A. Newman, F. Bréhard, N.K. Le

Second Session Final Exam: May 17, 2017

Duration 2 hours. One cheat sheet allowed.

Exercise 1.

$$
\begin{array}{lrrl}
\text{Maximize} & 3x_1 & +x_2 & \\
\text{Subject to} & x_1 & -x_2 & \leq -1 \\
 & -x_1 & -x_2 & \leq -3 \\
 & 2x_1 & +x_2 & \leq 4 \\
 & x_1, & x_2 & \geq 0.
\end{array}
\qquad (P_1)
$$

1. Solve the linear program (P_1) using the simplex algorithm.
2. Certify the optimality of you solution for (P_1) via a linear sum of constraints.
3. Write the dual of (P_1).
4. Find an optimal solution for the dual.

Exercise 2.

Let $\mathcal{U}$ be a universe of n elements, i.e. $\mathcal{U} = \{e_1, e_2, \ldots, e_n\}$. Let $S_1, S_2, \ldots, S_m$ be m sets, where each $S_j \subseteq \mathcal{U}$. Our goal is to assign a nonnegative weight to each element so that the total weight of the elements in each set is 1.

a. Find an instance where the desired weight assignment does not exist.

b. Write a linear program for this problem.

c. In the case when the desired assignment does not exist, define a certificate that can be used to prove this.

d. Describe a procedure for finding such a certificate.

Exercise 3.

Let $G = (V, E)$ be a cubic, 2-edge connected graph. (Every vertex has degree three, and removing any single edge from G does not disconnect the graph.) A *cycle cover* is a set of cycles $\{C_1, C_2, \ldots, C_k\}$ such that each vertex in V is contained in exactly one cycle.

Suppose each edge $ij \in G$ has weight $w_{ij} > 0$, and let $W = \sum_{ij \in E} w_{ij}$. Show that G has a cycle cover with weight at most $\frac{2W}{3}$.

Exercise 4.

We are given n items, $e_1, e_2, \ldots, e_n$. Each item e_i has a size s_i, a value v_i, and a weight w_i.

1. Write an integer program for the problem of finding a subset of elements such that the total size of the subset is at least S, the total value of the subset is at least V, and the total weight of the subset is at most W.

2. Write a linear program for the relaxed version of this problem in which we are allowed to include fractional amounts of each item in the subset.

3. Give an efficient algorithm to find a feasible subset that contains at most four fractional items.

Exercise 5.

Given a directed graph $G = (V, A)$, the goal is to find a minimum cardinality subset of the edges $S \subset A$ such that S contains at least two edges from each cycle. Consider the following linear programming relaxation:

$$\begin{aligned} &\min \sum_{ij \in A} x_{ij} \\ &\sum_{ij \in C} x_{ij} \geq 2, \quad \text{for all directed cycles } C \in A, \\ &0 \leq x_{ij} \leq 1, \quad \text{for all edges } ij \text{ in } A. \end{aligned} \tag{P_2}$$

Give a polynomial-time separation oracle for (P_2).

Exercise 6.

For any integer $g \in \mathbb{Z}_+$ and any small constant $\epsilon \in (0, \frac{1}{1000})$, there exists an $n \in \mathbb{Z}_+$ such that there is a graph $G = (V, E)$ on $n = |V|$ vertices with girth g and maximum cut at most $(\frac{1}{2} + \epsilon)|E|$ edges.[1]

1. Show that there is no mapping from the vertices of V to unit vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^n$ such that for each edge $ij \in E$, $v_i \cdot v_j \leq -\frac{1}{2}$.

2. Prove that G is not 3-colorable.

3. What is the largest integrality gap you can find for the following relaxation of the maximum cut problem?

$$\begin{aligned} &\max \sum_{ij \in E} x_{ij} \\ &\sum_{ij \in C} x_{ij} \leq |C| - 1, \quad \text{for all odd cycles } C \in E, \\ &0 \leq x_{ij} \leq 1, \quad \text{for all edges } ij \text{ in } E. \end{aligned}$$

[1] S. Poljak. *Polyhedral and eigenvalue approximations of the max-cut problem.*