TD4: Linear Programming Duality

Exercise 1.
Let (P) be the following linear program:

$$\begin{array}{lrrrcr} \text{maximize} & -x_1 & -3x_2 & -x_3 & & \\ \text{subject to :} & 2x_1 & -5x_2 & +x_3 & \leq & -5 \\ & 2x_1 & -x_2 & +2x_3 & \leq & 4 \\ & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

a. Solve (P) using the simplex algorithm.

b. Use the dual to prove, if applicable, the optimality of the solution found.

Exercise 2. ***The Case Infeasible/Infeasible.***
Propose a linear program such that neither the primal nor the dual admit a solution.

Exercise 3.
Find the duals for the following linear programs and solve:

a.

$$\begin{array}{lrrl} \text{maximize} & 3x_1 & +x_2 & \\ \text{subject to :} & x_1 & -x_2 & \leq -1 \\ & -x_1 & -x_2 & \leq -3 \\ & 2x_1 & +x_2 & \leq 4 \\ & x_1, & x_2 & \geq 0 \end{array} \qquad \begin{array}{lrrl} \text{maximize} & 3x_1 & +x_2 & \\ \text{subject to :} & x_1 & -x_2 & \leq -1 \\ & -x_1 & -x_2 & \leq -3 \\ & 2x_1 & +x_2 & \leq 2 \\ & x_1, & x_2 & \geq 0 \end{array}$$

$$\begin{array}{lrrl} \text{maximize} & -x_1 & -2x_2 & \\ \text{subject to :} & -3x_1 & +x_2 & \leq -1 \\ & x_1 & -x_2 & \leq 1 \\ & -2x_1 & +7x_2 & \leq 6 \\ & 9x_1 & -4x_2 & \leq 6 \\ & -5x_1 & +2x_2 & \leq -3 \\ & 7x_1 & -3x_2 & \leq 6 \\ & x_1, & x_2 & \geq 0 \end{array}$$

Exercise 4.
Consider the following linear program:

$$\begin{array}{lrrrrcr} \text{maximize} & -3x_1 & +2x_2 & -2x_3 & -x_4 & & \\ \text{subject to :} & 4x_1 & -2x_2 & +x_3 & -x_4 & \leq & 2 \\ & -x_1 & & -x_3 & & \leq & 2 \\ & & & & x_1, x_2, x_3, x_4 & \geq & 0 \end{array} \qquad (P_1)$$

a. Solve (P_1) using the simplex algorithm.

b. Write the dual (D) of the program (P_1).

c. Solve (D) with graphical method. Is the solution found compatible with that of (P_1)?

Exercise 5.

Consider the following linear program:

$$
(P_2)\quad
\begin{array}{lrcrcrcrcr}
\text{minimize} & -2x_1 & - & 3x_2 & - & 2x_3 & - & 3x_4 & & \\
\text{subject to :} & -2x_1 & - & x_2 & - & 3x_3 & - & 2x_4 & \geq & -8 \\
 & 3x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & \leq & 7 \\
 & & & & & & & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}
$$

a. Write (P_2) in canonical form.

b. Write the dual (D) of the program (P_2).

c. Give a graphical solution of the dual (D).

d. Compute the first iteration of the simplex on the program (P_2). After three iterations, the optimal solution found is $x_1 = 0$, $x_2 = 2$, $x_3 = 0$ and $x_4 = 3$.

e. Verify that the solution of (D) obtained for question c) is optimal.

Exercise 6.

In the following problem, the goal is to assign one agent to each task. The objective is to maximize the skills of the agents with respect to their assigned tasks.

Skill	task 1	task 2	task 3	task 4
agent 1	5	3	4	5
agent 2	4	5	4	6
agent 3	5	6	5	7
agent 4	4	4	3	5

The final dictionary obtained when solving the program is :

$x_{13} =$	1	$-a_1$		$+t_2$			$-a_2$	$-x_{23}$		$+t_1$	$-a_3$	$-x_{33}$	$+t_4$		$-a_4$	$-x_{43}$	
$x_{24} =$	1					$-x_{21}$	$-a_2$	$-x_{23}$	$-x_{22}$								
$x_{32} =$	1		$-x_{31}$			$-x_{21}$	$-a_2$	$-x_{23}$	$-x_{22}$		$-a_3$	$-x_{33}$	$+t_4$	$+x_{14}$			$+x_{44}$
$x_{42} =$	0		$+x_{31}$	$-t_2$	$-x_{12}$	$+x_{21}$	$+a_2$	$+x_{23}$			$+a_3$	$+x_{33}$	$-t_4$	$-x_{14}$			$-x_{44}$
$x_{41} =$	1		$-x_{31}$	$+t_2$	$+x_{12}$	$-x_{21}$	$-a_2$	$-x_{23}$			$-a_3$	$-x_{33}$	$+t_4$	$+x_{14}$	$-a_4$	$-x_{43}$	
$x_{11} =$	0			$-t_2$	$-x_{12}$		$+a_2$	$+x_{23}$		$-t_1$	$+a_3$	$+x_{33}$	$-t_4$	$-x_{14}$	$+a_4$	$+x_{43}$	
$t_3 =$	0	$+a_1$		$-t_2$			$+a_2$			$-t_1$	$+a_3$		$-t_4$		$+a_4$		
$x_{34} =$	0					$+x_{21}$	$+a_2$	$+x_{23}$	$+x_{22}$				$-t_4$				$-x_{44}$
$z =$	20	$-4a_1$	$-x_{31}$	$-t_2$	$-2x_{12}$	$-x_{21}$	$-4a_2$	$+0x_{23}$	$+0x_{22}$	$-t_1$	$-5a_3$	$+0x_{33}$	$-2t_4$	$-x_{14}$	$-3a_4$	$+0x_{43}$	$+0x_{44}$

a. From this dictionary, find an optimal solution for this assignment problem.

b. Using the dictionary, find an optimality certificate. Explain this certificate carefully.

Exercise 7.
Consider the following linear program:

$$\begin{array}{lrcrcrcl} \text{maximize} & x_1 & - & 3x_2 & + & 3x_3 & & \\ \text{subject to :} & 2x_1 & - & x_2 & + & x_3 & \leq & 4 \\ & -4x_1 & + & 3x_2 & & & \leq & 2 \\ & 3x_1 & - & 2x_2 & - & x_3 & \leq & 5 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Is the solution $x_1^* = 0$, $x_2^* = 0$, $x_3^* = 4$ optimal?

Exercise 8.
Consider the following linear program:

$$\begin{array}{lrcrcrcrcrcl} \text{maximize} & 7x_1 & + & 6x_2 & + & 5x_3 & - & 2x_4 & + & 3x_5 & & \\ \text{subject to :} & x_1 & + & 3x_2 & + & 5x_3 & - & 2x_4 & + & 2x_5 & \leq & 4 \\ & 4x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & + & x_5 & \leq & 3 \\ & 2x_1 & + & 4x_2 & + & 4x_3 & - & 2x_4 & + & 5x_5 & \leq & 5 \\ & 3x_1 & + & x_2 & + & 2x_3 & - & x_4 & - & 2x_5 & \leq & 1 \\ & & & & & & & & & x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

Is the solution $x_1^* = 0, x_2^* = \frac{4}{3}, x_3^* = \frac{2}{3}, x_4^* = \frac{5}{3}, x_5^* = 0$, optimal?

Exercise 9.
Is the solution $x_1 = 1/7$, $x_2 = 0$, $x_3 = 4/7$, $x_4 = 0$ an optimal solution of the following linear program? Justify your answer.

$$\begin{array}{lrrrrcl} \text{maximize} & 6x_1 & & +8x_3 & +4x_4 & & \\ \text{subject to :} & 7x_1 & +8x_2 & +7x_3 & +2x_4 & \leq & 5 \\ & 4x_1 & +x_2 & +6x_3 & +10x_4 & \leq & 4 \\ & 9x_1 & +5x_2 & +2x_3 & +10x_4 & \leq & 3 \\ & 3x_1 & +10x_2 & +3x_3 & +4x_4 & \leq & 6 \\ & & & & x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

Exercise 10.
Is the solution $x_1 = 2/13$, $x_2 = 0$, $x_3 = 8/13$, $x_4 = 0$ an optimal solution of the following linear program? Justify your answer.

$$\begin{array}{lrrrrcl} \text{maximize} & x_1 & +x_2 & +x_3 & +x_4 & & \\ \text{subject to :} & 5x_1 & +6x_2 & +2x_3 & +3x_4 & \leq & 2 \\ & x_1 & +3x_2 & +3x_3 & +5x_4 & \leq & 2 \\ & 2x_1 & +6x_2 & +4x_3 & +2x_4 & \leq & 3 \\ & 6x_1 & +5x_2 & +4x_3 & +x_4 & \leq & 6 \\ & & & & x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

Exercise 11. ***Morra Game.***
This game consists of two opposing players: Alice and Bob. At every turn, each player hides one or two coins and then tries to guess aloud the number of coins hidden by the other player. If at the end of the turn, only one player correctly guessed, this player receives from the other player as many coins as the total number of hidden coins. In all other cases, the game is null. For example:

- If Alice hides 1 and guess 2 and if Bob hides 2 and guess 1, the game is null.
- If Alice hides 1 and guess 2 and if Bob hides 2 and guess 2, then Bob gives 3 coins to Alice.

The goal of this exercise is to find a mixed strategy for Alice, *i.e.* a probability distribution on the four possible moves that ensure Alice a non-negative expected gain against any strategy of Bob.

a. Model the problem with a linear program.

b. Solve it.

c. Given that Alice plays the previous solution, is it possible for Bob to adapt in order to get a better expected gain than Alice? Is it possible for both players find a probability distribution such that any change over their distribution would dimish their gain? (For more information, see Nash equilibriums and the Lemke-Howson algorithm).

Exercise 12.

In the SET-COVER (SC) problem, there is a universe U of n elements. There is also a family of m subsets $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$, where $S_i \subset U$. Each set S_i is assigned a non-negative weight, w_i. The goal is to find a subset $C \subset \mathcal{S}$ of minimum weight such that $U = \bigcup_{S_i \in C} S_i$.

a. Give a fractional relaxation (R-SC) of SC .

b. Write its dual.

c. When the weight function $w_i = 1$ for all S_i, give interpretations of the integer solutions for the primal and for the dual.