TD6: Integer Solutions

Exercise 1. Finding Integer Points in Polytopes

Show that the problem of deciding whether or not a polytope contains an integer point is NP-complete.

Exercise 2. Carathéodory's Theorem

Define the polytope $P \in \mathbb{R}^n$ as the convex hull of points $x_1, x_2, \ldots, x_m \in \mathbb{R}^m$, where $m > n$. Show that any point $y \in P$ can be written as a convex combination of at most $n+1$ extreme points of P.

Exercise 3. Perfect Matching Polytope

Use the perfect matching polytope to show that a d-regular bipartite graph has d perfect matchings.

Exercise 4. Totally Unimodular Matrices

Let $\mathbf{A}$ be an $n \times n$ square matrix. Show that the following statements are equivalent.

(i) $\mathbf{A}$ has determinant ± 1.

(ii) $\mathbf{A}$ has integer entries and its inverse has integer entries.

Exercise 5. Maximum Flows

Recall the linear program ($\mathcal{P}_{max-flow}$) for the maximum flow problem from Lecture 6. Show that if the capacities $\{c_{uv}\}$ are integral, then there is an integral optimal solution. (Hint: Show the incidence matrix is totally unimodular.)

Exercise 6. Transportation Problem

Recall the linear program ($\mathcal{P}_{transport}$) for the transportation problem from Lecture 6. Show that if the values for the supplies and demands $\{d_, s_i\}$ and costs $\{c_{ij}\}$ are integral, then there is an integral optimal solution.

Exercise 7. Shortest Path

The shortest s-t-path problem is defined as follows. Given a graph $G = (V, E)$ with non-negative edge lengths and two designated vertices, s and t, find the minimum length path from s to t. Write a linear program for the shortest s-t-path problem. Does the feasible region correspond to an integer polytope?

Exercise 8. Berge-Fulkerson Conjecture

A famous unsettled conjecture in graph theory is that every bridgeless, cubic graph contains six perfect matchings such that each edge is contained in exactly two of them. We now consider a relaxation of this conjecture.

Let $G = (V, E)$ be a bridgeless, cubic graph. Define $n = |V|$.

1. Show there exists a finte set of perfect matchings $\mathscr{M} = \{M_1, M_2, \ldots, M_k\}$ such that each edge $e \in E$ appears in exactly $\frac{k}{3}$ matchings.

2. What is the smallest value of k for which you can find such a set of perfect matchings? Note that k may be a function of n.