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THE SCHRIJVER SYSTEM OF ODD JOIN POLYHEDRA

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Graphs for which the set of t-joins and t-cuts has "the max-flow-min-cut property", i.e. for which the minimal defining system of the t-join polyhedron is totally dual integral, have been characterized by Seymour. An extension of this problem is *to characterize the (uniquely existing) minimal totally dual integral defining system (Schrijver-system) of an arbitrary t-join polyhedron.* This problem is solved in the present paper. The main idea is to use t-borders, a generalization of t-cuts, to obtain an integer minimax formula for the cardinality of a minimum t-join. (A t-border is the set of edges joining different classes of a partition of the vertex set into t-odd sets.) It turns out that the (uniquely existing) "strongest minimax theorem" involves just this notion.

Introduction

A system of inequalities $Ax \leq b$ is called *totally dual integral* (TDI) if for any $w \in \mathbf{Z}^n$ the linear programming problem max $\{wx\colon Ax \leq b\}$ has an all integer dual optimum solution: provided the optimum exists ($\mathbf{Z}$ is the set of integers, A is an $m \times n$ rational matrix and b is an m dimensional rational vector).

A. Schrijver has proved in [23] that every full dimensional rational polyhedron has a unique minimal TDI defining system with integer left hand sides, which was called in [4] the *Schrijver-system* of the polyhedron. The Schrijver-system can be interpreted to yield the "strongest minimax theorem" through the duality theorem of linear programming. In [22] Schrijver proved that the "first Chvátal closure" of polyhedra arises by rounding the right hand sides of its Schrijver-system. The Schrijver-system also shows whether a polyhedron is "integral" or not: it is integral if and only if its Schrijver-system has integer right hand sides. These results (cf. [25]) show that TDI-ness is not just one of the tools for proving integrality of polyhedra, but a notion that lies in the heart of integer programming. The intriguing problem of determining the Schrijver-systems of some known polyhedra arises.

For matching polyhedra this problem is settled by the result of Cunningham and Marsh ([5], see also [21], [3]) which says that the minimal defining system of matching polyhedra is TDI, i.e. coincides with the Schrijver-system. The most well-known non-TDI classes of polyhedra are perhaps the generalizations of matching

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polyhedra. Cook and Pulleyblank ([20], [21], [1], [2], [3], [4]) determine the Schrijver-system of b-matching polyhedra. In the present paper we determine the Schrijver-system of another generalization of matching polyhedra: that of t-join polyhedra.

Let G be an undirected graph and $V(G)$, $E(G)$ its vertex-set and edge-set respectively. Let t: $V(G) \to \mathbf{Z}$. A t-join is a set $F \subset E(G)$ with $d_F(x) \equiv t(x) \bmod 2 \, \forall x \in V(G)$, where $d_F(x)$ is the number of edges in F adjacent to x. If $t = d_E$ there is a one to one correspondance between t-joins and "postman tours".

t-joins were first studied in [19] and [7]. Edmonds and Johnson's results imply the defining system of the dominant (convex hull plus the nonnegative orthant) of the characteristic vectors of t-joins (cf. [7], [24], [18]). Such a polyhedron is called *t-join polyhedron.* Total dual half integrality of t-join polyhedra follows from Edmonds and Johnson's result on generalized matching polyhedra [6]. A first proof appeared in [16], and in [32] it was proved that an integral dual solution exists provided the total weight of any circuit is even. Seymour [31] characterized those graphs for which the minimal polyhedral defining system is TDI. For a summary of results see e.g. Lovász and Plummer [18], and for improved algorithmic proofs cf. Korach [14]. A structure theorem sharpening these results with new proof and algorithm can be found in [9] and [26], [27], [28].

Since the t-join polyhedron is full dimensional it has a Schrijver-system.

The first question that arises is to characterize those (G, t) pairs for which the minimal defining system of the t-join polyhedron is TDI. Since the TDI-ness of the minimal defining system of a t-join polyhedron (cf. Section 2) is equivalent to the max-flow-min-cut property of the clutters of t-joins and t-cuts (i.e. to the existence of an integer maximum packing of t-cuts for an arbitrary weight function), this is the well-known graphic case of Seymour's celebrated characterization [31].

In this paper we are solving the more general question of characterizing the inequalities in the Schrijver-system of the t-join polyhedron of G, for arbitrary (G, t)-pairs. We do not use Seymour's theorem in our proof, and do not even know any simpler way of proving the main result using Seymour's theorem. On the other hand, in Section 3, we deduce a characterization of the max-flow-min-cut property, and point out its connection to Seymour's theorem.

Since the t-join polyhedron is full dimensional it has a Schrijver-system.

In order to obtain the Schrijver-system of a polyhedron first a strong "integer minimax theorem" is needed which is conjectured to yield a Schrijver-system. Then the minimality of the system has to be proved.

Since for t-joins the known minimax theorems may have half-integer values on the right hand side, we will first have to find an appropriate *integer minimax theorem* to start with. Section 1 is devoted to this task. Then, in Section 2 we sharpen this minimax theorem in order to get the *strongest possible* TDI *defining system.* Finally, in Section 3 we prove that the TDI *description found in Section* 2 *is minimal,* i.e. it is the Schrijver-system, and point out the connection of the results to Seymour's characterization of max-fiow-min-cut binary clutters in the graphic case.

1. An integer minimax theorem

The key-notion of this paper is the notion of t-borders defined below.

Let t: $V(G) \to \mathbf{Z}$. We shall always suppose $t(V(G)) \equiv 0 \bmod 2$, since it is necessary (and sufficient) for the existence of t-joins. (If $X \subset V(G)$ then $t(X) :=$

$:=\sum\{t(x): x\in X\}$). If $X\subset V(G)$, $t(X)\equiv 1 \bmod 2$ then X is called a *t*-odd set and the *coboundary* $\delta(X):=\{xy\in E(G): y\notin X, x\in X\}$ of X is called a *t-cut*. Clearly, if $\delta(X)$ is a t-cut and F is a t-join then $|F\cap\delta(X)|\geqq 1$.

If $\mathcal{P}$ is a partition of $V(G)$ and $t(P)\equiv 1 \bmod 2$ for every $P\in\mathcal{P}$ then it will be called a *t-partition* and $\delta(\mathcal{P}):=\cup\{\delta(P): P\in\mathcal{P}\}$ will be called a *t-border* (i.e. $\delta(\mathcal{P})$ is the set of edges that go between different classes of $\mathcal{P}$; t cuts are those t-borders $\delta(\mathcal{P})$ for which $|\mathcal{P}|=2$). Clearly, if $\mathcal{P}$ is a t-partition then $|\mathcal{P}|$ is even, and for every t-join F:

$$|F\cap\delta(\mathcal{P})|\geqq\frac{|\mathcal{P}|}{2}. \tag{1.1}$$

Let $\tau(G,t)=\min\{|F|: F \text{ is a } t\text{-join}\}$ and

$$\beta(G,t)=\max\left\{\sum_{i=1}^{k}\frac{|\mathcal{P}_i|}{2}: \mathcal{P}_i \text{ is a } t\text{-partition } (i=1,\dots,k), \text{ and } \delta(\mathcal{P}_i)\cap\delta(\mathcal{P}_j)=\emptyset\ (i\neq j)\right\}$$

$\sum_{i=1}^{k}\frac{|\mathcal{P}_i|}{2}$ will be called the *value* of $\{\mathcal{P}_1,\dots,\mathcal{P}_k\}$. (1.1) immediately implies $\tau(G,t)\geqq\beta(G,t)$. The main result of this section is:

Theorem 1.1. *For every* (G,t) *pair* $(t(V(G))\equiv 0 \bmod 2)$

$$\tau(G,t)=\beta(G,t).$$

In the proof we shall need the following notation. For $x,y\in V(G)$, $t^{x,y}$ denotes the functions defined by

$$t^{x,y}(v):=\begin{cases} t(v) & \text{if } v\notin\{x,y\},\\ t(v)+1 & \text{if } v\in\{x,y\}.\end{cases}$$

The following statements are easy to see: If F_1 is a t_1-join and F_2 is a t_2-join then $F_1\triangle F_2$ is a t_1+t_2-join. Specially, the symmetric difference of a minimum t-join and a minimum $t^{x,y}$-join or of a minimum $t^{a,x}$-join and a minimum $t^{a,y}$-join is the union of an (x,y) path and disjoint circuits, where the circuits have the same number of edges in the two joins. It follows, that for an arbitrary minimum $t^{x,y}$-join $F^{x,y}$ there exists an (x,y) path P (i.e. a simple path between x and y) such that $F^{x,y}\triangle P$ is a minimum t-join. These facts will be often used in the proofs.

Proof. Let $a,b\in V(G)$ satisfy $\tau(G,t^{a,b})=\min\{\tau(G,t^{x,y}): x,y\in V(G)\}$. Let B be the vertex set of a component of the graph induced by the set

$$\{x\in V(G): \tau(G,t^{a,x})=\tau(G,t^{a,b})\}.$$

Let $\mathcal{P}=\{\{V(G)\setminus B\}\}\cup\{\{x\}: x\in B\}$. We shall prove (I) and (II) below:

(I) *For every* $P\in\mathcal{P}$ *and every minimum t-join* F: $|F\cap\delta(P)|=1$.

Clearly, (I) implies

$$t(P)\equiv 1 \bmod 2\ \forall P\in\mathcal{P} \text{ i.e. } \mathcal{P} \text{ is a } t\text{-border,}$$
$$\text{and } |\delta(\mathcal{P})\cap F|=\frac{|\mathcal{P}|}{2} \text{ for every minimum } t\text{-join } F. \tag{1.2}$$

Contracting an edge $xy\in E(G)$ in (G, t) means identifying them and defining $t(v_{xy}):=t(x)+t(y)$ where v_{xy} is the new vertex arising with the identification. Contracting a set of edges means contracting all edges of the set.

(II) *For the graph* (G^*, t^*) *arising after the contraction of* $\delta(\mathscr{P})$:

$$\tau(G^*, t^*) \geqq \tau(G, t)-\frac{|\mathscr{P}|}{2}.$$

Using (I) and (II) our theorem is implied as follows: We have already seen $\tau(G, t)\geqq\beta(G, t)$. We prove now by induction on $|V(G)|$ that there exists pairwise disjoint t-borders $\delta(\mathscr{P}_i)$ $(i=1, ..., k)$ such that

$$\tau(G, t) \leqq \sum_{i=1}^{k}\frac{|\mathscr{P}_i|}{2}.$$

Applying the induction hypothesis for (G^*, t^*) we get pairwise disjoint t^*-borders $\delta(\mathscr{P}_1), ..., \delta(\mathscr{P}_{k^*})$ in G^* with

$$\tau(G^*, t^*) \leqq \sum_{i=1}^{k^*}\frac{|\mathscr{P}_i|}{2}. \tag{1.3}$$

Let $k:=k^*+1$ and $\mathscr{P}_k:=\mathscr{P}$. (I) implies that $\delta(\mathscr{P}_k)$ is a t-border in G (cf. (1.2)) and $\delta(\mathscr{P}_i)$ $(i=1, ..., k-1)$ are also t-borders in G. Comparing (II) and (1.3) we get that

$$\tau(G, t) \leqq \sum_{i=1}^{k}\frac{|\mathscr{P}_i|}{2} \leqq \beta(G, t),$$

and $\mathscr{P}_k$ is obviously disjoint from $\mathscr{P}_i$ $(i=1, ..., k-1)$ as it was to be proved.

In order to prove (I) we need Claim 1 and Claim 2:

Claim 1. *Suppose* $b_1b_2\in E(G)$ $b_1, b_2\in B$, F_i *is a minimum* t^{a,b_i}*-join* $(i=1, 2,)$ *and* $P\subset F_1\triangle F_2$ *is a* (b_1, b_2) *path. Then* $V(P)\subset B$, *and* $F_1\triangle P$ *is a minimum* t^{a,b_2}*-join.*

Proof. Since $|F_1|=|F_2|$, the circuits of $F_1\triangle F_2$ and P have the same number of edges in F_1 and F_2, and this implies already that $F_1\triangle P$ is a minimum t^{a,b_2}-join. Let $C=P\cup b_1b_2$. C is a circuit and $|C\setminus F_1|=|C\cap F_1|+1$.

Let now $x\in V(P)$ be arbitrary. The above equality implies that for one of the (b_1, x) paths on C, denote it by $C(b_1, x)$, $|C(b_1, x)\setminus F_1|\leqq|C(b_1, x)\cap F_1|$ holds. Thus $|F_1\triangle C(b_1, x)|\leqq|F_1|=\tau(G, t^{a,b})$. $F_1\triangle C(b, x)$ is a $t^{a,x}$-join. By the minimality of $\tau(G, t^{a,b})$ equality holds here, i.e. $\tau(G, t^{a,x})=\tau(G, t^{a,b})$ for every $x\in V(P)$ that is $V(P)\subset B$. ▌

Claim 2. *If* $x\in B$ *is arbitrary, and* F^x *is a minimum* $t^{a,x}$*-join then* $F^x\cap\delta(x)=\emptyset$ *and* $F^x\cap\delta(B)=\emptyset$.

Proof. Suppose indirectly that $xy\in F^x\cap\delta(x)$. Then $F^x\setminus xy$ is a $t^{a,y}$-join in contradiction with the minimality of $|F^x|=\tau(G, t^{a,b})$. If, on the other hand $\alpha\beta\in F^x\cap\delta(B)$ $(\alpha\notin B, \beta\in B)$ then let Q be an (x, β) path $V(Q)\subset B$. We prove that for *every vertex* $q\in V(Q)$, *there exists a minimum* $t^{a,q}$*-join* F^q *with* $\alpha\beta\in F^q$. Applying this for $q=\beta$ we get a minimum $t^{a,\beta}$-join F^β with $\alpha\beta\in F^\beta$, which is a contradiction, since $F^\beta\setminus\alpha\beta$ is a $t^{a,\alpha}$-join, $|F^\alpha|=|F^\beta|-1=\tau(G, t^{a,b})-1$ contradicting the minimality of $\tau(G, t^{a,b})$.

To prove the above statement we use induction on $|Q(x,q)|$, $(Q(x,q)$ is the sequence of Q between x and $q)$. If $q=x$ we have it by assumption. Suppose we know the statement for $q_1 \in V(Q)$ and let us prove if for its neighbor $q_2 \in V(Q)$. Let F_i $(i=1,2)$ be minimum t^{a,q_i} joins. Applying Claim 1 we get a (q_1, q_2) path $P \subset F_1 \triangle F_2$, $V(P) \subset B$ and $F_1 \triangle P$ is a minimum t^{a,q_2}-join. $V(P) \subset B$ implies $\alpha\beta \in F_1 \triangle P$. ∎

We now prove (I). Let F be a minimum t-join, $x \in B$ be arbitrary and let $F = F^x \triangle P$ where F^x is a minimum $t^{a,x}$-join and P is an (a,x) path. Since $d_{F^x}(x)=0$ by Claim 2 and $d_p(x)=1$, we have $d_F(x)=1$. This proves (I) for $\{x\} \in \mathscr{P}$ $(x \in B)$. In order to prove (I) we still have to prove $|F \cap \delta(B)|=1$. Let p be the first vertex of P in B, starting from a. We show now, that $F \triangle P(a,p)$ is a minimum $t^{a,p}$-join. This will finish the proof of (I), since then by Claim 2 we have $[F \triangle P(a,p)] \cap \delta(B) = \emptyset$. Suppose indirectly, that $|F \triangle P(a,p)| > |F^p|$ where F^p is a minimum $t^{a,p}$-join. By Claim 2 neither F^p nor F^x intersect $\delta(B)$, and consequently any (p,x) path $Q \subset F^p \triangle F^x$ is entirely in B, whence it is disjoint from $P(a,p)$. Now it is straightforward to check that $F^x \triangle (Q \cup P(a,p))$ is a t-join and by the indirect assumption its cardinality is strictly less than $|F|$. This contradiction shows that (I) is true.

In order to prove (II) we need the following fact:

Claim 3. *The graph $G(B)$ induced by B is factorcritical (i.e. $G(B)-x$ has a perfect matching for every $x \in B$).*

Proof. Let $x \in B$ be arbitrary, and $F^x = F \triangle P$ where F is a minimum t-join F^x is a minimum t^x-join and P is an (a,x) path. By (I) we have $d_F(b)=1$ for all $b \in B$, and it follows that $d_{F^x}(b)=1$ for $b \in B$, $b \neq x$. By Claim 2 $d_{F^x}(x)=0$, and $F^x \cap \delta(B)=\emptyset$. Thus F^x matches the vertices of $B-x$. ∎

Let now $b \in B$ and let F^b be a minimum $t^{(a,b)}$-join. Denote by $(F^b)^*$ the edge-set that arises from F^b after the contraction of $\delta(\mathscr{P})$. We first show that $(F^b)^*$ is a *minimum* $(t^{a,b})^*$*-join of* G^*, where $(G^*, (t^{a,b})^*)$ arises by contracting $\delta(\mathscr{P})$ in $(G, t^{a,b})$. Suppose indirectly that there is a circuit $C^* \subset G^*$ such that $|(F^b)^* \triangle C^*| < (F^b)^*$. C^* becomes in G an (x,y)-path $x, y \in \Gamma(B)$, $(\Gamma(B)$ denotes the set of neighbors of $B.)$ Let $yb' \in E(G)$, $b' \in B$, and let $F^{b'}$ be a perfect matching of $G(B)-b'$, which exists by Claim 3.

Clearly, $[(F^b)^* \triangle C^*] \cup \{yb'\} \cup F^{b'} \subset E(G)$ is a $t^{a,x}$-join of G and has cardinality strictly smaller than $(F^b)^*+1+(|B|-1)/2$, i.e. at most $(F^b)^*+(|B|-1)/2 = \tau(G, t^{a,b})$, contradicting $x \notin B$. Thus $(F^b)^*$ is a minimum $(t^{a,b})^*$-join.

Now let $P^* \subset E(G^*)$ be an (a, b^*) path for which $(F^b)^* \triangle P^*$ is a minimum t^*-join, where b^* is the new vertex that arises after contracting $\delta(\mathscr{P}) = E(G(B)) \cup \delta(B)$. P^* becomes in G an (a,z) path $z \in \Gamma(B)$. Let $zb'' \in E(G)$, $b'' \in B$, and let $F^{b''}$ be a maximum matching of $G(B)-b''$. $[(F^b)^* \triangle P^*] \cup \{zb''\} \cup F^{b''} \subset E(G)$ is a t-join, and has cardinality $\tau(G^*, t^*)+1+(|B|-1)/2 = \tau(G^*, t^*)+|\mathscr{P}|/2$. Thus $\tau(G,t) \leqq \tau(G^*, t^*)+|\mathscr{P}|/2$ whence (II) and Theorem 1.1 are proved. ∎

Remark. Theorem 1.1 is a straightforward consequence of [26, Theorem 3.1, 4.4 or 5.8]. An algorithmic proof of these theorems is given in [27]. Since the presentation of this theorem and the reduction to it are quite lengthy to describe precisely we have chosen to use the *method* of [27] instead. The above proof too can straightforwardly be turned to an algorithm that determines a minimum t-join and a maximum packing of t-borders. The same method has been used in [29] to prove Seymour's theorem shortly.

2. Sharpening the minimax theorem

In this section we show that there exists a packing of t-borders with value $\beta(G, t)$ with t-borders having special properties.

Let $\mathcal{P}$ be a t-partition. We shall say that $\mathcal{P}$ is a *reduced t-partition* and $\delta(\mathcal{P})$ is a *reduced* t-border if (2.1) and (2.2) are satisfied:

(2.1) For every $P \in \mathcal{P}$ the graph induced by P is *connected.*

(2.2) Contracting every edge that has both endpoints in the same $P \in \mathcal{P}$ the resulting graph G^* is *bicritical.*

A graph is called *bicritical* if deleting any set of two different vertices the resulting graph has a perfect matching. Note that in light of (2.1) the contraction in (2.2) just means "shrinking" every $P \in \mathcal{P}$. The result of the present section is that there exists always a $\beta(G, t)$ element packing of *reduced* t-borders. In order to obtain this result, our main tool will be a powerful decomposition method of Lovász and Plummer (cf. [15], [17], [18]).

Let

$$\bar{\beta}(G,t) = \max\Big\{\sum_{i=1}^{k} \frac{|\mathcal{P}_i|}{2}: \mathcal{P}_i \text{ is a \textit{reduced} } t\text{-partition } (i = 1, \dots, k) \text{ and}$$
$$\delta(\mathcal{P}_i) \cap \delta(\mathcal{P}_j) = \emptyset \quad (i \neq j)\Big\}.$$

The result we shall prove can be written in the form $\beta(G, t) = \bar{\beta}(G, t)$, i.e. combining it with Theorem 1.1:

Theorem 2.1. $\tau(G, t) = \bar{\beta}(G, t)$.

Proof. We shall suppose that G is connected (otherwise we apply the theorem for every component). We shall also suppose that (2.3) holds:

(2.3) Every edge of G is contained in a minimum t-join.

(2.3) can be supposed without loss of generality, for if it does not hold we contract edges not contained in any minimum t-join (one by one), until (2.3) holds: τ does not change, and obviously disjoint reduced t-borders of the resulting graph correspond to disjoint reduced t-borders of G.

Let now $\mathcal{P}_1, \dots, \mathcal{P}_k$ be t-patitions that satisfy

(2.4) $$\delta(\mathcal{P}_i) \cap \delta(\mathcal{P}_j) \neq \emptyset, \quad \sum_{i=1}^{k} \frac{|\mathcal{P}_i|}{2} = \tau(G, t).$$

(By Theorem 2.1 this choice can be made.) Note that (2.4) implies

(2.5) If $P \in \mathcal{P}_i$ $(i \in \{1, \dots, k\})$ and F is a minimum t-join: $|F \cap \delta(P)| = 1$.

First we prove that $\mathscr{P}_i$ $(i=1, \ldots, k)$ satisfies (2.1) (provided (2.3) holds). Suppose indirectly that say the graph induced by some $P \in \mathscr{P}_1$ has at least two components. Let V_1 and V_2 be the vertex set of two of them. Clearly, $\delta(V_1) \subset \delta(P)$, $\delta(V_2) \subset \delta(P)$ and $\delta(V_1) \cap \delta(V_2) = \emptyset$. Both $t(V_1)$ and $t(V_2)$ cannot be odd, because then for an arbitrary t-join: $|F \cap \delta(V_1)| \geqq 1$, $|F \cap \delta(V_2)| \geqq 1$, whence $|F \cap \delta(P)| \geqq 2$ contradicting (2.5). So, say $t(V_1)$ is even and therefore for any t-join F, $|F \cap \delta(V_1)|$ is even. Let $e \in \delta(V_1)$ be arbitrary and let F be a minimum t-join, $e \in F$. (By (2.3) such a t-join exists.) We have now that $F \cap \delta(V_1)$ is nonempty and has even cardinality, whence $|F \cap \delta(P)| \geqq |F \cap \delta(V_1)| \geqq 2$, again contradicting (2.5). Thus (2.1) holds for $\mathscr{P}_i$ $(i=1, \ldots, k)$.

Consider now the t-borders $\mathscr{P}_1, \ldots, \mathscr{P}_k$ which satisfy (2.4) and k is *maximum* under this condition. We prove that for these t-borders (2.2) holds as well. Consider e.g. $\mathscr{P}_1$ and let H be the graph that arises after shrinking its classes. By (2.5), for every minimum t-join F, $F \cap \delta(\mathscr{P}_1)$ becomes a perfect matching of H after the shrinking. Suppose indirectly, that H is not bicritical, i.e. there exists $x, y \in V(H)$: $H - \{x, y\}$ has no perfect matching. By Tutte's theorem this means that there exists an $X \subset V(H) \setminus \{x, y\}$ such that $H - (X \cup \{x, y\})$ has more than $|X|$ odd components. Let $R = X \cup \{x, y\}$. Since $|V(H)|$ is even, the number r of odd components of $H-R$ has the same parity as $|R|$ or $|X|$. Thus $r \geqq |X| + 2 = R$. Since H has a perfect matching, $r = |R|$. $H-R$ has no even component, since by (2.3) an edge in the coboundary of such a component would be contained in some perfect matching (by (2.3)), although every perfect matching must match R to the odd components of $H-R$. For the same reason, R must be independent. Thus we have:

(2.6) $|R| \geqq 2$, R is independent and $H-R$ has $|R|$ components, all fo them odd.

Let $C_1, \ldots, C_r$ be the components of $G-R$, and $\mathscr{P}_j^* := \{\{x\}: x \in C_j\} \cup \cup \{\{V(H) \setminus V(C_j)\}\}$ $(j=1, \ldots, r := |R|)$. Recall that the vertices of H correspond to t-odd sets of G. Since $|V(H) \setminus V(C_j)|$ is odd $(j=1, \ldots, r)$ the partitions $\mathscr{P}_j'$ of $V(G)$ defined in the natural way from the partitions $\mathscr{P}_j^*$ $(j=1, \ldots, r)$ have t-odd classes. Thus $\delta(\mathscr{P}_j')$ $(j=1, \ldots, r)$ are t-borders. Clearly, $\delta(\mathscr{P}_i^*) \cap \delta(\mathscr{P}_j^*) = \emptyset$ $(i \neq j)$,

$$E(H) =$$

$$\bigcup_{j=1}^{r} \delta(\mathscr{P}_j^*) \text{ (since } R \text{ is independent) and } \sum_{j=1}^{r} \frac{|\mathscr{P}_j'|}{2} = \frac{1}{2} \sum_{j=1}^{r} (|C_j|+1) = \frac{|V(H)|}{2}.$$

Hence $\delta(\mathscr{P}_i') \cap \delta(\mathscr{P}_j') = \emptyset$ $(i \neq j)$,

$$\delta(\mathscr{P}_1) = \bigcup_{j=1}^{r} \delta(P_j') \quad \text{and} \quad \sum_{j=1}^{r} \frac{|\mathscr{P}_i'|}{2} = \frac{|\mathscr{P}_1|}{2}.$$

Thus, the set of t-partitions $\{\mathscr{P}_1', \ldots, \mathscr{P}_r'\} \cup \{\mathscr{P}_2, \ldots, \mathscr{P}_k\}$ satisfies (2.4) and has $k+r-1 > k$ classes. This contradicts the choice of k whence shows that H is bicritical. ∎

Now we extend Theorem 2.1 to the weighted case. Let w: $E(G) \to \mathbf{Z}_+$ ($\mathbf{Z}_+$ is the set of nonnegative integers), and $\tau(G, t, w) := \min \{w(F): F \text{ is a } t\text{-join}\}$. We shall say that $\{\delta(\mathscr{P}_1), \ldots, \delta(\mathscr{P}_k)\}$ is a *w-packing* of t-borders if every $e \in E(G)$ is contained in at most $w(e)$ of its elements. Eg. pairwise disjoint t-borders make up a 1-packing where 1 is the all 1 function on $E(G)$. The value of this w-packing is $\sum_{i=1}^{k} \frac{|\mathscr{P}_i|}{2}$. Let $\bar{\beta}(G, t, w)$ denote the maximum value of a w-packing of reduced t-borders.

Theorem 2.2. *For an arbitrary graph G, and functions t: $V(G)\to \mathbf{Z}$, $\left(t(V(G))\equiv 0 \bmod 2\right)$ and w: $E(G)\to \mathbf{Z}_+$ we have:*

$$\tau(G, t, w) = \bar{\beta}(G, t, w).$$

Proof. Replace each edge $e\in E(G)$ by a path of (edge-)cardinality $w(e)$ and define t on the new vertices to be 0. Denote the result by (G', t'). Clearly, $\tau(G, t, w)= =\tau(G', t')$ and $\bar{\beta}(G, t, w)=\bar{\beta}(G', t')$. Applying Theorem 2.1 to (G', t') we get our theorem. ∎

Theorem 2.2 immediately implies:

Theorem 2.3. *The system of inequalities*

(A) $x(\delta(\mathscr{P}))\geqq \frac{|\mathscr{P}|}{2}$ *for every reduced t-partition $\mathscr{P}$ of $V(G)$.*

(B) $x(e)\geqq 0$ *for every $e\in E(G)$ except if $G-e$ has (two) t-odd components.*
is a TDI *defining system of the t-join polyhedron of the graph G.*

Note that every inequality in (A) is a half-integer sum of inequalities of the form

(C) $x(\delta(A))\geqq 1$, *A for every $A\subset V(G)$ for which A is t-odd and A induces a connected graph.*

Thus the minimal defining system (B), (C) of t-join polyhedra and its half total dual integrality follows (cf. [6], [7], [16]). If (B) and (C) constitute a TDI system of inequalities, then (G, t) is said to have the *max-flow-min-cut* property (cf. Seymour [32]). This is equivalent to saying that for an arbitrary weight function there exists a "maximum w-packing" of t-cuts which has the same cardinality as the minimum t-joins. From Theorem 2.3 we get immediately that *for (G, t) to have the max-flow-min-cut property, it is sufficient not to contain any reduced t-border $\mathscr{P}$ with $|\mathscr{P}|\geqq 4$.* Theorem 3.1 below trivially implies that this condition is necessary as well (cf. Theorem 3.3 later).

3. The Schrijver-system

In this Section we prove the final result of the paper:

Theorem 3.1. *The inequalities* (A) *and* (B) *constitute the Schrijver-system of the t-join polyhedron of the graph G.*

Remark. In light of Theorem 2.3 we only have to prove:

(3.1) Every inequality in (A) and (B) must be present in any TDI defining system of the t-join polyhedron of G.

W. Cook [3] pointed out what are the statements to be proved in order to check a claim of the type of (3.1) for polyhedra which are the convex hulls of characteristic vectors of independence systems. ($\mathscr{F}$ is an independence system if $F_1\in\mathscr{F}$

$F_2 \subset F_1$ implies $F_2 \in \mathscr{F}$.) His framework has been used for matching and b-matching polyhedra ([3], [4]). It can also be used for t-join polyhedra, for a t-join polyhedron is the convex hull of the characteristic vectors of edge-sets containing t-joins, which edge-sets make up a "coindependence system". Below (in "Proof of Theorem 3.1"), we are making use of Cook's *method* even without explicitly using his notions and Lemma.

This method reduces the polyhedral question to questions about bicritical graphs. In the proof of some properties of bicritical graphs we shall repeatedly use the "brick decomposition" method of Lovász and Plummer (cf. [15], [17], [18], [8]), in order to reduce the statements to 3-connected bicritical graphs. Hence the following deep result of [8] on 3-connected bicritical graphs will play a crucial role:

Lemma 1. *If G is a 3-connected bicritical graph then it does not contain a nontrivial tight cut.*

(G is 3-connected if for every $x, y \in V(G)$, $G - \{x, y\}$ is connected. $\delta(X)$ $(X \subset V(G))$ is called a *tight cut* if every matching of G intersects $\delta(X)$ in exactly 1 edge. If $|X| = 1$ then the tight cut is called *trivial*, otherwise *nontrivial*.) For a proof of Lemma 1 we refer to [8]. For the generalization of this statement to the weighted case with a new proof cf. [30].

If G is a connected graph and $\{x, y\} \subset V(G)$ is a cutset, then there exist connected graphs G_1, G_2 with

(3.2) $$V(G_1) \cap V(G_2) = \{x, y\}, E(G_1) \cup E(G_2) = E(G).$$

If moreover G is bicritical, then it is straightforward to see (cf. [18]) that

(3.3) $\bar{G}_i := G_i \cup \{xy\}$ are bicritical and $|V(\bar{G}_i)| = |V(G_i)| \geqq 4$ $(i = 1, 2)$. Moreover, if M_1 is an arbitrary perfect matching of $\bar{G}_1$ and M_2 is an arbitrary perfect matching of $\bar{G}_2$ and xy is contained is exactly 1 of M_1 and M_2 then $(M_1 \cup M_2) \setminus \{xy\}$ is a perfect matching of G, and every perfect matching of G arises in this way.

(3.3) and the following lemma play a technical role in reducing Theorem 3.1 to Lemma 1. We shall say that $X \subset V(G)$ *separates* $x \in V(G)$ and $y \in V(G)$ if $x \in X$ and $y \notin X$ or $y \in X$ and $x \notin X$.

Lemma 2. *Suppose that G is bicritical and $\{x, y\} \subset V(G)$ is a cutset. Let G_1, G_2 be as in* (3.2), *and let $\delta(X)$ $(X \subset V(G))$ be a tight cut of G. Then exactly one of the following possibilities holds.*

a) *X separates x and y and $|X \cap V(G_i)|$ $(i = 1, 2)$ are both odd.*

b) *X does not separate x and y. Then if $x, y \notin X$: $X \subset V(G_1) \setminus \{x, y\}$, or $X \subset V(G_2) \setminus \{x, y\}$, and if $x, y \in X$: $X \supset V(G_1)$ or $X \supset V(G_2)$.*

Proof. Note that $|X|$ is odd, for $\delta(X)$ is tight. Suppose first $x, y \notin X$ and suppose indirectly $X \cap V(G_1) \neq \emptyset$ and $X \cap V(G_2) \neq \emptyset$. One of $|X \cap V(G_i)|$ $(i = 1, 2)$ is even, say $|X \cap V(G_1)|$. Clearly, $\emptyset \neq \delta(X \cap V(G_1)) \subset \delta(X)$. $(\delta(X) = \delta(X \cap V(G_1)) \cup \cup \delta(X \cap V(G_2)))$. Since G is bicritical $e \in \delta(X \cap V(G_1))$ is contained in a perfect matching, which perfect matching must contain another edge of $\delta(X \cap V(G_1))$.

This contradicts the tightness of $\delta(X)$. Thus $X \subset V(G_1) \setminus \{x, y\}$ or $X \subset V(G_2) \setminus \{x, y\}$. If $x, y \in X$ then $x, y \notin V(G) \setminus X$ and applying the just proved statement for $V(G) \setminus X$ we get the desired result claimed in *b*.

Suppose now that $x \in X$ and $y \notin X$ and suppose indirectly that $|X \cap V(G_1)|$ and $|X \cap V(G_2)|$ are even. Let $\bar{G}_i = G_i \cup \{xy\}$ $(i=1, 2)$. For quantities and sets in $\bar{G}_i$ we shall write subscript $\bar{G}_i$, (eg. $\delta_{\bar{G}_1}(X)$ $(X \subset V(\bar{G}_1))$ means the coboundary of X in $\bar{G}_1$). Since $\delta(X)$ is a tight cut, every perfect matching M with $M \cap \delta_{\bar{G}_1}(X \cap V(\bar{G}_1)) \neq \emptyset$ must contain exactly 2 edges in $\delta_{\bar{G}_1}(X \cap V(\bar{G}_1))$ one of which is xy. We shall prove below that if $\bar{G}_1$ is bicritical, then this is not possible.

If H is a graph which has a perfect matching, $X \subset V(H)$, $|X|$ is even, and $e \in \delta(X)$ then $(\delta(X), e)$ will be called a *tight pair* if every perfect matching M with $M \cap \delta(X) \neq \emptyset$ intersects $\delta(X)$ in exactly 2 edges one of which is e.

Claim 1. *Let $A \subset V(H)$, $e \in \delta(A)$, $e = ab$, $a \in A$, $b \notin A$. The following statements are equivalent:*

(i) *$(\delta(A, e)$, is a tight pair.*

(ii) *$\delta(A \setminus \{a\})$ is a tight cut and $\delta(A) \cap \delta(a) = \{e\}$.*

(iii) *For every perfect matching M of H: $|M \cap \delta(A) \setminus e| \leqq 1$.*

(i)⇒(iii), (iii)⇒(ii) and (ii)⇒(i) are all immediate. Above we have reduced the proof of Lemma 2 to the following claim:

Claim 2. *In a bicritical graph H, $|V(H)| \geqq 4$, there is no tight pair.*

To prove Claim 2 we use induction on $|V(H)|$. Suppose indirectly that $(\delta(A), e)$ is a tight pair, $e = ab$, $a \in A$, $b \notin A$. H is not 3-connected, for by Claim 1 (ii) $\delta(A \setminus a)$ is a strict cut, and if H was 3-connected we would have by Lemma 1 $|A \setminus a| = 1$. Thus $A = \{a, c\}$ for some $c \in V(H)$ would hold. By the second part of (ii) the only neighbors of a can be b and c. So a would be an isolated vertex in $H - \{b, c\}$ contradicting the 3-connectedness (and bicriticality) of H. Thus there exists a cutset $\{x, y\}$ $(x, y \in V(H))$ in H. Let G_1 and $\bar{G}_1$, G_2 and $\bar{G}_2$ be as in (3.2), (3.3). Since $e \in \delta(A) = \bigcup_{i=1}^{2} \delta(A \cap V(G_i))$, $e \in \delta(A \cap V(G_1))$ say.

Case 1. A does not separate x and y, say $x, y \notin A$. (Fig. 1.)

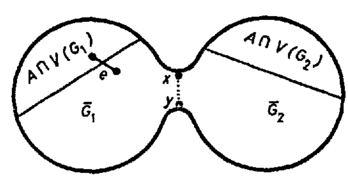

Fig. 1

If $A \cap V(G_j)$ $(j=1, 2)$ are odd, then a matching containing $e' \neq e$, $e' \in \delta_{\bar{G}_1}(A \cap V(G_1))$ contradicts the tightness of $(\delta(A), e)$. So, suppose $|A \cap V(G_j)|$ $(j=1, 2)$ are even. In this case we have from the induction hypothesis that $(\delta_{\bar{G}_1}(A \cap V(G_1), e)$ is not a tight pair, and it follows that $(\delta(A), e)$ is not a tight pair either.

Case 2. A separates x and y, say $x\in A$, $y\notin A$. (Fig. 2.)

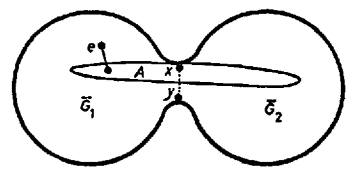

Fig. 2

Then $|A\cap V(G_1)|$ and $|A\cap V(G_2)|$ have different parity. If $|A\cap V(G_1)|$ is even then we get from the induction hypothesis, that $(\delta_{\bar{G}_1}(A\cap V(G_1)), e)$ is not strict, and if $|A\cap V(G_2)|$ is even we get that $(\delta_{\bar{G}_2}(A\cap V(G_2)), xy)$ is not strict. In the first case we get by Claim 1 (iii) a perfect matching $M_1\subset E(\bar{G}_1)$: $|M_1\cap\delta_{\bar{G}_1}(A\cap V(G_1))\setminus e|\geqq 2$ and in the second case a perfect matching $M_2\subset E(\bar{G}_2)$: $|M_2\cap\delta_{\bar{G}_2}(A\cap V(G_2))\setminus\{xy\}|\geqq 2$. Extending M_1 and M_2 resp. to a perfect matching of G (cf. (3.3)) we get a perfect matching M of G: $|M\cap\delta(A)\setminus e|\geqq 2$, a contradiction. Thus Claim 2 and Lemma 2 are proved. ∎

Proof of Theorem 3.1. For every inequality in (A) and (B) we shall define a weight function w on the edges of G in such a way that this specified inequality will be needed in order to get an integer optimum for the dual of

$$\max\{wx\colon x \text{ satisfies (A) and (B)}\}.$$

Suppose that $f\in E(G)$ is such that $x(f)\geqq 0$ is in (B). Let $w(e):=1$ if $e=f$ and $w(e):=0$ otherwise. Since $G-f$ has a t-join, $\tau(G, t, w)=0$. But then any optimal dual solution must use the dual variable corresponding to $x(f)\geqq 0$.

Suppose now that $\mathscr{P}$ is a reduced t-border (i.e. $x(\delta(\mathscr{P}))\geqq|\mathscr{P}|/2$ is in (A)). Let now $w(e)=1$ if $e\in\delta(\mathscr{P})$ and $w(e)=0$ otherwise. We can suppose without loss of generality that $\mathscr{P}=\{\{x\}, x\in V(G)\}$ and thus $\delta(\mathscr{P})=E(G)$, G is bicritical, and t is the all 1 function (otherwise we shrink the classes of $\mathscr{P}$). Clearly, $\tau(G, t, w)=|V(G)|/2$. Since every $e\in\delta(\mathscr{P})$ is contained in an optimal t-join (i.e. matching), by complementary slackness it follows that the positive dual variables of any integer optimal dual solution correspond to inequalities in (A), that is they determine a set of t-borders $\delta(\mathscr{P}_1), \ldots, \delta(\mathscr{P}_k)$ such that

$$(3.4)\quad E(G)=\delta(\mathscr{P}_1)\cup\ldots\cup\delta(\mathscr{P}_k)\quad \delta(\mathscr{P}_i)\cap\delta(\mathscr{P}_j)=\emptyset\quad (i\neq j),\ \sum_{i=1}^{k}\frac{|\mathscr{P}_i|}{2}=\frac{V(G)}{2}.$$

All we have to prove is that $k=1$ in (3.4).

We proceed by induction on $|V(G)|$. If indirectly $k\geqq 2$ then there exists an i and $P\in\mathscr{P}_i$ such that $|P|\geqq 2$. $\delta(P)$ is obviously strict (by complementary slackness), thus, by Lemma 1 G cannot be 3-connected. Let $\{x, y\}$ be a cutset, and let $G_1, G_2, \bar{G}_1, \bar{G}_2$ be as in (3.2), (3.3). If $\mathscr{P}$ is a partition of $V(G)$ define the partition $\mathscr{P}\cap G_i$ on $V(G_i)$ as follows:

$$\mathscr{P}\cap G_i:=\{P\cap V(G_i)\neq\emptyset\colon P\in\mathscr{P}\}\quad (i=1,2).$$

Furhermore let $\mathscr{I}_1=\{i\in\{1,\ldots,k\}\colon \mathscr{P}_i\cap G_1\neq\{V(G_1)\}\}$ and $\mathscr{I}_2=\{i\in\{1,\ldots,k\}\colon \mathscr{P}_i\cap G_2\neq\{V(G_2)\}\}$ and let $s=|\mathscr{I}_1\cap\mathscr{I}_2|$. By Lemma 2 $\mathscr{P}_i\cap G_1$ and $\mathscr{P}_i\cap G_2$ ($i=1,\ldots,k$) are t-borders of G_1 and G_2 resp., and $i\in\mathscr{I}_1\cap\mathscr{I}_2$ if and only if x and y are in different classes of $\mathscr{P}_i$ (see Figure 3).

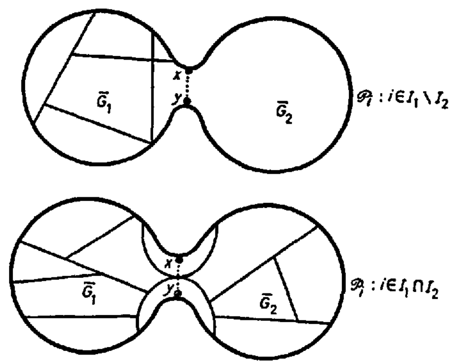

Fig. 3

Thus,

$$\sum_{i\in\mathscr{I}_1}\frac{|\mathscr{P}_i\cap G_1|}{2}+\sum_{i\in\mathscr{I}_2}\frac{|\mathscr{P}_i\cap G_2|}{2}=\sum_{i=1}^{k}\frac{|\mathscr{P}_i|}{2}+s=\left|\frac{V(G)}{2}\right|+s. \tag{3.5}$$

We prove that $s=1$. $s\geqq 1$, for s is the number of $\mathscr{P}_i$-s in which x and y are in different classes, and if they were in the same class in every $\mathscr{P}_i$ ($i=1,\ldots,k$), then

$$\tau(G-\{x,y\},t,w)\geqq\sum_{i=1}^{k}\frac{|\mathscr{P}_i|}{2}=\tau(G,t,w)$$

would hold, since then $\mathscr{P}_i$ would determine a t-partition of $V(G)-\{x,y\}$ ($i=1,\ldots,k$). But this is a contradiction, because $\tau(G-\{x,y\},t,w)=\tau(G,t,w)-1$. In order to prove $s\leqq 1$ consider the following weighting on $E(G_i)$ ($i=1,2$). The weight $\bar{w}(e)$ of every edge is 1 except for $\bar{w}(xy):=s$. Since $\bar{G}_i$ ($i=1,2$) are bicritical, they have a perfect matching not containing xy, and thus $\tau(\bar{G}_i,t,\bar{w})=|V(G_i)|/2$. Consequently, by (the trivial part of) Theorem 1.1:

$$\sum_{i\in\mathscr{I}_1}\frac{|\mathscr{P}_i\cap G_1|}{2}\leqq\frac{|V(G_1)|}{2},\quad\sum_{i\in\mathscr{I}_2}\frac{|\mathscr{P}_i\cap G_2|}{2}\leqq\frac{|V(G_2)|}{2}. \tag{3.6}$$

Adding the two inequalities in (3.6) we have:

$$\sum_{i\in\mathscr{I}_1}\frac{|\mathscr{P}_i\cap G_1|}{2}+\sum_{i\in\mathscr{I}_2}\frac{|\mathscr{P}_i\cap G_2|}{2}\leqq\frac{|V(G)|}{2}+1. \tag{3.7}$$

Comparing (3.5) and (3.7) we get that $s=1$ and equality holds everywhere in (3.5), (3.6) and (3.7). This means that (3.4) is satisfied for $\bar{G}_1$ and $\bar{G}_2$, whence by the induction hypothesis $|\mathscr{I}_1|=|\mathscr{I}_2|=1=|\mathscr{I}_1\cap\mathscr{I}_2|$. This implies $k=1$ as it was claimed. ∎

Note that Theorem 3.1 provides a *polynomial algorithm for deciding whether a given inequality is in the minimal* TDI *defining system for the t-join polyhedron of a given graph.* (The properties in the definition of (A) and (B) can be checked in polynomial time.) This problem is NP-complete in general, say for "solvable" classes of polyhedra (in the sense of [13], as it was recently proved by É. Tardos [33]. It is not difficult to combine the results of the present paper with those of [26] to get an integer primal and dual solution of linear programming problems defined by (A), (B) (cf. [30]).

Also note that by [11] this system is "locally strongly unimodular" as well.

Finally let us relate the Schrijver system (A), (B) to binary clutters with the max-flow-min-cut property. (G, t) is said to have the *max-flow-min-cut property* if the system of inequalities (B), (C) is TDI (cf. [31]). Noting that the only bicritical graph on 4 vertices is K_4, Seymour's characterization [31] of (G, t) pairs with the max-flow-min-cut property can be written in the following form:

Theorem 3.2. [31] *(G, t) has the max-flow-min-cut property if and only if there does not exist a reduced t-partition $\mathcal{P}$ with $|\mathcal{P}|=4$.*

Theorem 3.1 immediately implies the following characterization:

Theorem 3.3. *(G, t) has not the max-flow-min-cut property if and only if there exists a reduced t-partition $\mathcal{P}$ with $|\mathcal{P}|\geqq 4$.*

Comparing Theorems 3.2 and 3.3 it is apparent that:

Theorem 3.4. *If G is a bicritical graph, $|V(G)|\geqq 4$ and t is the all 1 function on $V(G)$, then there exists a reduced t-partition $\mathcal{P}$ with $|\mathcal{P}|=4$.*

(This can easily be proved using Theorem 3.2 only.)

In other words every bicritical graph can be contracted to K_4, in such a way that the number of vertices "falling" to any $x\in V(K_4)$ is odd. Note that conversely Theorem 3.3 and Theorem 3.4 imply Theorem 3.2. Bert Gerards and László Lovász have remarked that both [18, Theorem 5.4.11, and 10, Theorem 2.1] immediately imply Theorem 3.4, furthermore Gerards [12] has recently found a direct elementary proof. Knowing this, Theorem 3.3 implies Theorem 3.2, thus providing a new proof of Seymour's characterization of the max-flow-min-cut (G, t) pairs.

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