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Final Exam: February 7, 2017

Duration 3 hours. No notes or electronic devices allowed.

Exercise 1. Minimizing Pollution (8 points.)

The inhabitants of Villeurbanne make 1000 trips per day, where a trip consists of one person using one vehicle. Each trip uses a vehicle: a bicycle, a motorcycle, a petrol car or a diesel car. Below is a chart of the units of pollution produced by each vehicle per trip.

	Bicycle	Motorcycle	Petrol Car	Diesel Car
Fine Particles	0	30	60	200
CO_2	0	10	60	15
NOx	0	25	50	100

The mayor of Villeurbanne wants to know what is the minimum amount of Fine Particles units that can be produced per day subject to the following constraints:

- Due to the limited space given to bike lanes, at most 100 trips per day can be made using a bicycle.
- At most 600 units of $C0_2$ can be produced due to EU regulations.
- The total amount of NOx allowed is 7000 units.

a. Write a linear program to model the problem of finding a trip allocation that produces the fewest units of Fine Particle pollution.

Solution: Let x_1, x_2, x_3, x_4 denote the amount of trips made by bicycle, motorcycle, petrol car and diesel car, respectively.

$$\min\ 30x_2 + 60x_3 + 200x_4$$
$$x_1 + x_2 + x_3 + x_4 = 1000$$
$$x_1 \leq 100$$
$$10x_2 + 60x_3 + 15x_4 \leq 600$$
$$25x_2 + 50x_3 + 100x_4 \leq 7000$$

b. Does there exist a distribution of the 1000 trips among vehicles that obeys the constraints and produces less than 100,000 units of Fine Particle pollution? Justify your answer.

Solution: The number of motorcycles allowed in unbounded and motorcycles are less polluting than petrol cars and diesel cars in every category. Thus, we can see that there is no feasible solution if we set $x_1 = 100$ and $x_3 = x_4 = 0$.

Exercise 2. (5 points.)

Polly has written a program that finds a feasible point in a polyhedron or concludes that the polyhedron is empty. In other words, given any polyhedron of the form $P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0\}$, Polly's program returns a point $\mathbf{y} \in P$ or says "P is empty".

Molly would like to use this program to solve the following problem on n variables:

$$\begin{aligned} \min \ & \mathbf{c}^\intercal \mathbf{x} \\ \text{subject to:} \ & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq 0. \end{aligned} \tag{P}$$

Molly knows that (P) has an optimal solution that is a nonnegative integer. Moreover, she can bound the optimal value of her program by the integer U, where $U \ll n$. Each time Molly wants to use Polly's program, she must send Polly an email.

a. Can Molly use Polly's program to solve (P)? (This question assumes Polly's implementation is correct.)

Solution:

Molly can guess an upper or lower bound on the objective value of an optimal solution (she knows that it is an integer between zero and U) and add this guess to her problem as a constraint.

For example, she can add the constraint $\mathbf{c}^\intercal \mathbf{x} \geq t$ to her problem, and if the solution she receives from Polly is feasible (infeasible), she knows that the objective value is actually at least (at most) t. Molly can continue guessing until she finds the optimal value of her program.

b. If so, how many emails does Molly need to send Polly in the worst case?

Solution: Since Molly knows that the optimal value of her program is an integer between zero and U, she can perform a binary search on the optimal value of her program. So Molly needs to send Polly at most $\log U$ emails to find an optimal solution for her problem.

Exercise 3. (5 points.)

Polly implements the simplex algorithm to solve problems in the form of (P). For a particular linear programming problem (P_{olly}), she finds that the optimal solution is 17. She takes the dual of (P_{olly}) to obtain (D_{olly}). She uses the solver to determine that (D_{olly}) is unbounded.

Is Polly's implementation of the simplex algorithm correct? Justify your answer.

Solution: The Strong Duality Theorem states that if the primal is feasible and has a bounded solution, then it is equal to the optimal solution of the dual. So Polly's implementation is incorrect.

Exercise 4. (10 points.)

We are given the following linear program (P_1).

$$\begin{aligned} \text{Maximize} \quad & 3x_1 - x_2 + 2x_3 - 4x_4 \\ \text{subject to:} \quad & 2x_1 + x_2 + 2x_3 - x_4 \leq 2 \\ & -x_2 - x_3 \leq 2 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned} \qquad (\mathrm{P}_1)$$

a. Find an initial basic feasible solution for (P_1)

Solution:

$x_i = 0$ for $i \in \{1, 2, 3, 4\}$ is a basic feasible solution.

b. Let x_5 and x_6 denote the slack variables corresponding to the first and second constraints, respectively. Write the initial dictionary for Phase II of the simplex algorithm.

Solution:

$$\begin{array}{rcl} x_5 & = & 2 - 2x_1 - x_2 - 2x_3 + x_4 \\ x_6 & = & 2 + x_2 + x_3 \\ \hline z & = & 0 + 3x_1 - x_2 + 2x_3 - 4x_4 \end{array}$$

c. Use x_1 as the entering variable and x_5 as the leaving variable to obtain the next dictionary.

Solution:

$$\begin{array}{rcl} x_1 & = & 1 - \frac{1}{2}x_2 - x_3 + \frac{1}{2}x_4 - \frac{1}{2}x_5 \\ x_6 & = & 2 + x_2 + x_3 \\ \hline z & = & 3 - \frac{3}{2}x_5 - \frac{5}{2}x_2 - x_3 - \frac{5}{2}x_4 \end{array}$$

d. What is the optimal basic feasible solution you obtain?

Solution:

$(x_1 = 1,\ x_2 = x_3 = x_4 = 0)$.

e. Certify the optimality of your solution in [d.] by finding a linear sum of constraints that provide a matching upper bound.

Solution:

Using the last dictionary, we multiply the first constraint by $\frac{3}{2}$ to obtain:

$$\begin{array}{rcl} 3x_1 - x_2 + 2x_3 - 4x_4 & \leq & \\ \frac{3}{2}(2x_1 + x_2 + 2x_3 - x_4) & = & \\ 3x_1 + \frac{3}{2}x_2 + 3x_3 - \frac{3}{2}x_4 & \leq & \frac{3}{2} \cdot 2 \ = \ 3. \end{array}$$

Thus, the objective is upper bounded by 3, which is the value of the solution we found. This certifies that our solution is optimal.

Exercise 5. (8 points.)

We are given the following linear program (P_2).

$$
\begin{aligned}
\text{Maximize} \quad & 3x_1 + 2x_2 + x_3 + x_4 \\
\text{subject to:} \quad & 2x_1 + x_2 + x_3 - 2x_4 \le 7 \\
& -x_1 - x_2 + 2x_3 + 2x_4 \le 5 \\
& x_1 + x_2 + 3x_3 - x_4 \le 9 \\
& x_1, x_2, x_3, x_4 \ge 0.
\end{aligned}
\qquad (P_2)
$$

a. Write the dual linear program.

Solution:

$$
\begin{aligned}
\text{Minimize} \quad & 7y_1 + 5y_2 + 9y_3 \\
\text{subject to:} \quad & 2y_1 - y_2 + y_3 \ge 3 \\
& y_1 - y_2 + y_3 \ge 2 \\
& y_1 + 2y_2 + 3y_3 \ge 1 \\
& -2y_1 + 2y_2 - y_3 \ge 1 \\
& y_1, y_2, y_3 \ge 0.
\end{aligned}
\qquad (D_2)
$$

b. Is $x_1 = 12, x_2 = 11, x_3 = 0, x_4 = 14$ an optimal solution for (P_2)? Justify your answer.

Solution:
Using complementary slackness, we can find a solution for the system of equations consisting of the first, second and third (tight) constraints. We obtain ($y_1 = 1,\ y_2 = 4,\ y_3 = 5$). The given solution for (P_2) has objective value 72, and the solution we found has objectiv value 72 for (D_2). Thus, the given solution is optimal.

Exercise 6. (6 points.)

Polly wants to solve the following linear program (P_3).

$$\begin{aligned} \text{Maximize} \quad & x_1 + x_2 + x_3 \\ \text{subject to:} \quad & x_1 + x_2 + x_3 \le 2 \\ & x_1 - 2x_2 + x_3 \le 4 \\ & 2x_1 - x_2 + 2x_3 \le 6 \\ & x_1, x_2, x_3 \ge 0. \end{aligned} \tag{P$_3$}$$

She implements the simplex algorithm and uses it to find the solution for (P_3). She obtains the solution ($x_1 = \frac{2}{3}$, $x_2 = \frac{2}{3}$, $x_3 = \frac{2}{3}$).

Answer True, False or Cannot Determine. Justify your answer.
Polly's implementation of the simplex algorithm is correct on this instance.

Solution: The given point ($x_1 = \frac{2}{3}$, $x_2 = \frac{2}{3}$, $x_3 = \frac{2}{3}$) is not an extreme point for either of two reasons: (i) There are three variables, but there is only one tight constraint, and (ii) Of the three constraints (not including the nonnegativity constraints), only two are linearly independent. Thus, any extreme point of (P_3) contains at most two nonzero values.

So Polly's implementation is incorrect, since the simplex algorithm returns an extreme point.

Exercise 7. Perfect Matchings (8 points.)

Let $G = (V, E)$ be a bipartite graph where each vertex has degree exactly four. Suppose that each edge $e \in E$ has edge weight $w_e \ge 0$. Show that G has a perfect matching with weight at least $W/4$, where $W = \sum_{e \in E} w_e$.

Solution: Let $\mathbf{x} \in \mathbb{R}^{|E|}$ be defined as follows: For all edges $ij \in E$, set $x_{ij} = \frac{1}{4}$. Then $\mathbf{x}$ belongs to the perfect matching polyope on the bipartite graph G, which consists of degree constraints (each vertex has degree 1) and nonnegativity constraints.

Since the perfect matching polytope on a bipartite graph has integer extreme points, there exists an extreme point with value at least that of $\mathbf{x}$ (i.e. at least $W/4$) and this extreme point corresponds to an integral solution, which is a perfect matching.

Exercise 8. (10 points.)

Given a directed graph $G = (V, A)$, a *feedback arc set* of G is a set of arcs $F \subset A$ such that $A \setminus F$ is acyclic, i.e. contains no directed cycles. Consider the following linear programming relaxation for the minimum feedback arc set problem.

$$\begin{aligned} \min \quad & \sum_{ij \in A} x_{ij} \\ \text{subject to:} \quad & \sum_{ij \in C} x_{ij} \geq 1, \quad \text{for all directed cycles } C \text{ in } A, \\ & x_{ij} \geq 0. \end{aligned} \tag{P$_{fas}$}$$

a. Give a polynomial-time separation oracle for (P$_{fas}$).

Solution: For each directed edge ij with value x_{ij}, we use a shortest path algorithm to see if the shortest path from j to i is at least $1 - x_{ij}$. If this is true for all edges, then it follows that $\mathbf{x}$ is a valid solution for (P$_{fas}$) on G. If it is not true for some edge, then we have a violated cycle constraint.

b. Is the integrality gap 1 for the graph in Figure 1? Justify your answer.

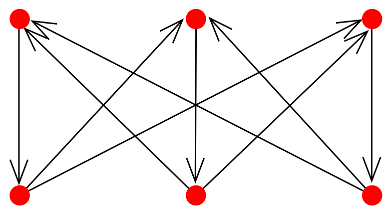

Figure 1: A directed graph with six vertices and nine edges.

Solution: Each down edge can be assigned $x_{ij} = \frac{1}{2}$ and every other edge can have $x_{ij} = 0$. This satisfies all cycle constraints, since every cycle uses at least two down edges. However, an integer solution requires at least two edges. Thus, the integrality gap is at least $\frac{4}{3}$.

c. Is (P$_{fas}$) an integer polytope? Justify your answer.

Solution: (P$_{fas}$) is not an integer polytope for the graph in Figure 1, since there is an extreme point with value at most $\frac{3}{2}$, and such an extreme point cannot be integer.

Exercise 9. Maximum Cut (10 points.)

Polly has an undirected graph $G = (V, E)$ and she wants to find a maximum cut of G. In other words, she wants to find a subset $S \subset V$ such that the number of edges crossing the cut $(S, V \setminus S)$ is maximized. She wants to use the following vector program for the maximum cut problem.

$$\begin{aligned} \max \sum_{(i,j)\in E} & \frac{1 - v_i \cdot v_j}{2} \\ \text{subject to: } & v_i \cdot v_i = 1 \\ & v_i \in \mathbb{R}^n \end{aligned} \qquad (\mathrm{P}_{cut})$$

Polly finds an optimal solution for (P_{cut}) in which for every edge $ij \in E$, it is the case that $v_i \cdot v_j = -\frac{1}{2}$.

a. What is an upper bound on the size of a maximum cut of G in terms of $|E|$? (Give the smallest upper bound you can find.)

Solution: Since (P_{cut}) is a *relaxation* of the maximum cut problem, it provides an upper bound on the size of a maximum cut of G. Each edge contributes $\frac{3}{4}$ to the objective function, so the maximum cut of G has size at most $\frac{3}{4}|E|$.

b. What is a lower bound on the size of a maximum cut of G in terms of $|E|$? (Give the largest lower bound you can find.)

Solution: Using random hyperplane rounding, we will cut edge ij with probability:

$$\begin{aligned} \frac{\arccos(v_i \cdot v_j)}{\pi} &= \frac{2\pi/3}{\pi} \\ &= \frac{2}{3}. \end{aligned}$$

Thus, we conclude that G contains a cut with size at least $\frac{2}{3}|E|$.

Exercise 10. Bin Packing. (10 points.)

Consider an instance of bin packing in which we are given n items that we want to pack into as few unit-capacity bins as possible. Each of the n items has a size in the set $S = \{\frac{1}{4} < s_1 < s_2 < \cdots < s_k < \frac{1}{2}\}$, where $k \leq n$. Each item size s_i has multiplicity b_i, i.e. there are b_i items with size s_i.

A pattern p is a pair or triple of items. A pattern p is *valid* if the sum of the sizes of items in p is at most 1. Let $\mathcal{P}$ denote the set of all valid patterns. For this instance of bin packing, $\mathcal{I}$, let $OPT_{LP}(\mathcal{I})$ denote the optimal value for the following linear program.

$$\begin{aligned} \min \sum_{p \in P} x_p & \\ \sum_{p : s_i \in p} x_p \geq b_i, & \quad \text{for all item sizes } s_i, \\ x_p \geq 0. & \end{aligned} \tag{P$_{bin}$}$$

a. Suppose that k (the number of distinct item sizes) is small compared to n. Let $\mathbf{x}^*$ denote an optimal extreme point of (P$_{bin}$). What is an upper bound on the number of nonzero variables that $\mathbf{x}^*$ contains in terms of k?

Solution: Since there are k constraints that are not nonnegativity constraints, the extreme point $\mathbf{x}^*$ contains at most k nonzero values.

b. Give a polynomial-time algorithm that uses at most $OPT_{LP}(\mathcal{I}) + k$ bins.

Solution: We can consider each $x_p^* > 0$ and round up, i.e. use $\lceil x_p^* \rceil$ bins. Since there are at most k nonzero values, the contribution to the objective function beyond the value of $\mathbf{x}^*$ is at most k.

Exercise 11. 2-Edge Connectivity (10 points.)

Given an undirected graph $G = (V, E)$, the 2-edge connectivity problem is to find a subset of edges $F \subseteq E$ such that F is 2-edge connected and $|F|$ is minimized. Recall that a graph is *2-edge connected* if there does not exist a single edge whose removal disconnects the graphs. For a set $S \subset V$, $\delta(S) \subset E$ denotes the edges with one endpoint in S and the other endpoint in $V \setminus S$.

$$\begin{aligned} & \min \sum_{ij \in E} x_e \\ \text{subject to:} \quad & \sum_{e \in \delta(S)} x_e \geq 2, \text{ for all } S \subset V,\ S \neq \emptyset, \\ & x_e \geq 0. \end{aligned} \qquad (\mathrm{P}_{2-ECSS})$$

Polly solves the linear program (P_{2-ECSS}) for the graph G' shown in Figure 2. She obtains the solution $\mathbf{y} \in \mathbb{R}^{|E|}$ shown.

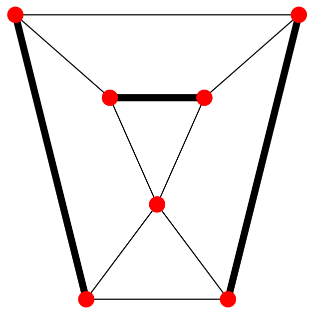

Figure 2: The graph G'. Each thick edge e has $y_e = 1$ and every other edge has $y_e = \frac{1}{2}$.

a. Find a maximal laminar family of tight cuts in G' for the solution $\mathbf{y}$.

Solution: The tight cuts around each of the seven vertices plus each of the four non-singleton cuts shown in Figure 3 form a maximal laminar family of 11 tight cuts.

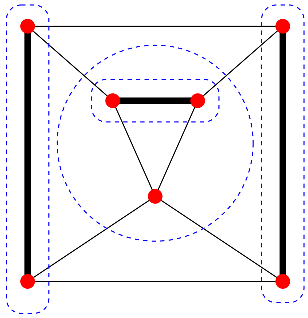

Figure 3: Four tight cuts are shown using dashed blue lines.

b. Is $\mathbf{y}$ a basic feasible solution for (P_{2-ECSS}) on G'? Justify your answer.

Solution:

There are 11 nonzero variables. If the 11 tight constraints in the laminar family are linearly independent, then we can conclude that $\mathbf{y}$ is a basic feasible solution.

We will now show that these 11 tight constraints are indeed linearly independent. Consider the labeling for the edges shown in Figure 4.

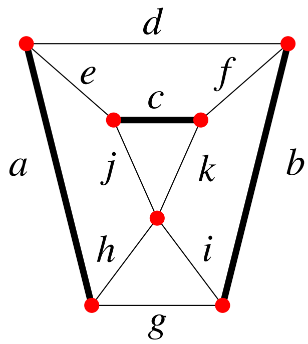

Figure 4: Labels for edges in G'.

Let c_i denote the coefficient of the i^{th} constraint. The 11 tight constraints can be written as:

$$\begin{aligned}
c_1 : \ a + d + e &= 2,\\
c_2 : \ a + g + h &= 2,\\
c_3 : \ b + d + f &= 2,\\
c_4 : \ b + g + i &= 2,\\
c_5 : \ c + f + k &= 2,\\
c_6 : \ c + e + j &= 2,\\
c_7 : \ i + j + h + k &= 2,\\
c_8 : \ e + j + f + k &= 2,\\
c_9 : \ e + f + h + i &= 2,\\
c_{10} : d + f + i + g &= 2,\\
c_{11} : \ d + e + g + h &= 2.
\end{aligned}$$

These 11 constraints are linearly independent if no constraint can be derived from linearly combination of the other constraints. In other words, if there is some vector $\mathbf{c} = \{c_1, c_2, \ldots, c_{11}\}$ such that the corresponding linear combination of the constraints is zero, then $\mathbf{c}$ must be uniformly zero. So, we will now try to show that such a $\mathbf{c} \neq 0$ exists, and obtain a contradiction.

Each variable (i.e. edge in G') yields a constraint on the coefficients. For example, the variable a corresponds to the constraint:

$$c_1 + c_2 = 0 \quad \Rightarrow \quad c_1 = -c_2.$$

Similarly, for the variable b and c, we derive:

$$c_3 = -c_4 \quad \text{and} \quad c_5 = -c_6.$$

To give one more example, for the variable d, we derive:

$$c_1 + c_3 + c_{10} + c_{11} = 0.$$

Solving the resulting system of eleven equations, we can verify that the only solution is when the vector $\{c_1, c_2, \ldots, c_{11}\}$ is uniformly zero.

Exercise 12. Triangle Packing (10 points.)

Let $G = (V, E)$ be an unweighted, undirected graph. Let T be the set of all triangles in E; a triangle is a triple of edges $\{e_1, e_2, e_3\}$ such that these three edges form a directed 3-cycle. The *triangle hitting set* problem is to find a set of edges $S \subseteq E$ such that $S \cap t \neq \emptyset$ for all $t \in T$ and $|S|$ is minimized.

$$\begin{aligned} \min \quad & \sum_{e \in E} x_e \\ \text{subject to:} \quad & \sum_{e \in t} x_e \geq 1, \text{ for all triangles } t \text{ in } T, \\ & x_e \geq 0. \end{aligned} \tag{$\mathrm{P}_{triangle}$}$$

Polly found a 3-approximation for this problem in Homework 2. She would like to find a 2-approximation. Towards this goal, she conjectures that the extreme points for ($\mathrm{P}_{triangle}$) are half-integral. Recall that a point $\mathbf{x} \in \mathbb{R}^{|E|}$ is half-integral if $x_e \in \{0, \frac{1}{2}, 1\}$ for all $e \in E$.

Let G be the graph K_5, i.e. the complete graph on five vertices. Are all the extreme points for ($\mathrm{P}_{triangle}$) on G half-integral? Justify your answer.

Solution 1: Let $x_e = \frac{1}{3}$ for each edge $e \in E$. Then $\mathbf{x}$ is a feasible solution for ($\mathrm{P}_{triangle}$) on G. The value of this solution is $\frac{10}{3}$. This implies that there exists an extreme point of ($\mathrm{P}_{triangle}$) on G with value at most $\frac{10}{3}$.[1] Can such a vertex be half-integral? If so, it must have value at most 3 (i.e. if it has value 3.5, then it would have value greater than that of $\mathbf{x}$). If we can show that there does not exist a feasible half-integral solution with value at most 3, then there must exist a vertex that is not half-integral.

Let us assume such a half-integral vertex $\mathbf{y}$ exists (where the value of $\mathbf{y}$ is at most 3) and consider the following cases. In each case, we will see that $\mathbf{y}$ is not feasible for ($\mathrm{P}_{triangle}$) on G.

1. $\mathbf{y}$ contains six edges with value $y_e = \frac{1}{2}$ and four edges with value $y_e = 0$ (i.e. $\mathbf{y}$ contains no 1-edges). The subgraph consisting of the four edges with value $y_e = 0$ must contain a vertex with degree two and these two edges are in some triangle (because G is a complete graph) whose total value is at most half, so $\mathbf{y}$ is not feasible.

2. $\mathbf{y}$ contains one edge with value $y_e = 1$, four edges with value $y_e = \frac{1}{2}$ and five edges with value $y_e = 0$. Suppose $y_f = 1$ for some $f \in E$. Consider the graph $G' = (V, E \setminus f)$. Let $\mathbf{y}'$ denote $\mathbf{y}$ restricted to edges in G'. Is $\mathbf{y}'$ a feasible solution for ($\mathrm{P}_{triangle}$) on G' ? G' contains a K_4, which requires four half-edges to cover each triangle. Let $v \in V$ be such that the induced graph on $V \setminus v$ is K_4. Then at least one edge adjacent to v must be a half-edge, so we cannot cover all the triangles in G' using only four half-edges.

3. $\mathbf{y}$ contains two edges with value $y_e = 1$, two edges with value $y_e = \frac{1}{2}$ and the rest with value $y_e = 0$. The two 1-edges can either be disjoint

[1] See Theorem 11, Lecture 2. This is also known as the Fundamental Theorem of Linear Programming.

or adjacent. In either case, removing the two 1-edges results in a graph with two edge disjoint triangles that require four half-edges rather than the two available.

4. $\mathbf{y}$ contains three edges with value $y_e = 1$ and $y_e = 0$ for the rest. If we remove all the 1-edges, the resulting graph contains a triangle, whose value is zero.

We can conclude that no feasible half-integral solution has value at most 3. Therefore, there must exist a vertex that is not half-integral.

(This problem was inspired by the solution of Ni Luh Dewi Sintiari to Exercise 4d) on Homework 2.)

Here is an **Alternate Solution** that is nicer than the one above and was found by most students who gave a correct solution.

Solution 2: Define $\mathbf{z}$ as follows: $z_e = \frac{1}{3}$ for all $e \in E$. Then the optimal value of ($\mathrm{P}_{triangle}$) on G is at at most $\frac{10}{3}$. So if there is a half-integral optimal extreme point, it must have value at most 3 (since it must be a multiple of $\frac{1}{2}$ and cannot exceed $\frac{10}{3}$). By the constraints in ($\mathrm{P}_{triangle}$), for each $t \in T$, we have:

$$\sum_{e \in t} x_e \geq 1.$$

By the fact that there are $\binom{5}{3} = 10$ triangles in G, we have:

$$\sum_{t \in T} \sum_{e \in t} x_e \geq 10.$$

Using the fact that each edge is in exactly three triangles, we have:

$$\sum_{t \in T} \sum_{e \in t} x_e = \sum_{e \in E} \sum_{t : e \in t} x_e = \sum_{e \in E} 3 \cdot x_e \geq 10.$$

So,

$$\sum_{e \in E} x_e \geq \frac{10}{3}.$$

So, in particular, an optimal extreme point of ($\mathrm{P}_{triangle}$) on G is not half-integral.

Here is another very nice **Alternate Solution** that was found by Quentin Guilmant and Nicolas Pinson.

Solution 3: Consider the solution $\mathbf{z}$ where $z_e = \frac{1}{3}$. If all extreme points of ($\mathrm{P}_{triangle}$) on G are half-integral, then we must be able to write $\mathbf{z}$ as a convex combination of half-integral extreme points:

$$\mathbf{z} = \sum_{i=1}^{k} \lambda_i \mathbf{x}^i, \quad \text{for} \sum_{i=1}^{k} \lambda_i = 1 \text{ and } \lambda_i > 0.$$

where $\mathbf{x}^i$ are half-integral extreme points. Since for each triangle $t \in$, we have the constraint $\sum_{e \in t} x_e \geq 1$ in ($\mathrm{P}_{triangle}$), and each triangle in $\mathbf{z}$ is tight, we can conclude that for each $t \in T$, $\sum_{e \in t} x_e^i = 1$. (In other words, all triangle constraints are tight for each extreme point $\mathbf{x}^i$.) Now we can show (by case analysis) that there is no half-integral solution for which every triangle is tight! So we have a contradiction.

Remark: Several students observed that there are ten triangles and ten nonzero variables. They then said that the constraints corresponding to these ten triangles are linearly independent, and therefore $\mathbf{z}$ (as defined above) is a basic feasible solution. However, it is not obvious that the ten triangle constraints are in fact linearly independent. So this argument is incomplete.