Problems about Uniform Covers, with Tours and Detours (Oberwolfach Report)

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(joint work with Yohann Benchetrit and Matěj Stehlík)

Is $\underline{1}$ a linear or integer combination of some combinatorially interesting vectors? Some examples, with detours:

1. Tours

A *tour* in the graph $G = (V, E)$ is an Eulerian $0 - 1 - 2$ function on the edges (even on stars, connected support). We adapt Wolsey's argument [16] to prove:

Fact: *If G is 3-edge-connected, the all 1 function $\underline{1}$ is in the convex hull of tours.*

Proof. $\underline{2}/3$ dominates a point in the spanning tree polytope (satisfies subtour elimination) ; $\underline{1}/3$ dominates a point in the T-join polyhedron, for all T. It is then easy to see that $\underline{1} = \underline{2}/3 + \underline{1}/3$ is in the convex hull of trees + an edge-set for each tree correcting the parities of its degrees. □

The same holds for T-tours, that is, connected T-joins, in particular $\{s, t\}$-tours.

Problem 1: Can this bound be improved for tours?

The answer is probably yes: by the '4/3 integrality gap conjecture' [8] $\underline{4}/3 \times \underline{2}/3 = \underline{8}/9$ is in the convex hull of tours. For $\{s, t\}$-tours $\underline{3}/2 \times \underline{2}/3 = 1$.

We make now a detour to a lower bound that is in some cases better than linear programming. The more there are degree 2 vertices the better it is.

Let $G = (V, E)$ be a graph, $m := |E|$, $n := |V|$. There is a unique graph $G^* = (V^*, E^*)$, $m^* := |E^*|$, $n^* := |V^*|$ without degree 2 vertices of which G is a subdivision. Let T_G be the set of odd degree vertices of G, τ the minimum size of a T_G-join, and OPT the minimum size of a tour.

Inequality: *Let G be a 2-edge-connected graph. Then $m+\tau-2k \le OPT \le m+\tau$, where $k = m-n+1 = m^*-n^*+1$ is the number of ears in an ear-decomposition.*

Proof. Consider a tour in $G = (V, E)$, and let F be the set of edges of multiplicity 2 or 0, and $F^* \subseteq F$ those of multiplicity 0; F is a T_G-join.

Since $E \setminus F^*$ is connected, $|E \setminus F^*| \ge n - 1$, that is, $|F^*| \le m - n + 1 = m^* - n^* + 1 = k$. The tour length is: $|E| + |F| - 2|F^*| \ge m + \tau - 2k$. □

Note that the upper bound is just the minimum of the Chinese Postman trail; F^* contains at most one edge of each series class; the inequality and its proof can be straightforwardly generalized to weights.

Corollary : *For the subdivisions of a given graph the solution of the Chinese Postman problem has a constant additive error for the smallest tour.*

Problem 2: When the lower bound is bad (k is large), the upper bound can also be replaced by a much smaller value! How to improve the bounds in a useful way?

2. H-perfect graphs

Given a graph G and a non-negative rational λ, the *fractional chromatic number* χ_f is the minimum of λ such that $\underline{1}/\lambda$ is in the stable set polytope. For *t-perfect* graphs [13] the maximum of $\underline{1}$ on $\{x \in \mathbb{R}^{V(G)} : x(S) \leq 1, \text{for all stable } S, x \geq 0\}$ is at most 3, so the optimum of the dual, $\chi_f \leq 3$.

Shepherd conjectured that the same is true for the chromatic number χ.

Laurent and Seymour [13] realized that the complement of the line graph of the prism (a prism is the complement of C_6) is a counterexample. This graph is the "t-minor" of a 3-colorable t-perfect graph, contradicting the integer round-up property of 3-colorable t-perfect graphs, conjectured by Shepherd [15]. It is then natural to conjecture 4-colorability. Actually more could be true:

Conjecture 3: Every h-perfect graph is $\omega + 1$ - colorable ($\omega :=$ clique-number).
Theorem: *If this conjecture is true for $\omega = 2$, then it is true in general.*

Proof. If $\omega > 2$, the optimal face is that of the ω-cliques so any stable set active in an optimal dual solution meets all ω-cliques. □

Benchetrit [1] found that the complement of the line graph of a 5-wheel is also a counterexample to Shepherd's conjecture. In some sense the two counterxamples are the only obstacles to the integer round-up property [1].

We make now a detour to the maximum number, β, of starting odd ears in an ear decomposition [3], related to h-perfect graphs, rounding, the matching polytope; expressing the complexity of the latter. This is joint work with Yohann Benchetrit.

Question 4: What is the complexity of computing β?

We call θ here a subgraph consisting of three edge-disjoint paths, two of which are odd, and one even, between two fixed vertices of a graph. A basis of the cycle space (over GF(2)) of a graph that consists only of odd cycles will be called an *odd cycle-basis.* The existence of an odd cycle basis of a non-bipartite graph immediately follows from the open ear-decomposition of 2-connected graphs, and the following easy and well-known fact [11]: *in a 2-vertex-connected non-bipartite graph there exist both an even and an odd path between any two vertices. The following theorem straightforwardly implies a characterization of h-perfect line graphs.*

Theorem *Let G be a 2-vertex-connected graph. The following are equivalent:*

(i) *There exists no θ in G.*
(ii) $\beta(G) \leq 1$.
(iii) *Any two simple odd cycles have an odd number of common edges.*
(iv) *In each odd cycle basis, any two cycles meet in an odd number of edges.*
(v) *There exists an odd cycle basis with the property stated in (iii).*

Proof. Any of (i) or (iii) imply (ii), since an odd cycle C completed by an open odd ear P is a θ, and contradicts (iii). These are known from [5], [6], the rest is from [3].

Supposing (ii) the proof of (iii) is a graph-theory exercise: if two cycles, Q_1 and Q_2 do not satisfy (iii) and $|V(Q_1) \cap V(Q_2)| \geq 2$, then $|E(Q_1) \setminus E(Q_2)|$ is odd, easily contradicting (ii). Otherwise Q_1 and Q_2 are edge-disjoint and one concludes using Menger's theorem.

Two implications are straightforward: (iv) is just a special case of (iii), and (v) is a special case of (iv). Last, but not least, if (v) holds, then any odd cycle is the mod 2 sum of an odd number of cycles, and then knowing (iii) for the basis, it follows for any pair of odd cycles. □

3. Hereditary hypergraphs

This section reports about joint work with Matěj Stehlík [14]. Let $H = (V, E)$ be a *hereditary hypergraph*: if $e \in E$ all subsets of e are in E.

Closed Problems:

1. Is $\underline{1}$ an integer sum of incidence vectors of $e \in E$, $|e| \geq 2$?
2. Compute the minimum size ρ of a cover of V by members of E.
3. Compute the maximum size μ of a set that can be partitioned into $e \in E$, $|e| \geq 2$. Such a set is called a *μ-matching.*

Theorem: *Problem 2. is NP-hard (*SET COVER*) but 1. and 3. are polynomially solvable. Furthermore, there exists a cover of size ρ containing a μ-matching.*

The polynomial algorithms are easy consequences of vertex-packing edges and triangles [7], whereas the last sentence follows from [11][Exercise 9.4], originating from Gallai's work [10]. Yet the connections provide a new insight into packing and covering: the difficult theorem of Gallai [10] is equivalent to the factor-critical version of [9], and relevant information is smuggled in about the NP-hard problem of minimum covers, and by transposition, about minimum transversals [14].

Problem 5: Study some conjectures about packing, covering and minimum transversals bearing in mind the connections mentioned above.

4. Triangles

Problem 6: [12] Characterize the graphs for which $\underline{1}$ is a nonnegative combination of triangles as edge-sets. In other words, can the system of linear inequalities describing the cone of triangles of a graph be described ?

The origins of this problem are in regular covers of edges by triangles, see [12].

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