A. Newman, F. Bréhard, N.K. Le

Homework 2

— Deadline: 3rd January 2017 in Class —

Instructions: Please write each problem on a separate sheet of paper. You may discuss the problems with your fellow students, but you must write your own solutions. Starred problems (e.g. * and **) are more challenging and will be counted as extra credit.

Exercise 1. Maximum Bipartite Subgraphs (*30 pts*)

The *maximum bipartite subgraph* problem is defined as follows. Given an undirected graph $G = (V, E)$, the goal is to find a subset of the edges $S \subseteq E$ such that (V, S) is bipartite and $|S|$ is maximized. Consider the following linear programming relaxation:

$$\max \sum_{ij \in E} x_{ij}$$
$$\sum_{ij \in C} x_{ij} \leq |C| - 1, \quad \text{for all odd cycles } C \in E,$$
$$0 \leq x_{ij} \leq 1, \quad \text{for all edges } ij \text{ in } E. \qquad (P_{bipartite})$$

a) (*15 pts*) Give a polynomial-time separation oracle for $(P_{bipartite})$.

b) (*5 pts*) Find a graph that exhibits an integrality of at least $\frac{10}{9}$. (The graph should contain at most ten vertices.)

c) (*10 pts*) There are graphs for which the polytope $(P_{bipartite})$ is integer. Such graphs are called *weakly bipartite*. Suppose G is a weakly bipartite graph. Prove that there is a partition of the vertices of G such that every triangle in G is partitioned by the cut. In other words, there exists a subset $U \subset V$ such that for every triangle t in G, $E(U, V \setminus U) \cap t \neq \emptyset$.

Exercise 2. Randomized Rounding (Problem 5.8 from Williamson-Shmoys) (*20 pts*)

Consider a variation of the maximum satisfiability problem in which all variables occur positively in each clause. Each clause has a nonnegative weight w_j and there is an additional nonnegative weight $v_i \geq 0$ for each boolean variable x_i. The goal is now to set the boolean variables to maximize the total weight of the satisfied clauses plus the total weight of variables set to be false.

a) (*5 pts*) Give an integer programming formulation for this problem, with 0-1 variables y_i to indicate whether or not x_i is set to true.

b) (*15 pts*) Consider the linear programming relaxation of the integer program in part a). Show that a randomized rounding of this linear program in which variable x_i is set to true with probability $1 - \lambda + \lambda y_i^*$ gives a $2(\sqrt{2} - 1)$-approximation algorithm for some appropriate setting of λ; note that $(2(\sqrt{2} - 1) \approx .828$. In other words, show that this randomized rounding results in a solution whose *expected* value is at least $2(\sqrt{2} - 1)$ times the value of an optimal solution.

Exercise 3. Traveling Salesman Problem (*30 pts*)

Let $G = (V, E)$ be a complete, weighted graph. Each edge $ij \in E$ has weight $w_{ij} \geq 0$ and these weights obey the triangle inequality: $w_{ij} + w_{jk} \geq w_{ik}$ for all $i, j, k \in V$. The *traveling salesman problem* is to find a minimum weight Hamiltonian tour in G. Consider the following integer program:

$$\min \sum_{ij \in E} w_{ij} x_{ij}$$
$$\sum_{j \in \delta(i)} x_{ij} = 2, \text{ for all } i \in V,$$
$$\sum_{\substack{ij \in E: \\ i \in S, j \notin S}} x_{ij} \geq 2, \text{ for all cuts } S \subset V,$$
$$x_{ij} \in \{0, 1\}. \qquad (P_{tsp})$$

a) (*10 pts*) Prove that the problem of finding an optimal solution for (P_{tsp}) is equivalent to that of finding a minimum weight Hamiltonian tour in G.

b) (*10 pts*) Relax the integrality requirement in (P_{tsp}) to obtain the constraint: $0 \leq x_{ij} \leq 1$. Let $OPT_f(G)$ denote the optimal value for the resulting linear programming relaxation.

   Let T denote a minimum weight spanning tree in G and let $w(MST(G))$ denote its weight. Let $J \subset V$ denote the vertices in V that have an odd-degree in T. Show that there is a minimum weight perfect matching on J with value at most $\frac{OPT_f(G)}{2}$.

c) (*10 pts*) Conclude that the traveling salesman problem has a solution with weight at most $w(MST(G)) + \frac{OPT_f(G)}{2}$.

Exercise 4. Triangle Packing (*20 pts + 20 pts extra credit*)

Let $G = (V, E)$ be an unweighted, undirected graph. Let T be the set of all triangles in E; a triangle is a triple of edges $\{e_1, e_2, e_3\}$ such that these three edges form a cycle with 3 edges. Our goal is to find a set of edges $S \subset E$ such that $S \cap t \neq \emptyset$ for all $t \in T$ and $|S|$ is minimized. We call this the *triangle hitting set problem.* Let $OPT(G)$ denote the size of an optimal triangle hitting set.

a) (*10 pts*) Give a 3-approximation for the triangle hitting set problem.

b) (*10 pts*) Consider the following linear programming relaxation for the problem:

$$\min \sum_{e \in E} x_e$$
$$\sum_{e \in t} x_e \geq 1 \text{ for all triangles } t \in T,$$
$$x_e \geq 0. \qquad (P_{triangle})$$

   Write the dual linear program.

c) * (*10 pts*) Consider an optimal solution x^* for $(P_{triangle})$ with value $OPT_f(G)$. Suppose $x^*_e > 0$ for all $e \in E$. Show that in this case, $OPT(G) \leq \frac{3}{2} OPT_f(G)$. (Hint: Use complementary slackness to bound $OPT_f(G)$ in terms of $|E|$, and then find a bound on $OPT(G)$ in terms of $|E|$.)

d) ** (*10 pts*) Prove that $OPT(G) \leq 2\ OPT_f(G)$.