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# Final Exam: February 7, 2017

## Duration 3 hours. No notes or electronic devices allowed.

**Exercise 1. Minimizing Pollution** (8 points.)

The inhabitants of Villeurbanne make 1000 trips per day, where a trip consists of one person using one vehicle. Each trip uses a vehicle: a bicycle, a motorcycle, a petrol car or a diesel car. Below is a chart of the units of pollution produced by each vehicle per trip.

| | Bicycle | Motorcycle | Petrol Car | Diesel Car |
|---|---|---|---|---|
| Fine Particles | 0 | 30 | 60 | 200 |
| $CO_2$ | 0 | 10 | 60 | 15 |
| NOx | 0 | 25 | 50 | 100 |

The mayor of Villeurbanne wants to know what is the minimum amount of Fine Particles units that can be produced per day subject to the following constraints:

- Due to the limited space given to bike lanes, at most 100 trips per day can be made using a bicycle.
- At most 600 units of $C0_2$ can be produced due to EU regulations.
- The total amount of NOx allowed is 7000 units.

a. Write a linear program to model the problem of finding a trip allocation that produces the fewest units of Fine Particle pollution.

b. Does there exist a distribution of the 1000 trips among vehicles that obeys the constraints and produces less than 100,000 units of Fine Particle pollution? Justify your answer.

**Exercise 2.** (5 points.)

Polly has written a program that finds a feasible point in a polyhedron or concludes that the polyhedron is empty. In other words, given any polyhedron of the form $P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0\}$, Polly's program returns a point $\mathbf{y} \in P$ or says "$P$ is empty".

Molly would like to use this program to solve the following problem on $n$ variables:

$$\begin{aligned} \min \quad & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to:} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq 0. \end{aligned} \tag{P}$$

Molly knows that (P) has an optimal solution that is a nonnegative integer. Moreover, she can bound the optimal value of her program by the integer $U$, where $U \ll n$. Each time Molly wants to use Polly's program, she must send Polly an email.

a. Can Molly use Polly's program to solve (P)? (This question assumes Polly's implementation is correct.)

b. If so, how many emails does Molly need to send Polly in the worst case?

**Exercise 3.** (5 points.)

Polly implements the simplex algorithm to solve problems in the form of (P). For a particular linear programming problem ($\text{P}_{olly}$), she finds that the optimal solution is 17. She takes the dual of ($\text{P}_{olly}$) to obtain ($\text{D}_{olly}$). She uses the solver to determine that ($\text{D}_{olly}$) is unbounded.

Is Polly's implementation of the simplex algorithm correct? Justify your answer.

**Exercise 4.** (10 points.)

We are given the following linear program ($\mathrm{P}_1$).

$$\begin{aligned} \text{Maximize} \quad & 3x_1 - x_2 + 2x_3 - 4x_4 \\ \text{subject to:} \quad & 2x_1 + x_2 + 2x_3 - x_4 \leq 2 \\ & -x_2 - x_3 \leq 2 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned} \tag{$\mathrm{P}_1$}$$

a. Find an initial basic feasible solution for ($\mathrm{P}_1$)

b. Let $x_5$ and $x_6$ denote the slack variables corresponding to the first and second constraints, respectively. Write the initial dictionary for Phase II of the simplex algorithm.

c. Use $x_1$ as the entering variable and $x_5$ as the leaving variable to obtain the next dictionary.

d. What is the optimal basic feasible solution you obtain?

e. Certify the optimality of your solution in [d.] by finding a linear sum of constraints that provide a matching upper bound.

**Exercise 5.** (8 points.)

We are given the following linear program ($P_2$).

$$\begin{aligned} \text{Maximize} \quad & 3x_1 + 2x_2 + x_3 + x_4 \\ \text{subject to:} \quad & 2x_1 + x_2 + x_3 - 2x_4 \leq 7 \\ & -x_1 - x_2 + 2x_3 + 2x_4 \leq 5 \\ & x_1 + x_2 + 3x_3 - x_4 \leq 9 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned} \tag{$P_2$}$$

a. Write the dual linear program.

b. Is $x_1 = 12, x_2 = 11, x_3 = 0, x_4 = 14$ an optimal solution for ($P_2$)? Justify your answer.

**Exercise 6.** (6 points.)

Polly wants to solve the following linear program ($\mathrm{P}_3$).

$$\begin{aligned} \text{Maximize} \quad & x_1 + x_2 + x_3 \\ \text{subject to:} \quad & x_1 + x_2 + x_3 \leq 2 \\ & x_1 - 2x_2 + x_3 \leq 4 \\ & 2x_1 - x_2 + 2x_3 \leq 6 \\ & x_1, x_2, x_3 \geq 0. \end{aligned} \tag{$\mathrm{P}_3$}$$

She implements the simplex algorithm and uses it to find the solution for ($\mathrm{P}_3$). She obtains the solution ($x_1 = \frac{2}{3},\ x_2 = \frac{2}{3},\ x_3 = \frac{2}{3}$).

*Answer True, False or Cannot Determine. Justify your answer.*
**Polly's implementation of the simplex algorithm is correct on this instance.**

**Exercise 7. Perfect Matchings** (8 points.)

Let $G = (V, E)$ be a bipartite graph where each vertex has degree exactly four. Suppose that each edge $e \in E$ has edge weight $w_e \geq 0$. Show that $G$ has a perfect matching with weight at least $W/4$, where $W = \sum_{e \in E} w_e$.

**Exercise 8.** (10 points.)

Given a directed graph $G = (V, A)$, a *feedback arc set* of $G$ is a set of arcs $F \subset A$ such that $A \setminus F$ is acyclic, i.e. contains no directed cycles. Consider the following linear programming relaxation for the minimum feedback arc set problem.

$$\begin{aligned} \min \quad & \sum_{ij \in A} x_{ij} \\ \text{subject to:} \quad & \sum_{ij \in C} x_{ij} \geq 1, \quad \text{for all directed cycles } C \text{ in } A, \\ & x_{ij} \geq 0. \end{aligned} \tag{P$_{fas}$}$$

a. Give a polynomial-time separation oracle for (P$_{fas}$).

b. Is the integrality gap 1 for the graph in Figure 1? Justify your answer.

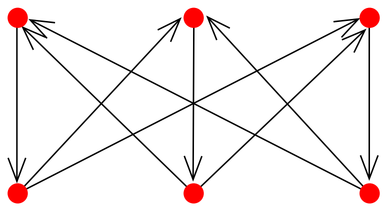

Figure 1: A directed graph with six vertices and nine edges.

c. Is (P$_{fas}$) an integer polytope? Justify your answer.

**Exercise 9. Maximum Cut** (10 points.)

Polly has an undirected graph $G = (V, E)$ and she wants to find a maximum cut of $G$. In other words, she wants to find a subset $S \subset V$ such that the number of edges crossing the cut $(S, V \setminus S)$ is maximized. She wants to use the following vector program for the maximum cut problem.

$$\begin{aligned} \max \sum_{(i,j)\in E} & \frac{1 - v_i \cdot v_j}{2} \\ \text{subject to: } & v_i \cdot v_i = 1 \\ & v_i \in \mathbb{R}^n \end{aligned} \qquad (\mathrm{P}_{cut})$$

Polly finds an optimal solution for $(\mathrm{P}_{cut})$ in which for every edge $ij \in E$, it is the case that $v_i \cdot v_j = -\frac{1}{2}$.

a. What is an upper bound on the size of a maximum cut of $G$ in terms of $|E|$? (Give the smallest upper bound you can find.)

b. What is a lower bound on the size of a maximum cut of $G$ in terms of $|E|$? (Give the largest lower bound you can find.)

**Exercise 10. Bin Packing.** (10 points.)

Consider an instance of bin packing in which we are given $n$ items that we want to pack into as few unit-capacity bins as possible. Each of the $n$ items has a size in the set $S = \{\frac{1}{4} < s_1 < s_2 < \cdots < s_k < \frac{1}{2}\}$, where $k \leq n$. Each item size $s_i$ has multiplicity $b_i$, i.e. there are $b_i$ items with size $s_i$.

A pattern $p$ is a pair or triple of items. A pattern $p$ is *valid* if the sum of the sizes of items in $p$ is at most 1. Let $\mathcal{P}$ denote the set of all valid patterns. For this instance of bin packing, $\mathcal{I}$, let $OPT_{LP}(\mathcal{I})$ denote the optimal value for the following linear program.

$$\begin{aligned} & \min \sum_{p\in P} x_p \\ \sum_{p: s_i \in p} & x_p \geq b_i, \quad \text{for all item sizes } s_i, \\ & x_p \geq 0. \end{aligned} \qquad (\mathrm{P}_{bin})$$

a. Suppose that $k$ (the number of distinct item sizes) is small compared to $n$. Let $\mathbf{x}^*$ denote an optimal extreme point of $(\mathrm{P}_{bin})$. What is an upper bound on the number of nonzero variables that $\mathbf{x}^*$ contains in terms of $k$?

b. Give a polynomial-time algorithm that uses at most $OPT_{LP}(\mathcal{I}) + k$ bins.

**Exercise 11. 2-Edge Connectivity** (10 points.)

Given an undirected graph $G = (V, E)$, the 2-edge connectivity problem is to find a subset of edges $F \subseteq E$ such that $F$ is 2-edge connected and $|F|$ is minimized. Recall that a graph is *2-edge connected* if there does not exist a single edge whose removal disconnects the graphs. For a set $S \subset V$, $\delta(S) \subset E$ denotes the edges with one endpoint in $S$ and the other endpoint in $V \setminus S$.

$$\begin{aligned} \min \quad & \sum_{ij \in E} x_e \\ \text{subject to:} \quad & \sum_{e \in \delta(S)} x_e \geq 2, \;\; \text{for all } S \subset V, \;\; S \neq \emptyset, \\ & x_e \;\; \geq 0. \end{aligned} \tag{$\mathrm{P}_{2-ECSS}$}$$

Polly solves the linear program ($\mathrm{P}_{2-ECSS}$) for the graph $G'$ shown in Figure 2. She obtains the solution $\mathbf{y} \in \mathbb{R}^{|E|}$ shown.

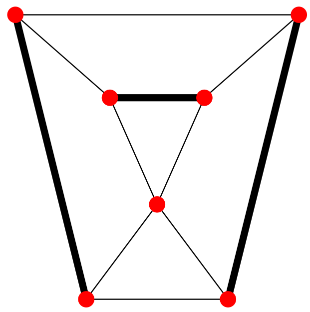

Figure 2: The graph $G'$. Each thick edge $e$ has $y_e = 1$ and every other edge has $y_e = \frac{1}{2}$.

a. Find a maximal laminar family of tight cuts in $G'$ for the solution $\mathbf{y}$.

b. Is $\mathbf{y}$ a basic feasible solution for ($\mathrm{P}_{2-ECSS}$) on $G'$? Justify your answer.

**Exercise 12. Triangle Packing** (10 points.)

Let $G = (V, E)$ be an unweighted, undirected graph. Let $T$ be the set of all triangles in $E$; a triangle is a triple of edges $\{e_1, e_2, e_3\}$ such that these three edges form a directed 3-cycle. The *triangle hitting set* problem is to find a set of edges $S \subseteq E$ such that $S \cap t \neq \emptyset$ for all $t \in T$ and $|S|$ is minimized.

$$\begin{aligned} \min \quad & \sum_{ij \in E} x_e \\ \text{subject to:} \quad & \sum_{e \in t} x_e \geq 1, \ \text{for all triangles } t \text{ in } T, \\ & x_e \ \geq 0. \end{aligned} \tag{P$_{triangle}$}$$

Polly found a 3-approximation for this problem in Homework 2. She would like to find a 2-approximation. Towards this goal, she conjectures that the extreme points for (P$_{triangle}$) are half-integral. Recall that a point $\mathbf{x} \in \mathbb{R}^{|E|}$ is half-integral if $x_e \in \{0, \frac{1}{2}, 1\}$ for all $e \in E$.

Let $G$ be the graph $K_5$, i.e. the complete graph on five vertices. Are all the extreme points for (P$_{triangle}$) on $G$ half-integral? Justify your answer.