Midterm Exam: October 25, 2016

Duration 2 hours. No notes or electronic devices allowed.

Exercise 1. (6 points.)

Suppose we are given the following nutrition table:

	Butter	Eggs	Cheese	Cream
Protein	1	1	2	1
Fat	3	1	2	2
Sugar	1	0	0	1
Calories	4	2	3	3

Polly wants to find a "good" diet, which is defined as a diet that contains:

- at least 10 units of protein,
- at least 15 units of fat,
- at least 6 units of sugar,
- at most 100 calories.

Answer True, False or Cannot Determine. Justify your answer.
There exists a good diet for Polly.

Exercise 2. (6 points.)

Polly wants to solve the following linear program (P_1).

$$\begin{aligned} \text{Maximize} \quad & 3x_1 - 2x_2 + 2x_3 - 4x_4 \\ \text{subject to:} \quad & 2x_1 + x_2 + 2x_3 - x_4 \leq 2 \\ & -2x_2 - x_3 \leq 2 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned} \tag{P_1}$$

Polly implements Phase II of the simplex algorithm using Bland's rule as the pivot rule. After transforming (P_1) into standard form, she uses her implementation to solve (P_1) using the solution $(0, 0, 0, 0, 2, 2)$ as an intitial basic feasible solution. She views the output of her program and sees that it has visited 33 dictionaries.

Answer True, False or Cannot Determine. Justify your answer.
Polly's implementation of the simplex algorithm is correct.

Exercise 3. (6 points.)

Polly wants to solve the following linear program (P_2).

$$\begin{aligned} \text{Maximize} \quad & \mathbf{c}'\mathbf{x} \\ \text{subject to:} \quad & \mathbf{Ax} \leq \mathbf{b} \\ & \mathbf{x} \geq 0. \end{aligned} \tag{P_2}$$

The matrix $\mathbf{A}$ consists of $m < n$ constraints on n variables and the polytope associated to the matrix $\mathbf{A}$ and vector $\mathbf{b}$ is non-degenerate.

Polly implements Phase II of the simplex algorithm using the maximum coefficient rule for the entering variable and the smallest subscript rule for the leaving variable. After finding an intial basic feasible solution, Polly runs her implementation on (P_2). She notices that the 7th and the 12th dictionaries are the same.

Answer True, False or Cannot Determine. Justify your answer.
Polly's implementation of the simplex algorithm is correct.

Exercise 4. (6 points.)

Polly finds the optimal solution to the following linear program on the graph $G = (V, A)$ (shown in Figure 1).

$$\begin{aligned} & \max \sum_{su \in A} x_{su} \\ \text{subject to:} \quad & \sum_{uv \in A} x_{uv} - \sum_{vw \in A} x_{vw} = 0, \quad \forall v \neq s, t, \\ & x_{uv} \leq c_{uv}, \quad \forall uv \in A, \\ & x_{uv} \geq 0. \quad \forall uv \in A. \end{aligned} \tag{P_3}$$

Polly obtains the solution 7.

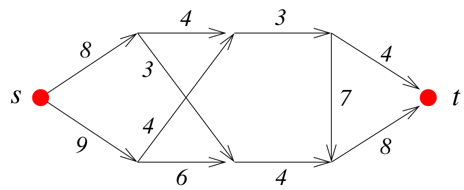

Figure 1: The graph $G = (V, A)$. The numbers shown correspond to the values c_{uv} for each edge.

Answer True, False or Cannot Determine. Justify your answer.
Polly's solution is correct.

Exercise 5. (6 points.)

Polly wants to solve the following linear program (P_4).

$$\begin{aligned} \text{Maximize} \quad & x_1 + x_2 \\ \text{subject to:} \quad & x_1 + x_2 \le 2 \\ & x_1, x_2 \ge 0. \end{aligned} \tag{P_4}$$

She implements the simplex algorithm and uses it to find the solution for (P_4). She obtains the solution $(x_1 = 1, x_2 = 1)$.

Answer True, False or Cannot Determine. Justify your answer.
Polly's implementation of the simplex algorithm is correct.

Exercise 6. (20 points.)

We are given the following linear program (P_5).

$$\begin{aligned} \text{Maximize} \quad & -3x_1 + 2x_2 - 2x_3 - x_4 \\ \text{subject to:} \quad & 4x_1 - 2x_2 + x_3 - x_4 \le -4 \\ & -x_1 + x_2 - x_3 \le -2 \\ & x_1, x_2, x_3, x_4 \ge 0. \end{aligned} \tag{P_5}$$

1. Find an intial basic feasible solution using Phase I of the simplex algorithm.
2. Find an optimal solution using Phase II of the simplex algorithm.
3. Certify the optimality of your solution. Explain your method carefully.

Exercise 7. (20 points.)

We are given the following linear program (P_6).

$$\begin{array}{lrl} \text{Maximize} & 3x_1 + 4x_2 + 2x_3 + x_4 & \\ \text{subject to:} & 3x_1 + x_2 + x_3 + 4x_4 & \leq 15 \\ & x_1 - 3x_2 + 2x_3 + 3x_4 & \leq 8 \\ & 2x_1 + x_2 + 3x_3 - x_4 & \leq 10 \\ & x_1, x_2, x_3, x_4 & \geq 0. \end{array} \qquad (P_6)$$

1. Write the dual linear program.
2. Show that $x_1 = 0, x_2 = 11, x_3 = 0, x_4 = 1$ is an optimal solution for (P_6). Explain your method carefully.

Exercise 8. Fractional Knapsack. (30 points.)

Polly has a knapsack with capacity B. She wants to fill it with items from the set $S = \{s_1, s_2, \ldots, s_n\}$. An item s_i has value v_i and takes up b_i units of space in the knapsack. There is only one copy of each item and each item fits in the knapsack, i.e. for each item s_i, $b_i \leq B$.

(i) Polly has an axe and can chop up the items into fractional amounts of arbitrary sizes. Write a linear program to find the maximum value (possibly fractional) set of items that Polly can fit in her knapsack.

(ii) Polly would like to minimize the use of her axe. She runs the simplex algorithm on the linear program from (i) and uses it to determine which items to pack in her knapsack. Some items wille be *completely* packed and some items will be *fractionally* packed. (We say an item s_i is fractionally packed if strictly more than zero units and strictly less than b_i units are packed.) What is the maximum possible number of items that will be fractionally packed in her knapsack? (Denote this integer by $K^\star$.)

(iii) Using insights from parts (i) and (ii), describe a combinatorial algorithm that outputs an optimal solution for the fractional knapsack problem. On any input instance, this algorithm should place at most $K^\star$ fractional items in the knapsack.