Strong parity vertex coloring of plane graphs*

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Abstract

A strong parity vertex coloring of a 2-connected plane graph is a coloring of the vertices such that every face is incident with zero or an odd number of vertices of each color. We prove that every 2-connected plane graph has a strong parity vertex coloring with 97 colors, even when restricted to proper colorings. This proves a conjecture of Czap and Jendroľ [Discuss. Math. Graph Theory 29 (2009), pp. 521–543.]. We also provide examples showing that eight colors may be necessary (ten when restricted to proper colorings).

Keywords: graph, strong parity vertex coloring, strong parity chromatic number, proper coloring, face, discharging

1 Introduction

The notions of strong parity vertex coloring and the strong parity chromatic number were defined by Czap and Jendroľ [3]. Let us recall their definition in an equivalent form. Let G be a nontrivial connected plane graph, and let f be one of its faces. An *(anticlockwise) facial walk* of f is a shortest closed walk in G with the embeddings of its edges defining a closed oriented curve $\mathcal{C}(t)$ such that for every t_0, a sufficiently small left-side neighborhood of $\mathcal{C}(t)$ at t_0 belongs entirely to the interior of f. Note that $\mathcal{C}(t)$ is precisely the topological

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boundary of f since G is connected, and that distinct facial walks of f differ only in the choice of their end vertex.

Consider a (possibly improper) vertex coloring of G. The face f satisfies the *strong parity vertex coloring condition* (*spv-condition* for short) with respect to the coloring if for each color c of the coloring, there is zero or an odd number of occurrences of vertices colored with c on a facial walk of f. The coloring is a *strong parity vertex coloring* (*spv-coloring* for short) if the spv-condition holds for every face of G. Assume now that G is 2-connected. Then the minimum number of colors in an spv-coloring of G is called the *strong parity chromatic number* of G and is denoted by $\chi_s(G)$.

We make three remarks regarding these definitions. First, one can obviously extend the notion of spv-coloring and χ_s to the class of connected and 2-connected plane multigraphs respectively. Second, the restriction to 2-connected graphs in the definition of χ_s is essential, since there are plane graphs of connectivity one that do not admit any spv-coloring (an example of Czap and Jendroľ [3] consists of two triangles sharing one vertex). On the other hand, if the facial walks of all faces in G are cycles (which is the case if G is 2-connected), then the coloring assigning a different color to each vertex of G is an spv-coloring. Third, the spv-condition for the face f with respect to a particular coloring could be modified to the requirement that for each color c of the coloring, there is zero or an odd number of vertices colored with c in the boundary of f. Then the spv-coloring as well as $\chi_s(G)$ would be well-defined for every plane multigraph G, and in the class of 2-connected plane multigraphs, both the notions would coincide with their current definition.

Czap and Jendroľ [3] conjectured that there is a constant bound K on χ_s in the class of 2-connected plane graphs. Furthermore, they suggested that the best possible bound equals 6, providing an infinite family of graphs with $\chi_s = 6$. The main result of our paper confirms the conjecture for the class of 2-connected plane multigraphs with an added restriction to proper colorings:

Theorem 1.1. *Every* 2*-connected plane multigraph has a proper spv-coloring with at most* 97 *colors.*

The proof is given in Section 2. (During the preparation of this paper, another proof—for a slightly worse constant—was independently found by Czap, Jendroľ, and Voigt [4].) In Section 3, we present examples showing that the best possible value of K in the above conjecture is at least 8, or at least 10 with the restriction to proper colorings.

It should be noted that prior to the introduction of parity vertex colorings, a related type of edge coloring was treated in [1, 2]. An edge variant of parity coloring (called facial parity edge coloring) was recently studied, e.g., in [5].

In the remainder of this section, we establish the basic notation used throughout the paper; the notions not mentioned here are standard in graph theory [6]. For brevity, we will always refer to multigraphs as graphs, except when this could be confusing. A graph is called *trivial* if it is empty or consists of a single vertex. Let G be a plane graph; let v be a vertex, e_1 and e_2 edges, and f a face of G. Then $F(v)$ or $F(e_1)$ denotes the set of faces incident with v and e_1 respectively. The *boundary vertices* and *boundary edges* of f are all the vertices and edges of G, respectively, incident with f. The sets of these vertices and edges are denoted by $V(f)$ and $E(f)$ respectively. We refer to $|V(f)|$ as the *length* of f.

The degree of v in G, i.e., the number of edges of G incident with v (where loops are counted twice), is denoted by $d(v)$. If $d(v) = 2$, the vertex v is a 2-*vertex*; when $d(v) > 2$, we call v a *high-degree* vertex or a vertex of *high* degree. The edges e_1 and e_2 are *parallel* if they are not loops and share their end vertices. When e_1 and e_2 are parallel and constitute the boundary of f, the face f is called a *digon*. A path P is *trivial* if it comprises a single vertex, that is, the length of P equals 0. Every vertex of P other than its end vertices is called an *internal vertex* of P. Finally, we remark that for all the notation defined, the relevant graph may be referred to by a subscript whenever necessary. For example, we write $F_G(v)$ or $d_G(v)$ if this graph is G.

2 Upper bound

This section is devoted to the proof of the main result, Theorem 1.1. In Section 2.1, we prove certain structural properties of a minimal counterexample. These are used in an application of the discharging method in Section 2.2.

We now introduce a graph operation to be used in the proof of Lemma 2.4 in Section 2.1. Let G be a plane graph and $v \in V(G)$ a vertex of degree $d \geq 2$, assume that there is no loop at v, and let the edges incident with v be enumerated in a clockwise order as $e_i = vv_i$, $i \in \mathbb{Z}_d$ (we write $\mathbb{Z}_d$ for the set $\{0, \dots, d-1\}$ with addition modulo d). Note that some of the vertices v_i may coincide. The *annihilation* of v is the construction of a plane graph G' from G defined as follows:

(1) add edges $e'_i = v_i v_{i+1}$, $i \in \mathbb{Z}_d$, embedded in the plane so that for each i, the edges e_i, e_{i+1}, and e'_i, in this order, constitute a facial walk;

(2) delete v together with all the edges e_i.

Intuitively, one may achieve the desired embeddings of the edges e'_i by drawing each e'_i 'close enough' to the curve consisting of the embeddings of e_i and e_{i+1}; see Figure 2.1 for an example of a properly conducted annihilation.

Regarding the faces of G and G', it is obvious that the following holds:

Observation 2.1. *Let G' be obtained from G by the annihilation of a vertex $v \in V(G)$. Then*

(1) *every face of G not in $F_G(v)$ is also a face of G';*

(2) *each face $g \in F_G(v)$ has its counterpart g' in G' such that a facial walk of g' may be obtained from a facial walk of g by replacing each of its subsequences of the form $e_i v e_{i+1}$ with e'_i, and hence $V(g') = V(g) - \{v\}$;*

(3) *there is precisely one more face in G', having the sequence $v_0 e'_{d-1} v_{d-1} e'_{d-2} \dots v_1 e'_0 v_0$ as its facial walk.*

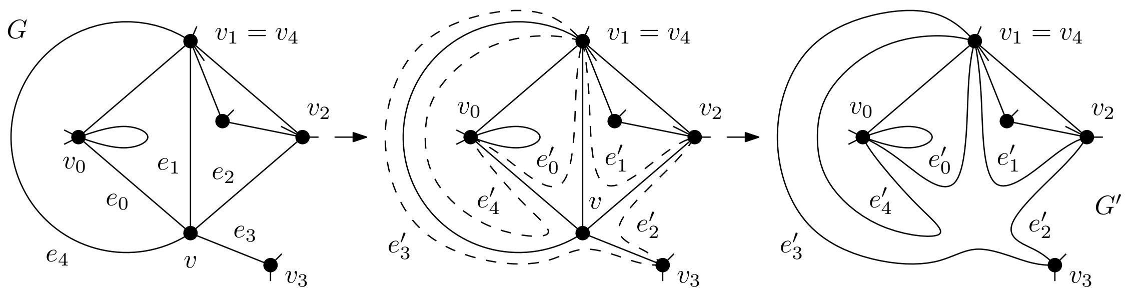

Figure 2.1. The annihilation of a vertex v. The original graph G is on the left, the resulting graph G' on the right.

If G is 2-connected, any of its vertices may be annihilated. Therefore, it makes sense to ask whether the class of 2-connected plane graphs is closed under the operation of the annihilation of a vertex. One can easily see that the answer is negative: if $v_i = v_j$ for some $i \neq j$ in the definition above, i.e., there is a pair of parallel edges incident with v in G, then the vertex v_i may become a cut vertex in G'. However, it turns out that 2-connectedness is preserved unless this situation occurs or G is too small.

Lemma 2.2. *Let v be a vertex of a 2-connected plane graph G, $|V(G)| \geq 4$, such that v is incident with no pair of parallel edges. Then the graph G' obtained from G by annihilating v is 2-connected.*

Proof. We use the well-known fact that a connected plane graph G on at least three vertices is 2-connected if and only if the facial walk of each of its faces is a cycle. (The 'only if' direction is Proposition 4.2.6 in [6]. Conversely, for any cutvertex v of G there is a face whose boundary contains two neighbors of v in different components of $G - v$. Since each cycle is contained within some block, the boundary of this face cannot be a cycle.)

We use this criterion for both G and G' in the following. (Note that G' is connected.) Take an arbitrary face f' of G', and let W be a facial walk of f'. We may assume that f' is not a face of G, otherwise there is nothing to show. Thus by Observation 2.1, W is either the walk $v_0 e'_{d-1} v_{d-1} e'_{d-2} \dots v_1 e'_0 v_0$ (up to the choice of the end vertex), or it arises from a facial walk of a face of G by replacing each of its subsequences of the form $e_i v e_{i+1}$ with e'_i. In both cases, it follows from the assumptions that W is a cycle. $\square$

Finally, we include a technical lemma that will greatly simplify the case analysis in the proof of Claim 1 in Section 2.2.

Lemma 2.3. *Let (l_i), (l'_i), $i = 0, \dots, k$, be tuples of positive integers such that $l_j \leq l'_j$ for every $j \neq k$, and $l'_k \geq l'_j$ for every $j \neq k$ with $l_j < l'_j$. Then $\sum_{i=0}^k l'_i \leq \sum_{i=0}^k l_i$ or $\sum_{i=0}^k 1/l'_i \leq \sum_{i=0}^k 1/l_i$.*

Proof. We first prove that if $\sum_{i=0}^k l'_i = \sum_{i=0}^k l_i$ and the tuples are distinct, then $\sum_{i=0}^k 1/l'_i < \sum_{i=0}^k 1/l_i$. We proceed by induction on the size of the set $J := \{j : j \neq k, l_j < l'_j\}$. From the assumption that (l'_i) and (l_i) are distinct and have the same sum (and $l_j \leq l'_j$ for $j \neq k$), it follows that J is nonempty. Now, fix j_0 as some index in J, and let $d := l'_{j_0} - l_{j_0}$. Consider a tuple (l''_i) such that $l''_{j_0} = l'_{j_0} = l_{j_0} + d$, $l''_k = l_k - d$, and $l''_j = l_j$ for each remaining index j. Clearly, $\sum_{i=0}^k l''_i = \sum_{i=0}^k l'_i$, $l''_k \geq l'_k$, and the number of $j \neq k$ such that $l''_j \neq l'_j$ equals $|J| - 1$. We have

$$\sum_{i=0}^{k} \frac{1}{l''_i} = \sum_{i=0}^{k} \frac{1}{l_i} + \left(\frac{1}{l''_{j_0}} - \frac{1}{l_{j_0}}\right) + \left(\frac{1}{l''_k} - \frac{1}{l_k}\right) = \sum_{i=0}^{k} \frac{1}{l_i} - d\left(\frac{1}{l_{j_0}(l_{j_0}+d)} - \frac{1}{l''_k(l''_k+d)}\right),$$

and since

$$l''_k \geq l'_k \geq l'_{j_0} > l_{j_0}$$

by the assumptions and the choice of j_0, it follows immediately that

$$\sum_{i=0}^{k} \frac{1}{l''_i} < \sum_{i=0}^{k} \frac{1}{l_i}. \tag{2.1}$$

When $|J| = 1$ (the base case of the induction), the tuple (l''_i) equals (l'_i) and there is nothing more to prove. Otherwise, we may apply the induction hypothesis to (l''_i) and (l'_i), in this order, obtaining $\sum_{i=0}^k 1/l'_i < \sum_{i=0}^k 1/l''_i$; this together with (2.1) gives the desired conclusion.

Now we prove the lemma. We may suppose that $\sum_{i=0}^k l'_i > \sum_{i=0}^k l_i$. If $l'_k \geq l_k$, then trivially $\sum_{i=0}^k 1/l'_i < \sum_{i=0}^k 1/l_i$. Otherwise, we claim that there exists a tuple (l''_i) such that $l_j \leq l''_j \leq l'_j$ for every $j \neq k$, $l''_k = l'_k$, and $\sum_{i=0}^k l''_i = \sum_{i=0}^k l_i$. Indeed, (l''_i) may be obtained from (l'_i) by replacing some of the values l'_j $(j < k)$ with smaller ones, using the fact that $\sum_{i=0}^{k-1}(l'_i - l_i) > l_k - l'_k$. Note that (l''_i) is distinct from both (l_i) and (l'_i). It follows from the above that $\sum_{i=0}^k 1/l''_i < \sum_{i=0}^k 1/l_i$. Furthermore, since $l''_i \leq l'_i$ for all $i \leq k$ and the tuples are not equal, $\sum_{i=0}^k 1/l'_i < \sum_{i=0}^k 1/l''_i$. The proof is now complete. □

2.1 Reducibility

Let G be a counterexample to Theorem 1.1 with the minimum number of vertices, and subject to this condition, with the minimum number of edges.

In Lemma 2.4 below, we infer several constraints applying to G. Based on these constraints, we derive bounds for the (reduced) face degree of a vertex in G in Lemma 2.5. As per standard terminology, a graph contradicting Lemma 2.4 is said to be *reducible*.

Before stating the lemma, we introduce some terminology. Let v be a vertex and f a face of G. The *(open) face-vertex neighborhood* of v of G, denoted by $N^F(v)$, is defined as $\left(\bigcup_{g\in F(v)} V(g)\right) - \{v\}$. Similarly, the *$f$-reduced (open) face-vertex neighborhood* of v, referred to as $N^F(v,f)$, is the set $\left(\bigcup_{g\in F(v),\, g\neq f} V(g)\right) - \{v\}$. We call the sizes of these sets the *face degree* of v and *f-reduced face degree* of v respectively, writing $d^F(v)$ and $d^F(v,f)$ respectively. As with the other notation, the graph G is included as a subscript if necessary. For instance, we may write $N^F_G(v)$ or $d^F_G(v,f)$.

Lemma 2.4. *The graph G has the following properties:*

(1) $|G| > 97$*;*

(2) *G does not contain parallel edges; in particular, G is without digons;*

(3) *no facial walk of a face of G contains four consecutive 2-vertices;*

(4) *for every vertex v of G, $d^F(v) > 96$;*

(5) *for every two vertices u and v of G such that $F(u) \cap F(v) = \{f\}$, $d^F(u, f) + d^F(v, f) > 95$.*

Proof. By assumption, G is a 2-connected graph. We prove each of the assertions by contradiction.

To see (1), consider an assignment of a different color to each vertex of G.

We proceed to show assertion (2); let e_1 and e_2 denote two parallel edges in G. We distinguish two cases. If e_1 together with e_2 delimit a digon f, we simply delete one of the two edges, say e_1, obtaining a (2-connected) graph G'. By the minimality of G, the graph G' has a proper spv-coloring c with at most 97 colors. As all faces of G except f have their counterparts in G' with the same sets of boundary vertices, and c is proper, c is also a proper spv-coloring of G.

When, on the other hand, the curve $\mathcal{C}$ comprising the embeddings of e_1 and e_2 is not the boundary of a digon, we produce two graphs G_1 and G_2 by deleting the interior and exterior of $\mathcal{C}$, respectively, from G. Both these graphs are clearly 2-connected (in particular, each has at least three vertices), and smaller than G with respect to the given ordering; thus each has a proper spv-colorings with at most 97 colors. The colorings can be chosen so that they coincide at the common vertices, i.e., the end vertices of e_1, and the number of colors in their union c is minimal. Then c is a proper spv-coloring of G using at most 97 colors.

Next, suppose $x_1x_2x_3x_4$ is a path contradicting statement (3). Let v_1 be the neighbor of x_1 in G other than x_2, and let v_4 be the neighbor of x_4 other than x_3. The vertices v_1 and v_4 are distinct and different from all x_i, $i = 1, \ldots, 4$, otherwise the facial walk would contain just four or five vertices, and by the 2-connectedness of G these would be the only vertices of G; a contradiction to assertion (1). We construct a graph G' by contracting the path $x_1 \ldots x_4v_4$ into a single vertex v_4'. It remains 2-connected due to statement (1), and hence by assumption, G' has a proper spv-coloring c with at most 97 colors. As v_1 and v_4' are adjacent in G', we obtain $c(v_1) \neq c(v_4')$. Now, we use the coloring for the corresponding vertices of G and assign the color $c(v_1)$ to x_2, x_4, and the color $c(v_4')$ to x_1, x_3, v_4. This way the occurrence of the colors $c(v_1)$ and $c(v_4')$ preserves the parity on the corresponding facial walks, and we obtain a proper spv-coloring of G with no more than 97 colors.

Now we focus on assertion (4). Suppose it does not hold. We perform the annihilation of v, obtaining a graph G'. By parts (1), (2) and Lemma 2.2, G' is 2-connected. By the minimality of G, G' has a proper spv-coloring c with at most 97 colors. Using c for G and assigning to v a color not used by c on any vertex in $N_G^F(v)$, but if possible present in c, we obtain a proper coloring of G of cardinality less than or equal to 97.

By Observation 2.1, the only faces of G whose sets of boundary vertices differ from those of their counterparts in G' are the elements of $F_G(v)$, but by the choice of the color of v, the spv-condition is maintained for them. Hence, the coloring of G is also an spv-coloring; a contradiction.

Finally we deal with statement (5). Suppose it does not hold. We construct a graph G'' by annihilating u. As above, G'' is 2-connected, and hence we may annihilate v in G'' to obtain the graph G'. Since u and v have precisely one common incident face in G, they are not adjacent; therefore, the annihilation of u does not create any new edges at v. This means, by part (2), that there is no pair of parallel edges incident with v in G'', and thus, considering statement (1) again, Lemma 2.2 can be applied to the annihilation of v. We conclude that G' is 2-connected. As it is also smaller than G with respect to our order, there is a proper spv-coloring c' of G' using at most 97 colors.

We extend the coloring to G as follows. If there exists a color used by c' on a vertex in $V_G(f) - N_G^F(u,f) - N_G^F(v,f)$ but no vertex in $N_G^F(u,f) \cup N_G^F(v,f)$, we assign this color to both u and v. Otherwise we color each of u and v with a different color not used by c' on any vertex in $N_G^F(u,f) \cup N_G^F(v,f)$, but if possible appearing in c'. Either case yields a coloring c of G with no more than 97 colors.

For the desired contradiction, it remains to show that c is a proper spv-coloring of G. As the neighbors of u and v belong to $N_G^F(u,f) \cup N_G^F(v,f)$, the coloring is indeed proper. Next, by Observation 2.1 and the assumption about $F(u) \cap F(v)$, we see that each face g of G has its counterpart g' in G' with $V_G(g)$ equal to $V_{G'}(g') \cup \{u,v\}$ if $f = g$, $V_{G'}(g') \cup \{u\}$ if $g \in F_G(u) - \{f\}$, $V_{G'}(g') \cup \{v\}$ if $g \in F_G(v) - \{f\}$, and $V_{G'}(g')$ otherwise. Considering the particular choice of the colors of u and v in either case, it is straightforward that the spv-condition holds for every face of G. □

Let f be a face of the graph G. The number of high-degree vertices (i.e., vertices of degree at least 3) on the boundary of f is called the *weight* of f and is denoted by $w(f)$. The face f is a *pseudodigon* if $w(f) = 2$ and f is not a digon. We say that f is *small* if $w(f) < 20$, and *large* if $w(f) \geq 20$.

The *configuration* of a vertex v of G is the tuple obtained by ordering the elements from the multiset $\{w(g) \colon g \in F(v)\}$ in a nondecreasing manner. If we remove (one copy of) the element $w(f)$ from this ordered multiset, we obtain the *f-reduced configuration* of v.

Lemma 2.5. *Let v be a vertex of G.*

(1) *Assume that (l_i), $i \in \mathbb{Z}_k$, is the configuration of v. Then:*

   (a) $d^F(v) \leq 4(l_0 + l_1) - 6$ *if* $k = 2$*;*

   (b) $d^F(v) \leq 4\sum_{i=0}^{2} l_i - 15 - 3|\{i : l_i + l_{i+1} \leq 25\}|$ *if* $k = 3$*;*

   (c) $d^F(v) \leq 4\sum_{i=0}^{k-1} l_i - 5k$ *if* $k \geq 4$*;*

(d) (l_i) *does not equal* $(2, 8, 19)$ *nor* $(3, 8, 19)$.

(2) *Let* f *be a face of* G *incident with* v. *Suppose that* (l_i), $i = 1, \ldots, k$, *is the* f*-reduced configuration of* v. *Then:*

(a) $d^F(v, f) \le 4\sum_{i=0}^{k-1} l_i - 5k + 4$;

(b) $d^F(v, f) \le 4(l_1 + l_2) - 9$ *if* $k = 2$ *and* $l_1 + l_2 \le 25$.

Proof. For convenience, we construct a graph G' from G by replacing each nontrivial path P in G having both end vertices of high degree and all internal vertices of degree 2 with an edge. (Note that P may be a single edge.) The correspondence between G and G' is then as follows. Since G is 2-connected and is not a cycle by Lemmas 2.4 (1) and 2.4 (3), the vertices of G' are precisely the high-degree vertices of G. Furthermore, for every edge e of G' there is a corresponding path P_e in G with the same end vertices and all internal vertices of degree 2. Every face f' of G' has its counterpart f in G (and vice versa) with a facial walk of f arising from a facial walk of f' by replacing each edge e with P_e; consequently, $w(f) = w(f')$. Note that by Lemma 2.4 (3), P_e has at most three internal vertices for every $e \in E(G')$.

We start with the following observation, which directly implies statement (1a).

Let $u \in V(G)$ *have degree* 2. *Denote by* f_1, f_2 *the faces incident with* u, *and by* f'_1, f'_2 *the corresponding faces in* G'. *Call* e_u *the edge of* G' *such that* $u \in V(P_{e_u})$, *and let* u' *be one of the end vertices of* P_{e_u}. *Finally, let* e_1, e_2 *denote the edges of* $E(f'_1)$, $E(f'_2)$, *respectively, that are incident with* u' *and different from* e_u. *Then* $d^F(u) \le 4(w(f_1) + w(f_2)) - 6$, *and if both* P_{e_1} *and* P_{e_2} *are single edges,* $d^F(u) \le 4(w(f_1) + w(f_2)) - 12$. (2.2)

To prove the observation, note that there are $|V(f'_1) \cup V(f'_2)| \le w(f'_1) + w(f'_2) - 2$ vertices and $|E(f'_1) \cup E(f'_2)| - 1 \le w(f'_1) + w(f'_2) - 2$ edges distinct from e_u incident with f'_1 or f'_2 in G'. Therefore in G the set $N^F(u)$ consists of at most $w(f'_1) + w(f'_2) - 2$ high-degree vertices, at most $3(w(f'_1) + w(f'_2) - 2)$ vertices of degree 2 contained in some P_e, $e \in E(f'_1) \cup E(f'_2) - \{e_u\}$, and at most two extra 2-vertices adjacent to u. In total, we obtain the desired bound for $d^F(u)$. Clearly, if neither P_{e_1} nor P_{e_2} have any internal vertices, the maximum number of 2-vertices in $N^F(u)$, and hence the upper bound in (2.2), drops by 6.

Statement (2a) for $k = 1$ now follows by focusing only on the vertices and edges incident with f'_1 or f'_2, say f'_1, in the proof of (2.2). First observe that v is of degree 2 as $k = 1$. Then the face incident with u, but distinct from f, has l_1 high-degree vertices, and at most $3l_1 - 1$ vertices of degree 2.

The rest of the proof is based on a similar, albeit generalized reasoning. Let v be of high degree d; hence it is also in $V(G')$. Let the edges incident with v in G' be enumerated in a clockwise order as $e_i = vv_i$, $i \in \mathbb{Z}_d$; let f'_i denote the face of G' whose facial walk contains $e_i v e_{i+1}$ as a subsequence and let f_i be the corresponding face of G; let E be the union of $E_{G'}(f'_i)$ and let E_j be the set $\bigcup_{i,\, i \ne j} E_{G'}(f'_i)$; finally, let E_v and V_v denote the set of all the edges e_i and vertices v_i respectively.

We claim that for every vertex v' in $N^F_{G'}(v)$, there exists an integer k such that $v' \in V(f'_k) - \{v, v_{k+1}\}$, and similarly, if $v' \in N^F_{G'}(v, f'_j) - \{v_j\}$ for a fixed j, then there exists an integer k distinct from j such that $v' \in V(f'_k) - \{v, v_{k+1}\}$. These assertions are trivially true when $v' \notin V_v$ or there is an integer k such that $e_k = vv'$ and f'_k is not a digon. Therefore assume the contrary. But then it is easy to see that all the faces f'_i are digons. By observation (2.2) and Lemma 2.4 (4), every P_{e_i} is a single edge; hence a pair of parallel edges occurs in G, contradicting Lemma 2.4 (2).

Thus $N^F_{G'}(v) = \bigcup_i (V(f'_i) - \{v, v_{i+1}\})$ and $N^F_{G'}(v, f'_j) = \bigcup_{i,\, i\neq j} (V(f'_i) - \{v, v_{i+1}\}) \cup \{v_j\}$. We conclude that $d^F_{G'}(v)$ and $d^F_{G'}(v, f'_j)$, i.e., the number of high-degree vertices in $N^F_G(v)$ and $N^F_G(v, f_j)$, respectively, is bounded from above by $\sum_i (w(f'_i) - 2)$ and $\sum_{i,\, i\neq j} (w(f'_i) - 2) + 1$, respectively. Clearly $|E - E_v| \leq \sum_i (w(f'_i) - 2)$ and $|E_j - E_v| \leq \sum_{i,\, i\neq j} (w(f'_i) - 2)$; therefore the number of 2-vertices in $N^F_G(v)$ and $N^F_G(v, f_j)$ not contained in any P_{e_i} is at most $3\sum_i (w(f'_i) - 2)$ and $3\sum_{i,\, i\neq j} (w(f'_i) - 2)$, respectively. Finally, there are up to three vertices of degree 2 in every P_{e_i}, hence at most $3d$ in total. Summing the respective bounds gives

$$d^F_G(v) \leq 4 \sum_i w(f'_i) - 5d, \tag{2.3}$$

$$d^F_G(v, f_j) \leq 4 \sum_{i,\, i\neq j} w(f'_i) - 5d + 9. \tag{2.4}$$

Statements (1c) and (2a) for $k \geq 2$ immediately follow.

Now we focus on assertions (1b) and (1d). By the hypotheses $d = 3$; up to the orientation of the plane, we may choose the labeling so that $w(f_i) = l_i$. Then for each j such that $l_j + l_{j+1} \leq 25$, the path $P_{e_{j+1}}$ is a single edge by (2.2) and Lemma 2.4 (4), and the right side of (2.3) decreases by 3. This proves statement (1b). Now let $(l_i) = (2, 8, 19)$ or $(3, 8, 19)$. Both P_{e_0} and P_{e_1} are single edges by the preceding analysis. We infer that P_{e_2} also has no internal vertex by (2.2) and Lemma 2.4 (4). Hence the bound in (2.3) drops from 105 by 9, which proves assertion (1d).

Finally, suppose the assumptions of statement (2b) hold. Again, up to the orientation of the plane, we may assume the labeling to be such that $w(f_j) = l_j$, $j = 1, 2$, and $f_0 = f$. As before, $l_1 + l_2 \leq 25$ implies that the path P_{e_2} has no internal vertices. The bound in (2.4) for $j = 0$ decreases by 3 and yields the desired inequality. □

2.2 Discharging

Having explored the properties of the graph G, we are ready to use the discharging method to arrive at a contradiction.

We assign an initial charge to the vertices and faces of G as follows:

- each vertex v receives $d(v) - 6$ units of charge;
- each face f receives $2|V(f)| - 6$ units of charge.

The following observation is a well-known consequence of Euler's formula.

Observation 2.6. *The sum of the charges defined above is* -12*.*

In the first phase, we redistribute the charges according to Rules 1–2:

Rule 1. *Every face that is not a pseudodigon sends two units of charge to each incident* 2*-vertex. Each pseudodigon does the same, except that one of the respective* 2*-vertices receives no charge.*

Observe that after the application of Rule 1, the charge of each face is nonnegative. In addition, the charge of a large face is at least $2 \cdot 20 - 6 = 34$.

Rule 2. *Every small face distributes its remaining charge evenly to all incident high-degree vertices (i.e., vertices of degree at least* 3*). Each large face (i.e., a face of weight at least* 20*) behaves in the same way, except that it retains a charge of* 4*.*

After applying the above rules the first phase is completed. In the second phase, Rule 3 is applied to the vertices that ended up with negative charge after the first phase.

Rule 3. *If a vertex has a negative charge of* c *and is incident with a large face* f*, then it receives the charge of* $-c$ *from* f*.*

We will show that the final charge of every vertex and face in G is nonnegative, contradicting Observation 2.6.

It will be convenient to alter the definition of the weight of a face f and the configuration of a vertex v as follows. The *modified weight* $w'(f)$ of f is defined as 3 if $w(f) = 2$, and $w(f)$ otherwise. Replacing the weight of each face by its modified weight in the definition of the configuration of v, we obtain the *modified configuration* of v. The *modified* f*-reduced configuration* of v is obtained from the f-reduced configuration of v in an analogous way.

First we analyze how much charge a vertex v of high degree d receives by Rule 2. Denote the faces incident with v by f_i, $i = 0, \ldots, d-1$, and let n_i be the number of 2-vertices incident with f_i. After applying Rule 1, each f_i has charge $2|V(f_i)| - 6 - 2n_i$ if f_i is not a pseudodigon, and $2|V(f_i)| - 6 - 2(n_i - 1)$ otherwise. By Lemma 2.4 (2), f_i is not a digon so in both cases the charge can be written as $2w'(f_i) - 6$. Hence, when f_i is a small face, it sends v the charge of

$$\frac{2w'(f_i) - 6}{w(f_i)} = 2 - \frac{6}{w'(f_i)}.$$

(The equality is true as $w(f_i) = w'(f_i)$ if f_i is not a pseudodigon, and $2w'(f_i) - 6 = 0$ otherwise.) On the other hand, when f_i is a large face, it sends v

$$\frac{2w'(f_i) - 6 - 4}{w(f_i)} = 2 - \frac{10}{w'(f_i)}$$

units of charge. Note that in both cases the charge received by v from f_i is nonnegative. In total, the vertex v obtains the nonnegative charge of

$$\begin{aligned}\sum_{\substack{i\\ w'(f_i)<20}} \left(2-\frac{6}{w'(f_i)}\right)+\sum_{\substack{i\\ w'(f_i)\geq 20}} \left(2-\frac{10}{w'(f_i)}\right)\\ =2d-6\sum_{\substack{i\\ w'(f_i)<20}} \frac{1}{w'(f_i)}-10\sum_{\substack{i\\ w'(f_i)\geq 20}} \frac{1}{w'(f_i)}.\end{aligned} \tag{2.5}$$

Next, we establish the following two essential claims. For convenience, we refer to the vertices with a negative charge after the first phase as *special* vertices. Since the initial charge of a vertex v is $d(v)-6$ and each vertex receives a nonnegative charge during the application of Rule 2, every special vertex has degree at most 5.

Claim 1. *Every special vertex is incident with a large face.*

Proof. We proceed by contradiction, assuming that v is a special vertex not incident with any large face. First suppose that $d(v)=2$; let f_1 and f_2 be the two faces incident with v. As the initial charge of v is -4, at least one of these faces, say f_1, is a pseudodigon by Rule 1. By assumption, $w(f_2)\leq 19$. Hence $d^F(v)\leq 84$ by Lemma 2.5 (1a), a contradiction to Lemma 2.4 (4).

Therefore, let v be a special vertex of high degree d. Summing its initial charge and the charge (2.5) received by Rule 2 gives

$$3d-6-6\sum_i \frac{1}{w'(f_i)}<0,$$

or equivalently,

$$\sum_i \frac{1}{w'(f_i)}>\frac{d}{2}-1. \tag{2.6}$$

We proceed by case analysis; let (l'_i) denote the modified configuration of v. (We write (l'_i) instead of (l_i) as a reminder that the configuration is a modified one.) Assume first that v is of degree 3. Then $l'_0\leq 5$, otherwise (2.6) fails since its left hand side is at most $3\cdot(1/6)$ which equals its right hand side $1/2$. Since v is not incident with any large face, we have $l'_1,l'_2\leq 19$. We aim to use Lemma 2.5 to bound $d^F(v)$. Although it is formulated for ordinary (non-modified) configurations, the monotonicity of the upper bounds ensures that the lemma remains valid if the configuration is a modified one. By Lemma 2.5 (1b), $d^F(v)\leq 4\sum_i l'_i-21$. Consequently, Lemma 2.4 (4) implies that

$$\sum_i l'_i\geq 30. \tag{2.7}$$

If $l'_0=3$ and $l'_1\leq 9$, then by (2.7) (l'_i) is one of the three tuples $(3,8,19)$, $(3,9,18)$, and $(3,9,19)$. The first of these is excluded by Lemma 2.5 (1d), and the remaining two contradict (2.6). If $l'_0 = 3$ and $l'_1 > 9$, then by Lemma 2.3 applied to the tuples $(3, 10, 15)$ and (l'_i), in this order, either $\sum_i l'_i \leq 28$ or $\sum_i 1/l'_i \leq 1/2$. However, that contradicts (2.7) or (2.6), respectively.

Hence $l'_0 \geq 4$. If $l'_0, l'_1 = 4$, then $\sum_i l'_i \leq 27$, which is impossible by (2.7). Otherwise we may use Lemma 2.3 for the tuples $(4, 5, 20)$ and (l'_i), and obtain a contradiction to (2.7) or (2.6) again.

Thus $d \geq 4$; as we have remarked above, $d \leq 5$. Lemmas 2.5 (1c) and 2.4 (4) imply (2.7) again. In particular, (l'_i) cannot be of the form $(3, 3, 3, x)$. If $d = 4$, then by Lemma 2.3 applied to the tuple $(3, 3, 4, 12)$, $\sum_i l'_i \leq 22$ or $\sum_i 1/l'_i \leq 1$, contradicting (2.7) or (2.6). For $d = 5$, we obtain a similar contradiction by applying Lemma 2.3 to $(3, 3, 3, 3, 6)$, which yields $\sum_i l'_i \leq 18$ or $\sum_i 1/l'_i \leq 3/2$. □

Claim 2. *Every large face has a nonnegative final charge.*

Proof. Let f be an arbitrary large face of G. We start by listing the possible f-reduced or modified f-reduced configurations of special vertices incident with f, and for each case we note a lower bound on the charge of these vertices. Take such a special vertex v.

If v is a 2-vertex, then by Rule 1 its f-reduced configuration is (2) and the charge equals -2.

Now suppose that v is of high degree d. Let (l'_i), $i = 1, \dots, d-1$, be its modified f-reduced configuration, and let d' denote the number of large faces incident with v. As noted earlier, $d \leq 5$. By considering the initial charge of v and (2.5), we see that after applying Rule 2, v has charge

$$3d - 6 - 6 \sum_{\substack{i \\ w'(f_i) < 20}} \frac{1}{w'(f_i)} - 10 \sum_{\substack{i \\ w'(f_i) \geq 20}} \frac{1}{w'(f_i)}.$$

As the charge is negative by assumption,

$$3d - 6 - 6 \sum_{\substack{i \\ w'(f_i) < 20}} \frac{1}{w'(f_i)} - \frac{d'}{2} < 0 \tag{2.8}$$

by the definition of large face. Furthermore, $w'(f_i) \geq 3$ always, and hence

$$3d - 6 - 2(d - d') - d'/2 < 0.$$

Since the left hand side equals $d - 6 + 3d'/2$, we deduce that $d' = 1$ (i.e., f is the only large face incident with v), and that $d \leq 4$.

Assume first that $d = 3$. Then (2.8) reduces to

$$\frac{5}{2} - 6\left(\frac{1}{l'_1} + \frac{1}{l'_2}\right) < 0.$$

We infer that either $l'_1 = 3$ and $l'_2 \leq 11$, or $l'_1 = 4$ and $l'_2 \leq 5$. The charge of v is at least $1/2 - 6/l'_2$ in the former, and at least $1 - 6/l'_2$ in the latter case.

(modified) f-reduced configuration	*charge*
(2)	-2
$(3,x)$, $x \le 11$	$\ge 1/2 - 6/x \ge -3/2$
$(4,x)$, $x \le 5$	$\ge 1 - 6/x \ge -1/2$
$(3,3,3)$	$\ge -1/2$

Table 1. The proof of Claim 2: the list of possible f-reduced (the first line) or modified f-reduced (the other lines) configurations of special vertices incident with the face f, together with the charge of these vertices.

Now let $d = 4$. By (2.8),

$$\frac{11}{2} - 6\sum_i \frac{1}{l'_i} < 0.$$

If some l'_i were greater than or equal to 4, this inequality would not hold. Hence $(l'_i) = (3,3,3)$ and the charge of v is at least $-1/2$. We summarize the results in Table 1.

Let S denote the set of all special vertices incident with f and R their total charge. We observe the following:

$$\text{\textit{Any two vertices } } u, v \in S \text{ \textit{ have at least two common incident faces.}} \tag{2.9}$$

Suppose the contrary. In view of the possible f-reduced or modified f-reduced configurations of u and v listed in Table 1, Lemma 2.5 (2a) and (2b) implies that both $d^F(u,f)$ and $d^F(v,f)$ are at most 47. On the other hand, Lemma 2.4 (5) and the assumption that f is the only face incident with both u and v imply that $d^F(u,f) + d^F(v,f) > 95$, a contradiction.

We proceed to prove Claim 2 by contradiction. Suppose that f has a negative charge after the application of Rule 3. Since the charge of f after the first phase is 4 units (by Rule 2 for large faces), this is equivalent to the condition

$$R < -4. \tag{2.10}$$

Considering the lower bounds for charges in Table 1, we see that there are at least three special vertices.

Let $v \in S$ be a 2-vertex; the other face f' incident with v is a pseudodigon. By Rule 1 and (2.9), every other vertex v' in S is one of the two high-degree vertices v_1, v_2 incident with f'. Therefore $S = \{v, v_1, v_2\}$; it follows that v_1 and v_2 are both of degree 3 by assumption (2.10). This means that $F(v_1) = F(v_2)$, and hence the configuration of both vertices is the same. Considering (2.10) again, the modified f-reduced configuration of both v_1 and v_2 is $(3,3)$.

At this point, we digress by making an auxiliary observation:

Let g be a face of G with $w(g) \leq 3$. The intersection of the boundaries of f and g consists of pairwise disjoint paths, of which at most one is nontrivial; the internal vertices of all these paths are of degree 2 in G. Furthermore, any two vertices of degree 3 incident with both f and g are precisely the end vertices of such a nontrivial path, and hence, there are at most two such vertices. (2.11)

To prove this, let us denote the intersection of the boundaries of f and g by H. The assertion about the structure of H is obvious by considering the 2-connectedness of G and the weights of both f and g. Let $\mathcal{P}$ denote the set of the respective paths. If v is a vertex of degree 3 incident with f and g, then H must contain an edge incident with v, i.e., v lies on—and hence is an end vertex of—a nontrivial path in $\mathcal{P}$. The second assertion easily follows.

Applying (2.11) to the face incident with v_1 different from f and f', and recalling that f' is a pseudodigon, we infer that $w(f) = 2$; a contradiction.

Thus all vertices in S are of high degree. Suppose that $S = \{v_1, v_2, v_3\}$. Then by assumption (2.10), the modified f-reduced configuration of v_1, v_2, and v_3 is $(3, 3)$. Hence by (2.11), no face other than f is incident with the three vertices. By (2.9), G contains three different faces f_{ij}, $1 \leq i < j \leq 3$, incident with both v_i and v_j and different from f. By (2.11), the boundary of f is precisely $\bigcup \mathcal{P}$; thus $w(f) = 3$, a contradiction to the assumption that f is large.

Therefore $|S| \geq 4$. We claim that G contains a face $f' \neq f$ incident with all the vertices in S. To prove this, take two vertices u, v from S nonadjacent in the boundary of f. By (2.9), there is a curve $\mathcal{C}_{uv}$ connecting u and v through a face $f_{uv} \neq f$ of G. Consider two other vertices x, y in S such that u, x, v, and y appear in a facial walk of f in this order. Again, there is a curve $\mathcal{C}_{xy}$ joining x and y contained in a face $f_{xy} \neq f$, with the exception of its ends. Clearly, $\mathcal{C}_{uv} \cap \mathcal{C}_{xy} \neq \emptyset$, and hence $f_{xy} = f_{uv}$; the assertion easily follows.

Now let $k := |S|$. Then $w(f') \geq k$, and consequently the f-reduced configuration of each v in S contains a number greater than or equal to k. From Table 1, we see that

$$R \geq k\left(\frac{1}{2} - \frac{6}{k}\right) = \frac{k}{2} - 6,$$

the right side of which is at least -4 by the condition on k. This contradicts assumption (2.10). □

With the help of the two preceding claims, we can easily finish the proof. By Claim 1 and Rule 3, every special vertex—and hence every vertex—ends up with a nonnegative charge. The final charge of every face is nonnegative as well; Rule 2 and Claim 2 guarantee this for small and large faces respectively. However, as already mentioned, this contradicts Observation 2.6.

3 Lower bound

In this section, we provide examples showing that the best possible constant bound on χ_s for the class of 2-connected plane simple graphs is at least 8, and the corresponding bound for proper spv-colorings is at least 10. Note that for the class of 2-connected plane multigraphs, the latter example implies a lower bound of 10 for general spv-colorings. Indeed, if we replace each of its edges by a digon bounding a face, then every spv-coloring of the resulting multigraph is necessarily a proper spv-coloring of the original graph.

First, we focus on the bound for proper spv-colorings. We construct a graph G_{55} on ten vertices by linking two disjoint cycles C^1, C^2 on five vertices with two additional edges whose endvertices in each cycle are adjacent. See Figure 3.1 (a).

By the spv-conditions for the two faces of G_{55} of length 5, every spv-coloring c must assign each vertex of C^1 a different color; the same holds for C^2. The spv-condition for the face of G_{55} of length 10 then implies that c uses each color precisely once.

Second, we consider the bound on χ_s (where the coloring is allowed to be improper). Take a three-sided prism G_3 embedded in the plane so that one of its triangular faces is the outer face f_3. As observed by Czap and Jendroľ [3, proof of Lemma 5.1], every coloring c of G_3 such that every face of G_3 distinct from f_3 satisfies the spv-condition colors each boundary vertex of f_3 with a different color.

Now construct a graph G_4 from a cycle on four vertices by replacing every other edge with a copy of G_3 in such a way that the outer face f_4 of G_4 is of length 4; see Figure 3.1 (b). Let c' be a coloring of G_4 satisfying the spv-condition for each face of G_4 other than f_4. When restricted to the vertices of any of the copies of G_3, c' has the property of the coloring c discussed above. This and the spv-condition for the face of G_4 of length 6 imply that c' assigns a different color to each boundary vertex of f_4.

Finally, we reproduce the construction of G_{55} with copies of G_4 in place of the cycles on five vertices. Thereby we obtain a graph G_{44} with the outer face f_{44} of length 8, such that every spv-coloring of G_{44} is injective on $V(f_{44})$. The graph G_{44} is shown in Figure 3.1 (c). □

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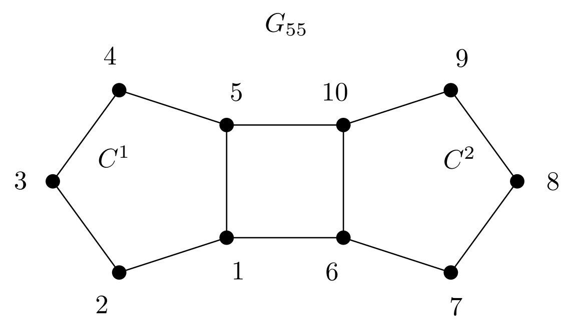
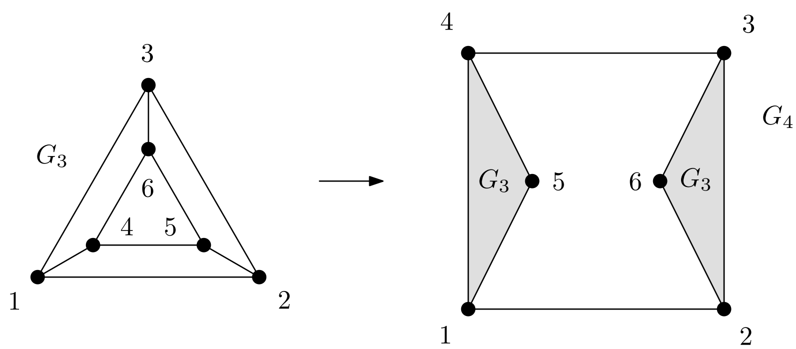
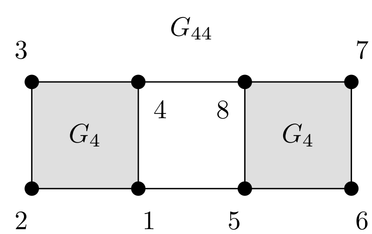


(a) The graph G_{55}.



(b) The graphs G_3 (left) and G_4 (right).



(c) The graph G_{44}.

Figure 3.1. Illustrations for Section 3. The labeled gray areas represent the respective subgraphs not depicted in detail. For each graph, the relevant coloring is unique up to symmetry; it is indicated by numerical labels.

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