



Layers and matroids for the traveling salesman's paths

Frans Schalekamp^a, András Sebő^b, Vera Traub^{c,*}, Anke van Zuylen^d

^a School of Operations Research and Information Engineering, Department of Computer Science, Cornell University, Ithaca NY, USA

^b Optimisation Combinatoire (G-SCOP), CNRS, Univ. Grenoble Alpes, Grenoble, France

^c Research Institute for Discrete Mathematics, University of Bonn, Bonn, Germany

^d Department of Mathematics, College of William & Mary, Williamsburg VA, USA

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ABSTRACT

Gottschalk and Vygen proved that every solution of the subtour elimination linear program for traveling salesman paths is a convex combination of more and more restrictive “generalized Gao-trees”. We give a short proof of this fact, as a *layered* convex combination of bases of a sequence of increasingly restrictive matroids. A strongly polynomial, combinatorial algorithm follows for finding this convex combination, which is a new tool offering polyhedral insight, already instrumental in recent results for the $s - t$ path TSP.

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1. Introduction

Gottschalk and Vygen [6] proved that every solution of the well-known subtour elimination linear program for traveling salesman paths is a convex combination of a set of more and more restrictive “generalized Gao trees” of the underlying graph, where a Gao-tree is a spanning tree that meets certain cuts in exactly one edge.

In this paper we provide *layered* convex combinations of bases of a sequence of more and more restrictive matroids for a larger set of points, which we call chain-points, generalizing the subtour elimination feasible solutions. This leads to a new connection of the TSP to matroids (observed in [8]), offering also a polyhedral insight, with specific algorithmic consequences, such as a strongly-polynomial combinatorial algorithm for finding this convex combination via the matroid partition theorem. In this paper we show how the technical difficulties for proving the existence of such a particular convex combination and of turning it into an algorithm can be avoided, and a simple proof follows.

The existence of such a convex combination has been used by Sebő and Van Zuylen [8] to prove the so far best upper bound on the integrality gap for the $s - t$ path traveling salesman problem. (We note that a recent result by Traub and Vygen [9] gives an approximation ratio of $3/2 + \varepsilon$ for any constant $\varepsilon > 0$, but this

result does not imply an improved bound on the integrality gap.) This convex combination and the method we exhibit now to prove this result may possibly be adapted for proving further results on versions of the traveling salesman problem. The statement itself and its connection to matroids is one more link of the TSP to matroid partition or intersection, a connection interesting for its own sake.

In Section 2 we introduce chain-points, a key notion of our proof, and layered convex combinations, the related matroids, some preliminaries about these and some simple assertions that enable us to “peel off” layers one by one. In Section 3 we execute the induction step of peeling off a layer. We finish this introduction by introducing some notation and terminology.

We assume throughout that we are given a graph $G = (V, E)$, and $s, t \in V$. For $S_1, S_2 \subseteq V$, we denote by $\delta(S_1, S_2) := \{i, j \in E : i \in S_1, j \in S_2\}$. If $S_1 = S_2$ we use $E(S_1) = \delta(S_1, S_1)$; if $S_1 = V \setminus S_2$ we use $\delta(S_1) = \delta(S_2) = \delta(S_1, V \setminus S_1)$. For $v \in V$, we let $\delta(v) = \delta(\{v\})$. We denote by $G(S) = (S, E(S))$ the subgraph induced by $S \subseteq V$. For $F \subseteq E$ and $x \in \mathbb{R}^E$, we denote $x(F) := \sum_{e \in F} x_e$. With an abuse of notation and terminology, sets and their incidence vectors will not be distinguished. We denote by $Sp(G) = \{x \in \mathbb{R}^E : x \geq 0, x(E(U)) \leq |U| - 1 \text{ for all } U \subseteq V, U \neq \emptyset, \text{ and } x(E) = |V| - 1\}$ the spanning tree polytope (convex hull of spanning trees) of G , and by $cone.Sp(G)$ their cone (vectors that are their non-negative combinations); $Sp(G) \subseteq cone.Sp(G)$.

We say that $x \in \mathbb{R}^E$ is a solution to the $(s - t)$ -path TSP subtour elimination LP if $x \in Sp(G)$, and x satisfies the degree constraints $x(\delta(v)) = 2$ for all $v \in V \setminus \{s, t\}$ and $x(\delta(s)) = x(\delta(t)) = 1$. This is

* Corresponding author.

E-mail addresses: fms9@cornell.edu (F. Schalekamp), andras.sebo@grenoble-inp.fr (A. Sebő), traub@or.uni-bonn.de (V. Traub), anke@wm.edu (A. van Zuylen).

equivalent to the “standard definition” of the subtour elimination linear program: for $\emptyset \subset U \subseteq V \setminus \{s, t\}$, the constraints $x(\delta(U)) \geq 2$ and $x(E(U)) \leq |U| - 1$ are equivalent if x satisfies the degree constraints, and for $\{s\} \subseteq U \subseteq V \setminus \{t\}$ similarly, $x(\delta(U)) \geq 1$ and $x(E(U)) \leq |U| - 1$ are equivalent if x satisfies the degree constraints.

2. Chain-points, layers and matroids

Let $\emptyset \neq V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_k \subsetneq V$, and $\mathcal{C} := \{\delta(V_0), \delta(V_1), \dots, \delta(V_k)\}$. We call spanning trees of G that meet each $C \in \mathcal{C}$ in exactly one edge *Gao-trees* for the chain $V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_k$. The *Gao-edges* of a cut $Q \in \mathcal{C}$ are the edges $e \in Q$ for which $e \notin C$ for all $C \in \mathcal{C} \setminus \{Q\}$.

We say that $x \in \mathbb{R}^E$ is a *chain-point* for the chain of sets $V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_k$, if

- (i) $x \in Sp(G)$,
- (ii) $x(\delta(V_0)) = x(\delta(V_k)) = 1, x(\delta(V_i)) < 2$ ($i = 1, \dots, k - 1$),
- (iii) $\sum_{v \in L_i} x(\delta(v)) = 2|L_i|$ ($i = 1, \dots, k$),

where $L_i := V_i \setminus V_{i-1} \neq \emptyset$ ($i = 0, \dots, k + 1$) are the *level-sets* (where $V_{-1} := \emptyset$ and $V_{k+1} := V$). Note that levels L_0 and L_{k+1} are not present in (iii).

Gao-edges can then be equivalently defined as those joining two consecutive level-sets.

When $V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_k$ are clear from the context, we use the terms chain-point and Gao-tree without mention of the chain. Gao [4] proved that there exists a Gao-tree in the support of a subtour elimination LP solution x . It is not hard to adapt Gao’s proof to chain-points; it also follows from Lemma 5 in this work (with $U = V$ and E the support of x). Observe that Gao-trees are the bases of the matroid that is formed as the direct sum of the cycle matroids for the level sets L_0, \dots, L_k , whose bases are the spanning trees for the levels sets, and the uniform matroids on the Gao-edges for each of the cuts $Q \in \mathcal{C}$, whose bases are the one element subsets.

As An, Kleinberg and Shmoys [1] noted, if x is a solution to the $s - t$ path TSP subtour elimination LP, then $\{V_0, V_1, \dots, V_k\} := \{A \subseteq V \setminus \{t\} : x(\delta(A)) < 2 \text{ and } s \in A\}$ form a chain, i.e., $V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_k \subsetneq V$, where $V_0 = \{s\}$ and $V_k = V \setminus \{t\}$. To check this for the sake of self-containedness, let $A, B \in \{V_0, V_1, \dots, V_k\}$, and suppose for a contradiction that $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$. It will be convenient to consider A and $C := V \setminus B$. Observe that $x(\delta(A)) + x(\delta(C)) = x(\delta(A)) + x(\delta(B)) < 2 + 2 = 4$, so we have

$$4 > x(\delta(A)) + x(\delta(C)) = x(\delta(A \cap C)) + x(\delta(A \cup C)) + 2x(\delta(A \setminus C, C \setminus A)) \tag{SUBMOD}$$

by a well-known identity. Our assumption that $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$ is equivalent to $A \cap C \neq \emptyset$ and $A \cup C \neq V$, where $A \cap C$ contains neither s nor t and $A \cup C$ contains both. Using that x is a solution of the $s - t$ -path subtour elimination LP, $x(\delta(A \cap C)) + x(\delta(A \cup C)) \geq 2 + 2 = 4$, a contradiction.

A solution x to the $s - t$ path TSP subtour elimination LP is a chain-point for the chain $\{A \subseteq V \setminus \{t\} : x(\delta(A)) < 2 \text{ and } s \in A\}$, since in this case $V_0 = \{s\}$, $V_k = V \setminus \{t\}$ and thus the conditions (ii) and (iii) are implied by the degree constraints.

Gottschalk and Vygen [6] showed that it is possible to write a solution x to the $s - t$ path TSP subtour elimination LP as a convex combination of spanning trees such that for every $\lambda \in [0, 1]$ the coefficients of Gao-trees for the chain $\{A \subseteq V \setminus \{t\} : x(\delta(A)) \leq 2 - \lambda \text{ and } s \in A\}$ sum to at least λ . We call such a convex combination *layered* (see below for details).

The advantage of the notion of chain-points rather than subtour elimination LP solutions is that chain-points are closed under “subtracting Gao-trees” (while the set of feasible solutions to the subtour elimination linear program does not have this property

due to the degree constraints). This allows “peeling off layers” one by one and proving the existence of a layered convex combination by induction (Lemma 2 and Theorem 3).

We now define layered convex combinations somewhat more generally for chain-points, and in terms of the introduced matroids, in more detail, and state our main result, the existence of such a convex combination. Then we state the lemma that allows to peel off layers one by one, that is, to deal with only two layers at a time.

Fix a chain $\emptyset \neq V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_k \subsetneq V$. For a chain-point x , we call the values of $x(\delta(V_i))$ for $i = 0, 1, \dots, k$ the *narrow cut sizes*. Let the different values of the narrow cut sizes be $2 - \lambda_1 > 2 - \lambda_1 - \lambda_2 > \dots > 2 - \lambda_1 - \dots - \lambda_\ell = 1$, where ℓ is the number of different sizes.

We consider ℓ different matroids (E, \mathcal{B}_j) whose set of bases \mathcal{B}_j are the Gao-trees for the chain of sets $\{V_i : x(\delta(V_i)) \leq 2 - \lambda_1 - \dots - \lambda_j\}$, ($j = 1, \dots, \ell$). We say that $\sum_{j=1}^{\ell} \lambda_j x_j$ is a *layered convex combination* for x if x_j is in the convex hull of bases in \mathcal{B}_j for $j = 1, \dots, \ell$ and $x = \sum_{j=1}^{\ell} \lambda_j x_j$.

Theorem 1. *If x is a chain-point, then there exists a layered convex combination for x .*

The following lemma enables us to “peel off” layers of chain-points one by one, and concentrate on the case when there are only two distinct narrow cut sizes.

Denote the convex hull of Gao-trees of G for the chain \mathcal{C} by $Sp_{\mathcal{C}}(G)$.

Lemma 2. *Let x be a chain-point, and $x = \varepsilon y + (1 - \varepsilon)x'$ with $y \in Sp_{\mathcal{C}}(G)$, $x' \in Sp(G)$, $\varepsilon \in [0, 1)$. Then x' is a chain-point for the chain $(V_i : x(\delta(V_i)) < 2 - \varepsilon, i \in \{0, 1, \dots, k\})$.*

Proof. Since $y \in Sp_{\mathcal{C}}(G)$, for all $i = 1, \dots, k$ we have $y(\delta(L_i)) = 2$ and $y(E(L_i)) = |L_i| - 1$; hence, $\sum_{v \in L_i} y(\delta(v)) = 2(|L_i| - 1) + 2 = 2|L_i|$. Since x is a chain-point, $\sum_{v \in L_i} x(\delta(v)) = 2|L_i|$ by (iii). Now (i) holds for x' by assumption; to check (ii) note $x'(\delta(V_i)) = \frac{x(\delta(V_i)) - \varepsilon}{1 - \varepsilon}$, whence $x'(\delta(V_0)) = x'(\delta(V_k)) = 1$ and $x'(\delta(V_i)) < 2$ if $x(\delta(V_i)) < 2 - \varepsilon$ ($i = 0, \dots, k$). Finally, $\sum_{v \in L_i} x'(\delta(v)) = \frac{2|L_i| - \varepsilon 2|L_i|}{1 - \varepsilon} = 2|L_i|$ ($i = 1, \dots, k$), so (iii) also holds for x' . \square

Theorem 3. *Let x be a chain-point, and let $\lambda = 2 - \max_{i=0, \dots, k} x(\delta(V_i))$. Then there exist $y \in Sp_{\mathcal{C}}(G)$, $x' \in Sp(G)$ such that*

$$x = \lambda y + (1 - \lambda)x'.$$

By Lemma 2 applied to $\varepsilon := \lambda$, the point x' provided by this theorem is a chain-point for $(V_i : x(\delta(V_i)) < 2 - \lambda, i \in \{0, \dots, k\})$, so by repeatedly applying Theorem 3 we get Theorem 1:

We prove this by induction on ℓ , the number of different narrow cut sizes. Let the different values of the narrow cut sizes be $2 - \sum_{j=1}^h \lambda_j$ for $h = 1, \dots, \ell$. If $\ell = 1$, then $\lambda_1 = 1$ by condition (ii), and thus $x \in Sp_{\mathcal{C}}(G)$, so x is itself a layered convex combination for x . If $\ell > 1$, by Theorem 3 there exist $y \in Sp_{\mathcal{C}}(G)$, $x' \in Sp(G)$ such that $x = \lambda_1 y + (1 - \lambda_1)x'$. Observe that if $x(\delta(V_i)) = 2 - \sum_{j=1}^h \lambda_j$ for some $h \in \{1, \dots, \ell\}$, then $x'(\delta(V_i)) = (2 - \sum_{j=1}^h \lambda_j - \lambda_1) / (1 - \lambda_1) = (2 - \sum_{j=2}^h \lambda_j) / (1 - \lambda_1)$ ($i = 0, \dots, k$). By Lemma 2, x' is a chain-point for the chain $(V_i : x(\delta(V_i)) < 2 - \lambda_1, i \in \{0, 1, \dots, k\})$, which thus has narrow cut sizes $(2 - \sum_{j=2}^h \lambda_j) / (1 - \lambda_1)$ for $h = 2, \dots, \ell$.

Let $\lambda'_j = \lambda_{j+1} / (1 - \lambda_1)$ for $j = 1, \dots, \ell - 1$, then by the inductive hypothesis, there exists a layered convex combination for x' , i.e., $x' = \sum_{j=1}^{\ell-1} \lambda'_j x'_j$, where x'_j is in the convex hull of bases in \mathcal{B}_{j+1} . We thus have $x = \lambda_1 y + (1 - \lambda_1) \sum_{j=1}^{\ell-1} \lambda'_j x'_j = \lambda_1 y + \sum_{j=2}^{\ell} \lambda_j x'_{j-1}$ where y is in the convex hull of bases in \mathcal{B}_1 and $x'_1, \dots, x'_{\ell-1}$ are in the convex hull of bases in $\mathcal{B}_2, \dots, \mathcal{B}_{\ell}$ respectively. Hence, there exists a layered convex combination for x .

We prove [Theorem 3](#) using the following fractional version of Edmonds' matroid partition theorem [[3](#)], which can be easily stated and proved for rational input from the well-known integer version by multiplying with the denominators of the occurring numbers. We include here a reduction to an explicitly stated version in the literature, a theorem on fractional polymatroids, pointed out to us by András Frank.

Lemma 4. *Let M_1 and M_2 be matroids on the same element set E , and denote by r_i the rank function of the matroid M_i ($i = 1, 2$). Let $w \in \mathbb{R}_{\geq 0}^E$, $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$ and let P_i be the convex hull of the independent sets of M_i ($i = 1, 2$). There exist $x_1 \in P_1, x_2 \in P_2$ such that $\lambda_1 x_1 + \lambda_2 x_2 = w$ if and only if $\lambda_1 r_1(X) + \lambda_2 r_2(X) \geq w(X)$ for all $X \subseteq E$.*

Proof. If $x_i \in P_i$ then $x_i(X) \leq r_i(X)$, hence $w = \lambda_1 x_1 + \lambda_2 x_2$ with $x_1 \in P_1, x_2 \in P_2$ implies $w(X) = \lambda_1 x_1(X) + \lambda_2 x_2(X) \leq \lambda_1 r_1(X) + \lambda_2 r_2(X)$.

To prove the reverse direction, apply [[7](#), Theorem 44.6] to the submodular set-functions on E , $f_i := \lambda_i r_i$ for which the conditions are satisfied, and let $w_i \in P_{f_i} = \lambda_i P_i$ ($i = 1, 2$), where P_f is the polymatroid associated with f .

Our conditions $\lambda_1 r_1(X) + \lambda_2 r_2(X) \geq w(X)$ for all $X \subseteq E$ and $w \geq 0$ express exactly that $w \in P_{f_1+f_2}$. So, by [[7](#), Theorem 44.6] $w = w_1 + w_2$, where $w_i \in P_{f_i} = \lambda_i P_i$ ($i = 1, 2$). \square

3. Peeling off a layer

Our only debt now is to show that one layer can be peeled off, that is, to prove [Theorem 3](#).

We use [Lemma 4](#) with $w := x$, M_1 the matroid whose bases are the Gao-trees and M_2 the cycle matroid of G , i.e., the matroid on E whose independent sets are forests in G . We denote by p the rank function of M_1 and by r the rank function of M_2 . Clearly, $p \leq r$. In the remainder of this section, we show that the condition of [Lemma 4](#) is satisfied for $\lambda_1 = \lambda, \lambda_2 = 1 - \lambda$, i.e. that $\lambda p(X) + (1 - \lambda)r(X) \geq x(X)$ for all $X \subseteq E$. [Lemma 4](#) then implies the existence of y and x' in the convex hull of independent sets of M_1 and M_2 respectively, and $x(E) = |V| - 1$ implies that they are in fact in the convex hull of bases of M_1 and M_2 , i.e., in $Sp_C(G)$ and $Sp(G)$ respectively, and [Theorem 3](#) follows.

To prove that the condition of [Lemma 4](#) is satisfied we need the following lemma.

Lemma 5. *Let x be a chain-point and $U \subseteq V$ such that $x(E(U)) = |U| - 1$. Then*

$$p(E(U)) = r(E(U)) = |U| - 1.$$

Proof. We need to show that there exists a spanning tree F in $G(U)$ that is an independent set of M_1 , i.e., there exists a Gao-tree containing F . We begin by proving two claims.

Claim 1. *The set $I(U) := \{i \in [0, k + 1] : L_i \cap U \neq \emptyset\}$ is the set of all integers of an interval.*

Indeed, suppose for a contradiction that there exists $j \in [0, k]$, $L_j \cap U = \emptyset$ such that $L_{j'} \cap U \neq \emptyset$ for some $j' < j$ and also for some $j' > j$; in other words, U is partitioned by $U \cap V_{j-1}$ and $U \setminus V_j$.

By (ii), and applying (SUBMOD) for $A := V_j, C := V \setminus V_{j-1}$, we get that:

$$4 > x(\delta(V_j)) + x(\delta(V \setminus V_{j-1})) = x(\delta(L_j)) + x(\delta(V)) + 2x(\delta(V_{j-1}, V \setminus V_j)).$$

To derive the desired contradiction, we will show that the right hand side is at least 4. First, $x(\delta(V)) = 0$, and (iii) implies that

$x(\delta(L_j)) = 2|L_j| - 2x(E(L_j))$ which is at least 2 since $x(E(L_j)) \leq |L_j| - 1$ for $x \in Sp(G)$. Further, observe that

$$\begin{aligned} x(\delta(V_{j-1}, V \setminus V_j)) &\geq x(E(U)) - x(E(U \cap V_{j-1})) - x(E(U \cap (V \setminus V_j))) \\ &\geq |U| - 1 - (|U \cap V_{j-1}| - 1) - (|U \setminus V_j| - 1) \\ &= 1, \end{aligned}$$

where the second inequality uses the fact that $x(E(U)) = |U| - 1$ and $x(E(A)) \leq |A| - 1$ for any A , and the equality uses the fact that $\{U \cap V_{j-1}, U \setminus V_j\}$ is a partition of U . The claim is proved.

The key claim we need now to prove the lemma is the following.

Claim 2. *Let $a, b \in \mathbb{N}, 0 \leq a \leq b \leq k + 1$, and $S := \bigcup_{i=a}^b L_i$. Then $E(S \cap U)$ is connected.*

Note that this was shown by Gao [[4](#)] for $U := V$ and is the key for the existence of a Gao-tree.

To prove [Claim 2](#), we first show that

$$|S| - 2 < x(E(S)) \leq |S| - 1. \tag{STRICT}$$

If $a = 0, b = k + 1$, then $S = V$ and by (i), $x(E(V)) = |V| - 1 > |V| - 2$; if exactly one of $a = 0$ or $b = k + 1$ holds, then, slightly more generally, subtracting from $x(E(V)) = |V| - 1$ the inequality $x(E(V \setminus S)) \leq |V \setminus S| - 1$ we get $x(E(S)) + x(\delta(S)) \geq |S|$ (again from (i)), and then by (ii) $x(\delta(S)) < 2$, and we are done again. Finally, suppose $1 \leq a \leq b \leq k$, and add up (iii) for $i = a, a + 1, \dots, b$. We get $x(E(S)) + \frac{1}{2}x(\delta(S)) = |S|$, where $\frac{1}{2}x(\delta(S)) \leq \frac{1}{2}x(\delta(V_{a-1})) + \frac{1}{2}x(\delta(V_b)) < 2$ by (ii), finishing the proof of (STRICT).

Now, by (i), x is a convex combination of spanning trees, and by (STRICT), at least one of these, denote it F , contains a spanning tree of S ; F , as all the spanning trees in the convex combination, also contains a spanning tree of U , because of $x(E(U)) = |U| - 1$. So F contains a spanning tree of $S \cap U$, and the claim is proved.

To finish the proof of the lemma, choose a spanning tree in $G(L_i \cap U)$ for each $i \in I(U)$, which is possible by [Claim 2](#) applied to $S := L_i$; then add a Gao-edge between $L_i \cap U$ and $L_{i+1} \cap U$ for each index i such that $i, i + 1 \in I(U)$, which is possible by applying [Claim 2](#) to $S := L_i \cup L_{i+1}$. In this way we get a spanning tree F of $G(U)$, which is an independent set of M_1 , since it can be completed to a Gao-tree of G by applying [Claim 2](#) to $U := V$ and $S := L_i$ for $i = 0, \dots, k + 1$ and then for $S := L_i \cup L_{i+1}$ for $i = 0, \dots, k$, if $\{i, i + 1\} \setminus I(U) \neq \emptyset$. \square

Proof of Theorem 3. First, observe that it suffices to prove the following claim.

Claim 3. *There exist $0 < \varepsilon \leq \lambda$ and $y \in Sp_C(G), x' \in Sp(G)$, such that $x = \varepsilon y + (1 - \varepsilon)x'$.*

Indeed, if this is true, then $x - \varepsilon y \in \text{cone.Sp}(G)$, and let ε_{\max} be the largest $\varepsilon \leq \lambda$ such that there exists $y \in Sp_C(G)$ such that $x - \varepsilon y \in \text{cone.Sp}(G)$. Note that the ε is well-defined, because $\{(\varepsilon, y) : y \in Sp_C(G), x - \varepsilon y \in \text{cone.Sp}(G), 0 \leq \varepsilon \leq \lambda\}$ is a polytope.

Defining $x' := \frac{x - \varepsilon_{\max} y}{1 - \varepsilon_{\max}}$, we see that $x' \in Sp(G)$. So to prove [Theorem 3](#) from [Claim 3](#) we have to prove $\varepsilon_{\max} = \lambda$.

Suppose for a contradiction that $\varepsilon_{\max} < \lambda$, then by [Lemma 2](#) applied to $\varepsilon := \varepsilon_{\max}$, x' is also a chain-point (for the same chain of sets). So then applying [Claim 3](#) to x' , there exist $\varepsilon' > 0$ and $y' \in Sp_C(G)$ such that $x' - \varepsilon' y' \in \text{cone.Sp}(G)$. However, then with $\varepsilon'' := (1 - \varepsilon_{\max})\varepsilon'$ we have

$$\begin{aligned} \text{cone.Sp}(G) \ni (1 - \varepsilon_{\max})(x' - \varepsilon' y') &= x - \varepsilon_{\max} y - \varepsilon'' y' \\ &= x - (\varepsilon_{\max} + \varepsilon'')z, \end{aligned}$$

where $z := \frac{\varepsilon_{\max} y + \varepsilon'' y'}{\varepsilon_{\max} + \varepsilon''} \in Sp_C(G)$. This contradicts that ε_{\max} is the largest $\varepsilon \leq \lambda$ such that there exists $y \in Sp_C(G)$ such that

$x' = x - \varepsilon y \in \text{cone.Sp}(G)$, finishing the proof of the theorem, provided that Claim 3 is true.

We prove now Claim 3 by checking the condition of the matroid partition theorem in the form of Lemma 4. Let us say that $\varepsilon \geq 0$ is suitable for $X \subseteq E$, if $\varepsilon > 0$ and

$$\varepsilon p(X) + (1 - \varepsilon)r(X) \geq x(X).$$

By Lemma 4, Claim 3 is equivalent to proving that there exists $\varepsilon > 0$ suitable for all $X \subseteq E$.

If $p(X) = r(X)$ then by (i) any $\varepsilon \in [0, 1]$ is suitable for X ; so assume $p(X) < r(X)$. If in addition $x(X) < r(X)$, then ε is suitable if and only if $0 < \varepsilon \leq \varepsilon_X := \frac{r(X) - x(X)}{r(X) - p(X)} (> 0)$. Since the number of such sets is finite, $\varepsilon_{\max} := \min\{\varepsilon_X : X \subseteq E, x(X) < r(X), p(X) < r(X)\} > 0$ is suitable for all of them.

It remains to show that there is no other case, that is, $x(X) = r(X)$ implies $p(X) = r(X)$. Denote V_X the set of vertices of (“covered by”) edges in X . We consider the components of (V_X, X) . If V_X is empty, $X = \emptyset$ and hence $r(X) = 0 = p(X)$. If (V_X, X) has only one component, then $r(X) = |V_X| - 1$. Apply Lemma 5 to the graph whose edge-set is the support of x , and $U := V_X$; then $X = E(U)$ and by the lemma, $p(X) = r(X)$ follows.

If (V_X, X) has multiple components, then $r(X)$ sums up over the components. The same holds for $p(X)$ as long as there are no two components that both contain edges in the same cut in \mathcal{C} . (This can be easily seen from the definition in Section 2 of the matroid in terms of a direct sum of cycle matroids on the level sets and uniform matroids on the Gao-edges of each cut.) Hence, the proof of the claim is completed by showing that it is not possible for two components A and B of (V_X, X) that both $E(A) \cap \delta(V_i) \neq \emptyset$ and $E(B) \cap \delta(V_i) \neq \emptyset$.

Indeed, if $E(A) \cap \delta(V_i) \neq \emptyset$, then $x(E(A) \cap \delta(V_i)) = x(E(A)) - x(E(A \cap V_i)) - x(E(A \setminus V_i)) \geq 1$, since $x(E(A)) = |A| - 1$ because of $x(X) = r(X)$, and $x(E(A')) \leq |A'| - 1$ for any $\emptyset \neq A' \subseteq A$, because $x \in \text{Sp}(G)$. By applying the same argument to B , if both $E(A) \cap \delta(V_i) \neq \emptyset$ and $E(B) \cap \delta(V_i) \neq \emptyset$, then $x(\delta(V_i)) \geq 2$, contradicting (ii). \square

Our proof implies that a layered convex combination can be found in strongly polynomial time with a combinatorial algorithm. One way of achieving this is through Edmonds’ matroid partition algorithm. The best-known fractionally weighted implementation of this is by Cunningham [2], which can be modified for different matroids. Note that for rational weights the multiplication with a common denominator allows a direct application of matroid union, that can be implemented in strongly polynomial time. The rank oracle of the occurring matroids is at hand from the above proofs.

Another way of achieving this is by using the algorithm for writing a point x in $\text{Sp}(G)$ as a convex combination of spanning trees in the proof of Theorem 51.5 of [7]. At every iteration of this

algorithm, a spanning tree T is chosen that has maximum weight for a certain weight function. The weight function is determined by the algorithm; it satisfies the property that a tree T is a maximum weight tree if and only if it satisfies $T(U) = |U| - 1$ for certain sets U for which $x(E(U)) = |U| - 1$. Given T , a value $\lambda \geq 0$ is computed so that $x' = (x - \lambda T)/(1 - \lambda) \in \text{Sp}(G)$. The algorithm then recurses on x' . Now, instead of choosing T to be an arbitrary maximum weight spanning tree, we choose a maximum weight spanning tree that is a Gao-tree for the chain $(V_i : x(\delta(V_i)) < 2, i \in \{1, \dots, k\})$. It follows from the results above that a Gao-tree with this property indeed exists.

It then follows from [7] that there exists a layered convex combination for x for which the number of spanning trees is at most linear in the size of the support of x , which is the same as for an arbitrary decomposition into spanning trees. If x is an extreme point solution of the subtour elimination linear program the support of x has size $O(|V|)$ (see [5]); so only $O(|V|)$ different trees are needed for a layered convex combination.

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