Homework 2
— Deadline: 3rd January 2017 in Class —

Instructions: Please write each problem on a separate sheet of paper. You may discuss the problems with your fellow students, but you must write your own solutions. Starred problems (e.g. * and **) are more challenging and will be counted as extra credit.

Exercise 1. Maximum Bipartite Subgraphs (30 pts)

The maximum bipartite subgraph problem is defined as follows. Given an undirected graph $G = (V,E)$, the goal is to find a subset of the edges $S \subseteq E$ such that $(V,S)$ is bipartite and $|S|$ is maximized. Consider the following linear programming relaxation:

$$
\max \sum_{ij \in E} x_{ij} \\
\sum_{ij \in C} x_{ij} \leq |C| - 1, \quad \text{for all odd cycles } C \in E, \\
0 \leq x_{ij} \leq 1, \quad \text{for all edges } ij \in E.
$$

(a) (15 pts) Give a polynomial-time separation oracle for $P_{\text{bipartite}}$.

(b) (5 pts) Find a graph that exhibits an integrality of at least $\frac{10}{7}$. (The graph should contain at most ten vertices.)

(c) (10 pts) There are graphs for which the polytope $P_{\text{bipartite}}$ is integer. Such graphs are called weakly bipartite. Suppose $G$ is a weakly bipartite graph. Prove that there is a partition of the vertices of $G$ such that every triangle in $G$ is partitioned by the cut. In other words, there exists a subset $U \subseteq V$ such that for every triangle $t$ in $G$, $E(U,V \setminus U) \cap t \neq \emptyset$.

Exercise 2. Randomized Rounding (Problem 5.8 from Williamson-Shmoys) (20 pts)

Consider a variation of the maximum satisfiability problem in which all variables occur positively in each clause. Each clause has a nonnegative weight $w_j$ and there is an additional nonnegative weight $v_i \geq 0$ for each boolean variable $x_i$. The goal is now to set the boolean variables to maximize the total weight of the satisfied clauses plus the total weight of variables set to be false.

(a) (5 pts) Give an integer programming formulation for this problem, with 0-1 variables $y_i$ to indicate whether or not $x_i$ is set to true.

(b) (15 pts) Consider the linear programming relaxation of the integer program in part a). Show that a randomized rounding of this linear program in which variable $x_i$ is set to true with probability $1 - \lambda + \lambda y_i^*$ gives a $2(\sqrt{2} - 1)$-approximation algorithm for some appropriate setting of $\lambda$; note that $(2(\sqrt{2} - 1) \approx .828$. In other words, show that this randomized rounding results in a solution whose expected value is at least $2(\sqrt{2} - 1)$ times the value of an optimal solution.
Exercise 3. Traveling Salesman Problem (30 pts)

Let \( G = (V, E) \) be a complete, weighted graph. Each edge \( ij \in E \) has weight \( w_{ij} \geq 0 \) and these weights obey the triangle inequality: \( w_{ij} + w_{jk} \geq w_{ik} \) for all \( i, j, k \in V \). The traveling salesman problem is to find a minimum weight Hamiltonian tour in \( G \). Consider the following integer program:

\[
\begin{align*}
\min & \sum_{ij \in E} w_{ij} x_{ij} \\
\text{subject to} & \sum_{j \in \delta(i)} x_{ij} = 2, \text{ for all } i \in V, \\
& \sum_{ij \in E: i \in S, j \notin S} x_{ij} \geq 2, \text{ for all cuts } S \subset V, \\
& x_{ij} \in \{0, 1\}. 
\end{align*}
\]

\((P_{\text{tsp}})\)

a) (10 pts) Prove that the problem of finding an optimal solution for \((P_{\text{tsp}})\) is equivalent to that of finding a minimum weight Hamiltonian tour in \( G \).

b) (10 pts) Relax the integrality requirement in \((P_{\text{tsp}})\) to obtain the constraint: \( 0 \leq x_{ij} \leq 1 \). Let \( \text{OPT}_f(G) \) denote the optimal value for the resulting linear programming relaxation.

Let \( T \) denote a minimum weight spanning tree in \( G \) and let \( w(\text{MST}(G)) \) denote its weight. Let \( J \subset V \) denote the vertices in \( V \) that have an odd-degree in \( T \). Show that there is a minimum weight perfect matching on \( J \) with value at most \( \text{OPT}_f(G) \).

c) (10 pts) Conclude that the traveling salesman problem has a solution with weight at most \( w(\text{MST}(G)) + \frac{\text{OPT}_f(G)}{2} \).

Exercise 4. Triangle Packing (20 pts + 20 pts extra credit)

Let \( G = (V, E) \) be an unweighted, undirected graph. Let \( T \) be the set of all triangles in \( E \); a triangle is a triple of edges \( \{e_1, e_2, e_3\} \) such that these three edges form a cycle with 3 edges. Our goal is to find a set of edges \( S \subset E \) such that \( S \cap t \neq \emptyset \) for all \( t \in T \) and \( |S| \) is minimized. We call this the triangle hitting set problem. Let \( \text{OPT}(G) \) denote the size of an optimal triangle hitting set.

a) (10 pts) Give a 3-approximation for the triangle hitting set problem.

b) (10 pts) Consider the following linear programming relaxation for the problem:

\[
\begin{align*}
\min & \sum_{e \in E} x_e \\
\text{subject to} & \sum_{e \in t} x_e \geq 1 \text{ for all triangles } t \in T, \\
& x_e \geq 0. 
\end{align*}
\]

\((P_{\text{triangle}})\)

Write the dual linear program.

c) *(10 pts)* Consider an optimal solution \( x^* \) for \((P_{\text{triangle}})\) with value \( \text{OPT}_f(G) \). Suppose \( x^*_e > 0 \) for all \( e \in E \). Show that in this case, \( \text{OPT}(G) \leq 2 \text{OPT}_f(G) \). (Hint: Use complementary slackness to bound \( \text{OPT}_f(G) \) in terms of \(|E|\).)

d) **(10 pts)** Prove that \( \text{OPT}(G) \leq 2 \text{OPT}_f(G) \).