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Midterm Exam: October 25, 2016

Duration 2 hours. No notes or electronic devices allowed.

Exercise 1. (6 points.)

Suppose we are given the following nutrition table:

	Butter	Eggs	Cheese	Cream
Protein	1	1	2	1
Fat	3	1	2	2
Sugar	1	0	0	1
Calories	4	2	3	3

Polly wants to find a "good" diet, which is defined as a diet that contains:

- at least 10 units of protein,
- at least 15 units of fat,
- at least 6 units of sugar,
- at most 100 calories.

Answer True, False or Cannot Determine. Justify your answer.
There exists a good diet for Polly.

Solution: Let x_1, x_2, x_3 and x_4 denote the units of butter, eggs, cheese and cream in Polly's diet. The following linear program models the given requirements.

$$\begin{aligned} 4x_1 + 2x_2 + 3x_3 + 3x_4 &\leq 100 \\ x_1 + x_2 + 2x_3 + x_4 &\geq 10 \\ 3x_1 + x_2 + 2x_3 + 2x_4 &\geq 15 \\ x_1 + x_4 &\geq 6 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

Since this linear program is non-empty (e.g. $\{x_1 = 10, x_2 = 0, x_3 = 0, x_4 = 0\}$ is feasible), the answer is **True**.

Exercise 2. (6 points.)

Polly wants to solve the following linear program (P_1).

$$\begin{aligned} \text{Maximize} \quad & 3x_1 - 2x_2 + 2x_3 - 4x_4 \\ \text{subject to:} \quad & 2x_1 + x_2 + 2x_3 - x_4 \le 2 \\ & -2x_2 - x_3 \le 2 \\ & x_1, x_2, x_3, x_4 \ge 0. \end{aligned} \tag{P_1}$$

Polly implements Phase II of the simplex algorithm using Bland's rule as the pivot rule. After transforming (P_1) into standard form, she uses her implementation to solve (P_1) using the solution $(0,0,0,0,2,2)$ as an intitial basic feasible solution. She views the output of her program and sees that it has visited 33 dictionaries.

Answer True, False or Cannot Determine. Justify your answer.
Polly's implementation of the simplex algorithm is correct.

Solution: There are at most $\binom{n+m}{m}$ basic feasible solutions (and dictionaries). Thus, there are at most $\binom{6}{2} = 15$ dictionaries. Since Bland's rule does not cycle, we can visit at most 15 dictionaries in a correct implementation. So the answer is **False**.

(Some people directly showed that using Bland's rule, the simplex algorithm uses only two dictionaries.)

Exercise 3. (6 points.)

Polly wants to solve the following linear program (P_2).

$$\begin{aligned} \text{Maximize} \quad & \mathbf{c}'\mathbf{x} \\ \text{subject to:} \quad & \mathbf{Ax} \le \mathbf{b} \\ & \mathbf{x} \ge 0. \end{aligned} \tag{P_2}$$

The matrix $\mathbf{A}$ consists of $m < n$ constraints on n variables and the polytope associated to the matrix $\mathbf{A}$ and vector $\mathbf{b}$ is non-degenerate.

Polly implements Phase II of the simplex algorithm using the maximum coefficient rule for the entering variable and the smallest subscript rule for the leaving variable. After finding an intial basic feasible solution, Polly runs her implementation on (P_2). She notices that the 7th and the 12th dictionaries are the same.

Answer True, False or Cannot Determine. Justify your answer.
Polly's implementation of the simplex algorithm is correct.

Solution: When the polytope is non-degenerate, the simplex algorithm always increases the solution as it goes from one dictionary to the next, regardless of pivot rule. Thus, each dictionary is unique. The statement is **False**.

Exercise 4. (6 points.)

Polly finds the optimal solution to the following linear program on the graph $G = (V, A)$ (shown in Figure 1).

$$\begin{aligned} \max \sum_{su \in A} x_{su} & \\ \text{subject to: } \sum_{uv \in A} x_{uv} - \sum_{vw \in A} x_{vw} &= 0, \quad \forall v \neq s, t, \\ x_{uv} &\leq c_{uv}, \quad \forall uv \in A, \\ x_{uv} &\geq 0. \quad \forall uv \in A. \end{aligned} \tag{P_3}$$

Polly obtains the solution 7.

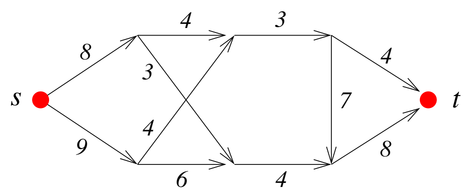

Figure 1: The graph $G = (V, A)$. The numbers shown correspond to the values c_{uv} for each edge.

Answer True, False or Cannot Determine. Justify your answer.
Polly's solution is correct.

Solution: The above linear program models the maximum flow problem. Polly has found a maximum flow with value 7 and the graph contains a minimum cut with value 7. Since the value of the maximum flow cannot exceed the value of a minimum cut, her solution is optimal.

Exercise 5. (6 points.)

Polly wants to solve the following linear program (P_4).

$$\begin{aligned} \text{Maximize} \quad & x_1 + x_2 \\ \text{subject to:} \quad & x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0. \end{aligned} \qquad (P_4)$$

She implements the simplex algorithm and uses it to find the solution for (P_4). She obtains the solution $(x_1 = 1, x_2 = 1)$.

Answer True, False or Cannot Determine. Justify your answer.
Polly's implementation of the simplex algorithm is correct.

Solution: The simplex algorithm always returns a basic feasible solution. Since there is only one constraint, a basic feasible solution contains only one non-zero value. Since the solution found by Polly's implementation of the simplex algorithm contains two non-zero values, the implementation is not correct.

Exercise 6. (20 points.)

We are given the following linear program (P_5).

$$\begin{aligned} \text{Maximize} \quad & -3x_1 + 2x_2 - 2x_3 - x_4 \\ \text{subject to:} \quad & 4x_1 - 2x_2 + x_3 - x_4 \leq -4 \\ & -x_1 + x_2 - x_3 \leq -2 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned} \qquad (P_5)$$

1. Find an intial basic feasible solution using Phase I of the simplex algorithm.

Solution: 1. Phase I of the simplex algorithm:

$$\begin{aligned} x_5 &= -4 - 4x_1 + 2x_2 - x_3 + x_4 + x_0 \\ x_6 &= -2 + x_1 - x_2 + x_3 + x_0 \\ z &= -x_0 \end{aligned} \qquad (D_0')$$

Use "illegal pivot" where x_0 enters the basis and x_5 leaves.

$$\begin{aligned} x_0 &= 4 + 4x_1 - 2x_2 + x_3 - x_4 + x_5 \\ x_6 &= 2 + 5x_1 - 3x_2 + 2x_3 - x_4 + x_5 \\ z &= -4 - 4x_1 + 2x_2 - x_3 + x_4 - x_5 \end{aligned} \qquad (D_1')$$

Now x_2 enters and x_6 leaves.

$$x_2 = \frac{1}{3}(2 + 3x_1 + 2x_3 - x_4 + x_5 - x_0)$$
$$x_0 = \frac{1}{3}(8 + 2x_1 - x_3 - x_4 + x_5 + 2x_0)$$
$$z = \frac{1}{3}(-8 - 2x_1 + x_3 + x_4 - x_5 - 2x_0) \qquad (D_2')$$

Now x_3 enters and x_0 leaves.

$$x_2 = 6 - 2x_0 + 3x_1 - x_4 + x_5 + x_6$$
$$x_3 = 8 - 3x_0 + 2x_1 - x_4 + x_5 + 2x_6$$
$$z = 0 - x_0 \qquad (D_3')$$

So the original linear program (P_5) is indeed feasible and an initial basic feasible solution is $(0, 6, 8, 0)$.

2. Find an optimal solution using Phase II of the simplex algorithm.

Solution:

2. Phase II of the simplex algorithm. Let us consider the last dictionary of Phase I:

$$x_2 = 6 - 2x_0 + 3x_1 - x_4 + x_5 + x_6$$
$$x_3 = 8 - 3x_0 + 2x_1 - x_4 + x_5 + 2x_6$$
$$z = -3x_1 + 2x_2 - 2x_3 - x_4$$
$$= -4 - x_1 - x_4 - 2x_6 \qquad (D_0)$$

We see that since we cannot increase z, Phase II is over and the optimal solution is -4.

3. Certify the optimality of your solution. Explain your method carefully.

Solution:

3. The last dictionary of Phase II of the simplex algorithm produces a certificate of optimality, indicated by the coefficients of the slack variables. If we take twice the second constraint in (P_5), we have:

$$2(-x_1 + x_2 - x_3 \leq -2) \quad \Rightarrow \quad -2x_1 + 2x_2 - 2x_3 \leq -4.$$

This leads to the upper bound on the objective function:

$$-3x_1 + 2x_2 - 2x_3 - x_4 \leq -2x_1 + 2x_2 - 2x_3 \leq -4.$$

Thus, there is no solution for (P_5) with value larger than -4, so -4 is the optimal solution.

Exercise 7. (20 points.)

We are given the following linear program (P_6).

$$\begin{array}{rrl} \text{Maximize} & 3x_1 + 4x_2 + 2x_3 + x_4 & \\ \text{subject to:} & 3x_1 + x_2 + x_3 + 4x_4 & \leq 15 \\ & x_1 - 3x_2 + 2x_3 + 3x_4 & \leq 8 \\ & 2x_1 + x_2 + 3x_3 - x_4 & \leq 10 \\ & x_1, x_2, x_3, x_4 & \geq 0. \end{array} \qquad (P_6)$$

1. Write the dual linear program.

Solution:

$$\begin{array}{rrl} \text{Minimize} & 15y_1 + 8y_2 + 10y_3 & \\ \text{subject to:} & 3y_1 + y_2 + 2y_3 & \geq 3 \\ & y_1 - 3y_2 + y_3 & \geq 4 \\ & y_1 + 2y_2 + 3y_3 & \geq 2 \\ & 4y_1 + 3y_2 - y_3 & \geq 1 \\ & y_1, y_2, y_3 & \geq 0. \end{array} \qquad (D_6)$$

2. Show that $x_1 = 0, x_2 = 11, x_3 = 0, x_4 = 1$ is an optimal solution for (P_6). Explain your method carefully.

Solution: This solution has a value of 45 for (P_6). By Weak Duality, every solution for (P_6) has value at most that of any solution for (D_6). Thus, we need to find a feasible solution for the dual with objective value 45 to prove optimality of the given solution. To do this, we can:

Use the simplex algorithm to determine that $y_1 = 1, y_2 = 0, y_3 = 3$ is such a solution. Or:

Observe that if the given solution is optimal, then it provides a combination of equations in (D_6) that achieves the (largest possible) lower bound of 45. Specifically,

$$\begin{array}{c} 11 \cdot (y_1 - 3y_2 + y_3 \geq 4) \quad \text{and} \quad 1 \cdot (4y_1 + 3y_2 - y_3 \geq 1) \quad \Rightarrow \\ 15y_1 + 8y_2 + 10y_3 \;\geq\; 15y_1 - 30y_2 + 10y_3 \;\geq\; 45, \end{array} \qquad (1)$$

and there should also exist a solution for (D_6) that achieves this bound. Thus, in addition to constraint (1) being tight (which implies $y_2 = 0$), the second and fourth constraints in (D_6) must be tight. This leads to the following system of equations:

$$\begin{aligned} y_1 + y_3 &= 4 \\ 4y_1 - y_3 &= 1, \end{aligned}$$

from which we deduce the feasible dual solution $y_1 = 1, y_2 = 0, y_3 = 3$.

Exercise 8. Fractional Knapsack. (30 points.)

Polly has a knapsack with capacity B. She wants to fill it with items from the set $S = \{s_1, s_2, \ldots, s_n\}$. An item s_i has value v_i and takes up b_i units of space in the knapsack. There is only one copy of each item and each item fits in the knapsack, i.e. for each item s_i, $b_i \leq B$.

(i) Polly has an axe and can chop up the items into fractional amounts of arbitrary sizes. Write a linear program to find the maximum value (possibly fractional) set of items that Polly can fit in her knapsack.

Solution: Let $x_i \in [0, 1]$ be an indicator variable for how much of item s_i is placed in the knapsack.

$$\begin{aligned} \text{Maximize} \quad & \sum_{i \in S} x_i v_i \\ \text{subject to:} \quad & \sum_{i \in S} x_i b_i \leq B, \\ & x_i \leq 1, \text{ for all } i \in S, \\ & x_i \geq 0, \text{ for all } i \in S. \end{aligned} \qquad (P_{knapsack})$$

(ii) Polly would like to minimize the use of her axe. She runs the simplex algorithm on the linear program from (i) and uses it to determine which items to pack in her knapsack. Some items wille be *completely* packed and some items will be *fractionally* packed. (We say an item s_i is fractionally packed if strictly more than zero units and strictly less than b_i units are packed.) What is the maximum possible number of items that will be fractionally packed in her knapsack? (Denote this integer by $K^\star$.)

Solution: The optimal solution for $(P_{knapsack})$ that Polly obtains from the simplex algorithm is a basic feasible solution. In a basic feasible solution for a linear program with n variables, there are at least n tight constraints. Thus, at least $n-1$ of the constraints of the form $x_i \geq 0$ or $x_i \leq 1$ are tight implying that at least $n-1$ variables have an integral x_i value. Therefore, at most one item will be fractionally packed in Polly's solution, and the value of $K^\star = 1$.

(iii) Using insights from parts (i) and (ii), describe a combinatorial algorithm that outputs an optimal solution for the fractional knapsack problem. On any input instance, this algorithm should place at most $K^\star$ fractional items in the knapsack.

Solution: Consider an optimal basic feasible solution x for $(P_{knapsack})$. We will show that this solution has the following properties:

$$\text{if } x_i > x_j, \text{ then } \frac{v_i}{b_i} \geq \frac{v_j}{b_j}. \qquad (\star\star)$$

Assume that property $(\star\star)$ does not hold for x. Then there is some i, j such that $x_i > x_j$ but

$$\frac{v_i}{b_i} < \frac{v_j}{b_j}.$$

The contribution of items s_i and s_j to the objective function is:

$$\frac{v_i}{b_i}(b_i x_i) + \frac{v_j}{b_j}(b_j x_j). \qquad (2)$$

Since $x_i > x_j$, there exists some $\epsilon : 0 < \epsilon \leq b_j x_j$ such that decreasing the space given to item s_i by ϵ and increasing the space given to item s_j by ϵ is still feasible, and the amount (3)

$$\frac{v_i}{b_i}(b_i x_i - \epsilon) + \frac{v_j}{b_j}(b_j x_j + \epsilon) \qquad (3)$$

is strictly larger than the amount in (2). This contradicts the assumption that x is optimal.

Property $(\star\star)$ suggests the following combinatorial algorithm. Sort the items in nonincreasing order according to $\frac{v_i}{b_i}$. Go through items in this order and add to the knapsack. At some point, the knapsack is full or there is an item s_i such that b_i is less than the remaining available space in the knapsack. For this item, place the largest possible fractional amount in the knapsack, e.g. use the axe to partition the item appropriately.

To prove this algorithm is optimal, we want to show that a solution with one fractional value for which property $(\star\star)$ holds is optimal. First we note that for any instance S and any solution $x \in \mathbb{R}^n$, we can assume that:

$$\frac{v_i}{b_i} \neq \frac{v_j}{b_j} \text{ for } i \neq j. \qquad (\bowtie)$$

If this is not the case, then we can use the following transformation. Consider the set $I = \{i_1, i_2, \ldots, i_\ell\}$ such that $\frac{v_p}{b_p} = \frac{v_q}{b_q}$ for all $p, q \in I$. Then set:

$$v_I = \sum_{i \in I} v_i; \quad b_I = \sum_{i \in I} b_i; \quad x_I = \frac{\sum_{i \in I} b_i x_i}{b_I}.$$

This transformation does not change the value of a solution (and thus does not change the optimal value) and in the case of a solution produced by our combinatorial algorithm, the transformed solution still has at most one fractional value.

Now let us consider an instance with property ($\bowtie$). and a solution $g \in \mathbb{R}^n$ produced by our combinatorial algorithm. Recall that the solution g has at most one fractional solution and by construction, g obeys property ($\star\star$). Additionally, consider an optimal solution $y \in \mathbb{R}^n$ to ($P_{knapsack}$) on this instance. Recall that y has at most one fractional solution and as shown above, it also obeys property ($\star\star$). Our goal is to show that g and y have the same value. Assume that the items in S have labels sorted in decreasing order according to $\frac{v_i}{b_i}$.

Let s_i be the first item for which $y_i < g_i$. Then there must be some item s_j such that $y_j > g_j$ for $j > i$. Since y does not contain two fractional values, either:

$$0 < y_i < 1 \text{ and } y_j = 1, \text{ or } y_i = 0 \text{ and } y_j > 0.$$

In either case, $y_i < y_j$. However, we have $\frac{v_i}{b_i} \geq \frac{v_j}{b_j}$. This is a contradiction to the fact that property ($\star\star$) holds for solution y.