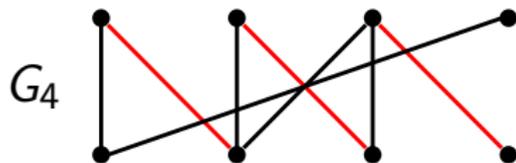
Exercise 6.8



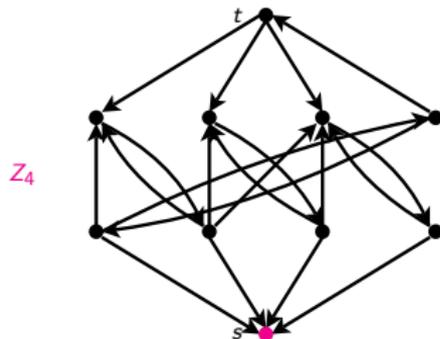
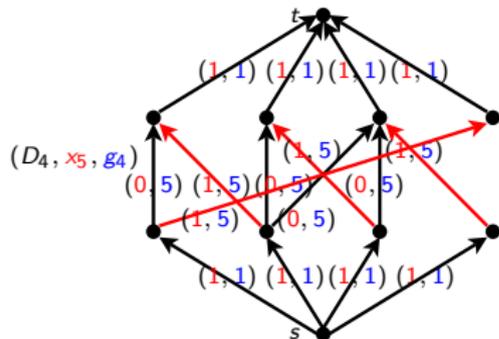
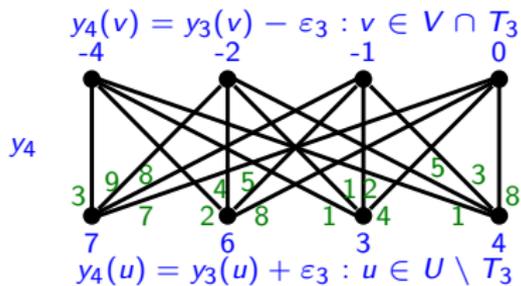
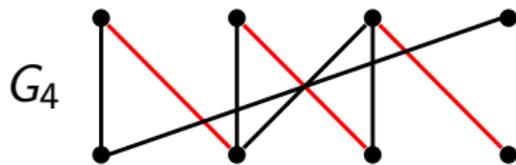
Exercise 6.8



Exercise 6.8

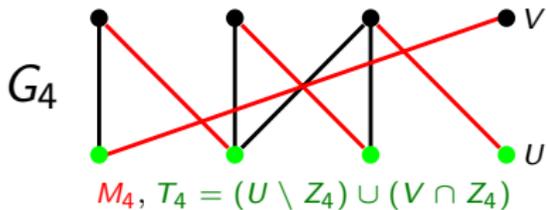


Exercise 6.8

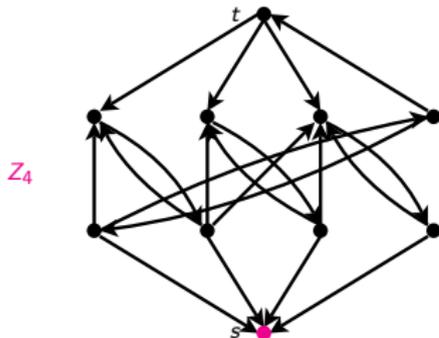
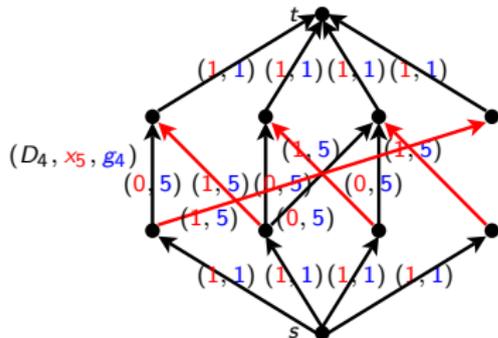
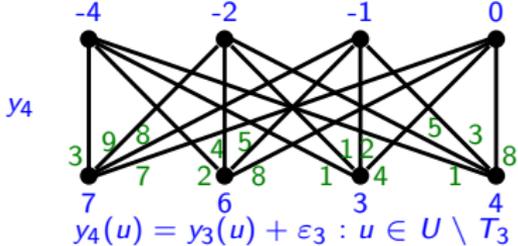


Exercice 6.8

couplage parfait de coût minimum



$$y_4(v) = y_3(v) - \varepsilon_3 : v \in V \cap T_3$$

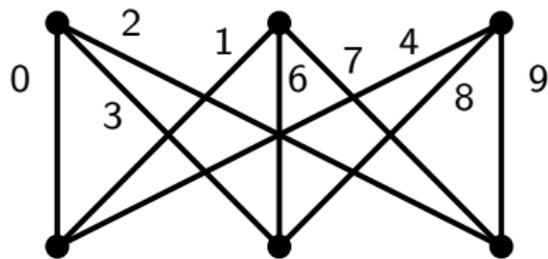


Énoncé

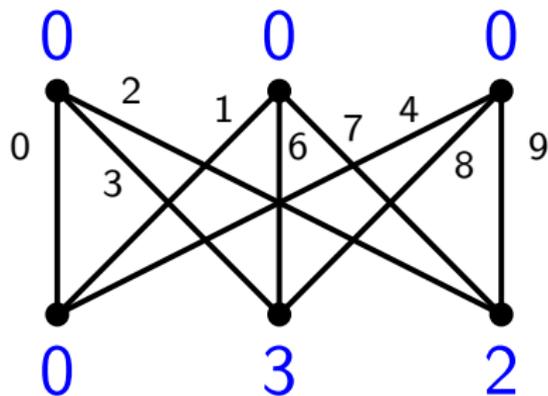
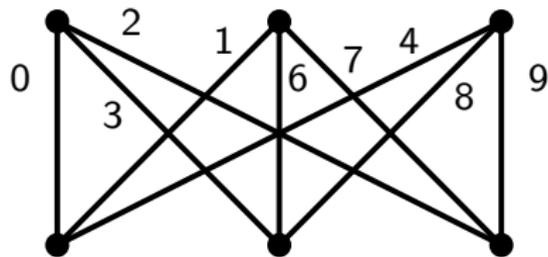
Exécuter l'algorithme de la méthode hongroise pour trouver un couplage parfait de coût minimum dans le graphe biparti $K_{3,3}$ avec des coûts sur les arêtes comme indiqué sur le tableau.

	v_1	v_2	v_3
u_1	0	1	4
u_2	3	6	8
u_3	2	7	9

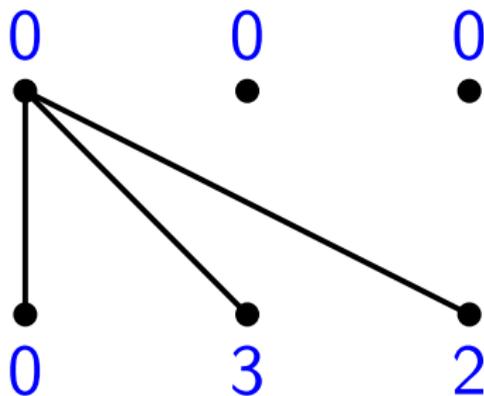
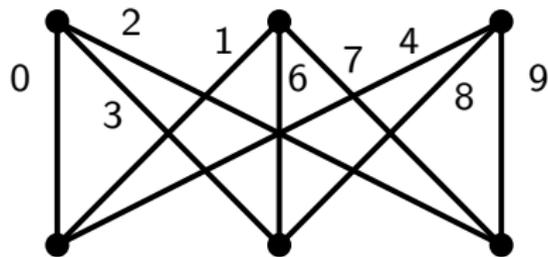
Solution de l'Exercice de l'examen 2019



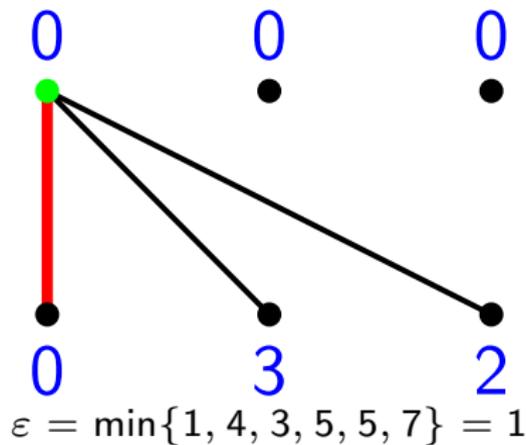
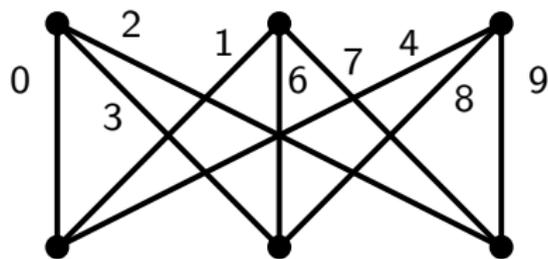
Solution de l'Exercice de l'examen 2019



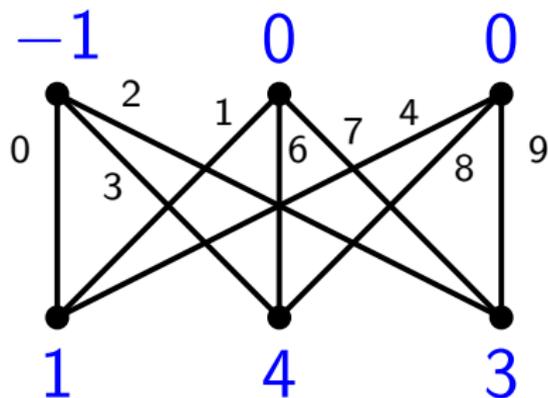
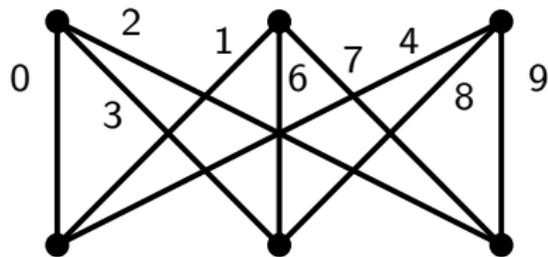
Solution de l'Exercice de l'examen 2019



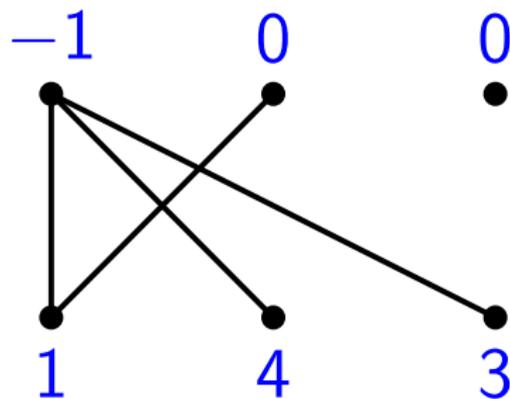
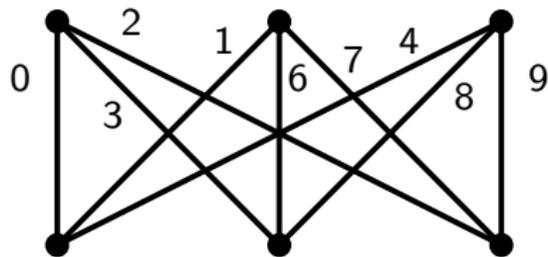
Solution de l'Exercice de l'examen 2019



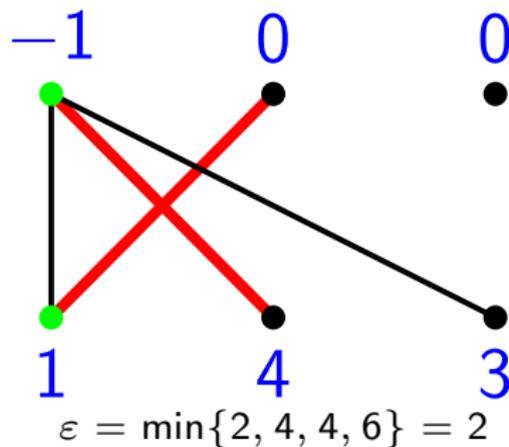
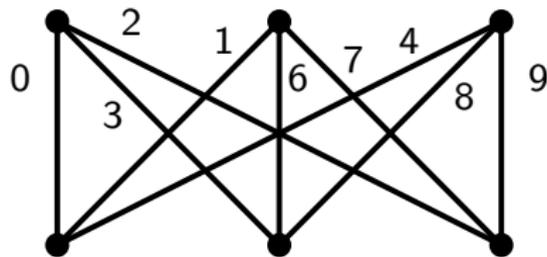
Solution de l'Exercice de l'examen 2019



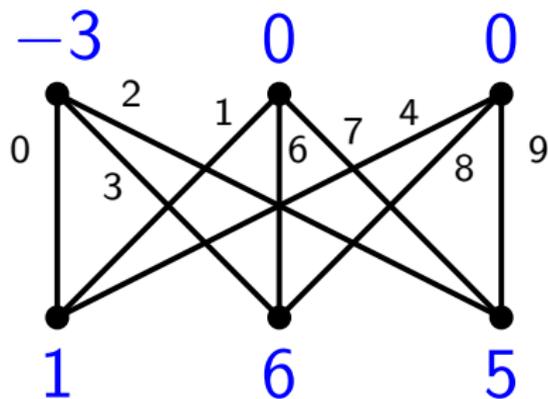
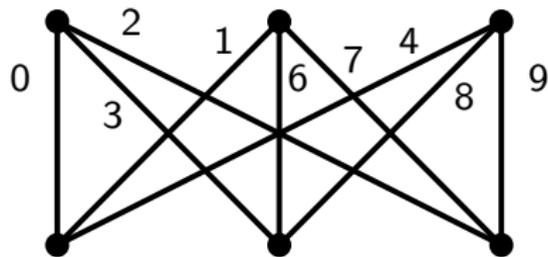
Solution de l'Exercice de l'examen 2019



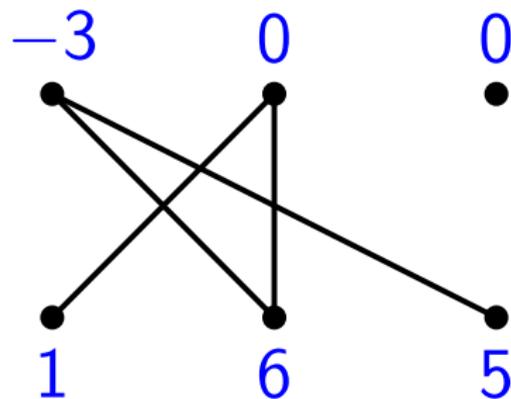
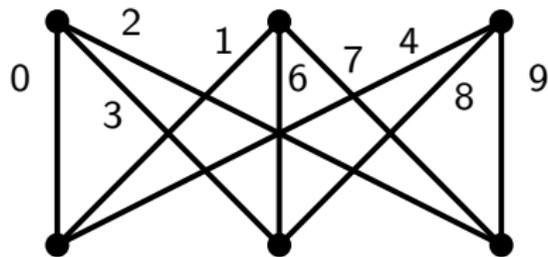
Solution de l'Exercice de l'examen 2019



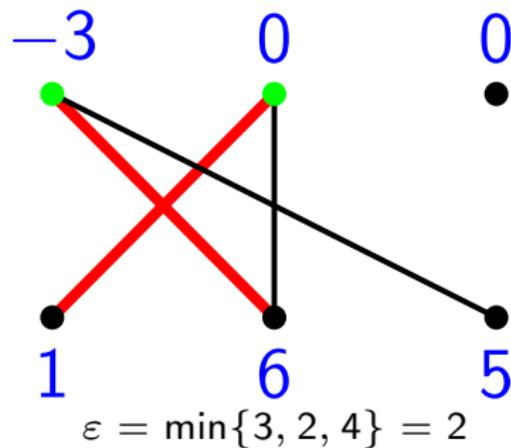
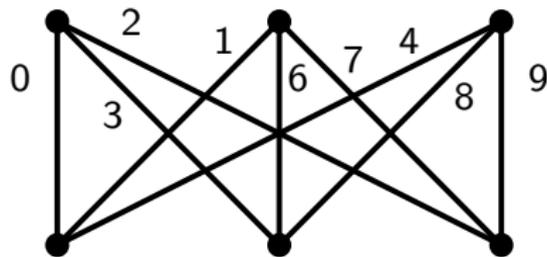
Solution de l'Exercice de l'examen 2019



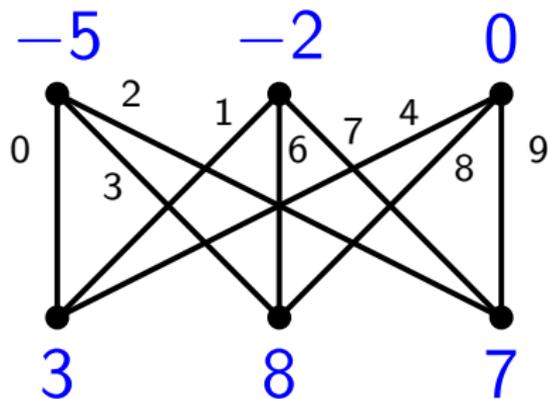
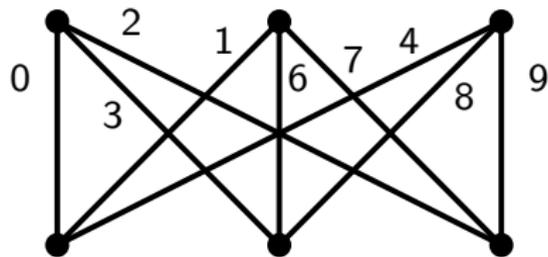
Solution de l'Exercice de l'examen 2019



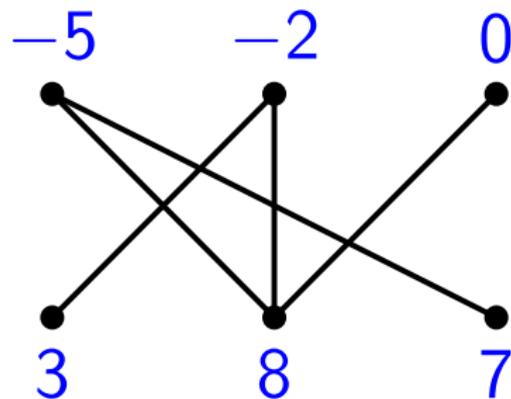
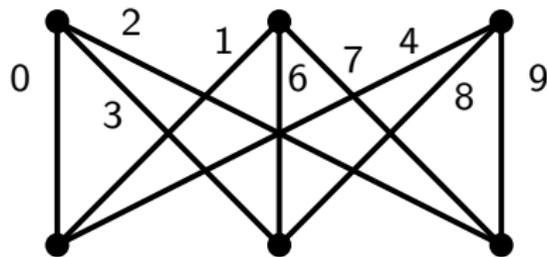
Solution de l'Exercice de l'examen 2019



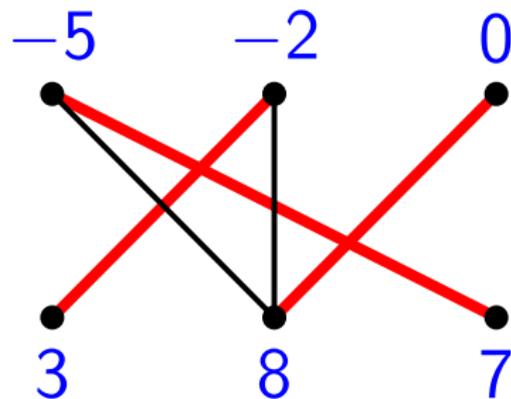
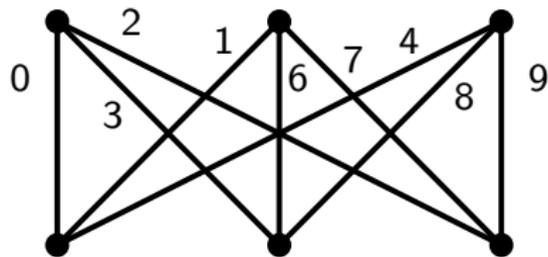
Solution de l'Exercice de l'examen 2019



Solution de l'Exercice de l'examen 2019



Solution de l'Exercice de l'examen 2019



Couplage parfait de coût minimum