TD3: More on the Simplex Algorithm

Exercise 1.
Write (without solving) an LP deciding if the following polyhedron is:

$$\begin{array}{rrrcr} -x_1 & +2x_2 & +2x_3 & \leq & 4 \\ -3x_1 & +x_2 & -x_3 & \leq & -5 \\ x_1 & -2x_2 & -x_3 & \leq & -1 \\ & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

1. empty?
2. bounded?

Exercise 2.
Solve the following linear programs using the simplex method:

$$\begin{array}{lrrl} \text{maximize} & 3x_1 & +x_2 & \\ \text{subject to :} & x_1 & -x_2 & \leq -1 \\ & -x_1 & -x_2 & \leq -3 \\ & 2x_1 & +x_2 & \leq 4 \\ & x_1, & x_2 & \geq 0 \end{array} \qquad \begin{array}{lrrl} \text{maximize} & 3x_1 & +x_2 & \\ \text{subject to :} & x_1 & -x_2 & \leq -1 \\ & -x_1 & -x_2 & \leq -3 \\ & 2x_1 & +x_2 & \leq 2 \\ & x_1, & x_2 & \geq 0 \end{array}$$

$$\begin{array}{lrrl} \text{maximize} & -x_1 & -2x_2 & \\ \text{subject to :} & -3x_1 & +x_2 & \leq -1 \\ & x_1 & -x_2 & \leq 1 \\ & -2x_1 & +7x_2 & \leq 6 \\ & 9x_1 & -4x_2 & \leq 6 \\ & -5x_1 & +2x_2 & \leq -3 \\ & 7x_1 & -3x_2 & \leq 6 \\ & x_1, & x_2 & \geq 0 \end{array}$$

Exercise 3.
For each possible instantiation of $\mathbf{a}, \mathbf{b} \in \{\textit{entering}; \textit{leaving}\}$ variable, prove or disprove the four following propositions:

A **a** variable in a pivot of the simplex cannot become a **b** variable in the next pivot.

Exercise 4.
Show on an example that choosing the entering variable having highest coefficient in z does not guarantee that the increase of the constant term of z is maximum amongst all possible pivots.

Exercise 5.
Find the necessary and sufficient conditions on the real values a, b and c for the program:

$$\begin{array}{lrcrcl} \text{Maximise} & x_1 & + & x_2 & & \\ \text{Subject to} & ax_1 & + & bx_2 & \leq & c \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

a. has an optimal solution.

b. is infeasible.

c. is unbounded.

Exercise 6.
A mobile phone constructor needs to honour a contract of 20000 phones in the next four weeks. The client will pay 20 € for each delivered before the end of the first week, 18 € before the end of the second week, 16 € before the end of the third week, and 14 € before the end of the fourth week. Each worker can build up to 50 phones per week. The company cannot honour the contract with its 40 employees, such that new temporary employees needs to be hired and trained. Each of the 40 permanent employees can be given the task to train a group of three temporary employees. After a week of training, those having followed the training can either build phone or train untrained employees.

There is at the moment no other contract going on, but each one of the employees, permanent as well as temporary, needs to be paid up to the end of the four week period (even though there are not doing anything).

An employee building phones, being inactive or training other employees is paid 200 € per week whereas an employee following a training is paid 100 € per week. The production cost (excluding the salaries) is 5 € per phone.

For example, the company can adopt the following building program:

Week 1	10 builders, 30 teachers, 90 trainees Actives salaries: 8000 € Trainees salaries: 9000 € Profit on the 500 phones: 7500 € Net loss: 9500 €
Week 2	120 builders, 10 teachers, 30 trainees Actives salaries: 26000 € Trainees salaries : 3000 € Profit on the 6000 phones: 78000 € Net profit: 49000 €
Week 3	160 builders Actives salaries: 32000 € Profit on the 8000 phones: 88000 € Net profit: 56000 €
Week 4	110 builders, 50 inactifs Actives salaries: 32000 € Profit on the 5500 phones: 49500 € Net profit: 17500 €

This planning, for which the company profit is 113000 € , is one of the many possible plannings. The company wishes to have the best possible profit: express this problem with a linear program (non necessarily using a canonical form). Solve it with your simplex algorithm.

Exercise 7. ***Polytope's Vertices.***

Show that if $x = (x_1, \ldots, x_n)$ is the solution associated with a feasible dictionary of a linear program (P), then x obeys the following property:

If $y = (y_1, \ldots, y_n), z = (z_1, \ldots, z_n)$ are solutions of (P) and $x = (y+z)/2$, then $x = y = z$.

Exercise 8.

Write the following linear program in canonical form.

$$\begin{array}{llll} \text{Minimise} & x_1 & -x_2 & \\ \text{Subject to} & x_1 & +x_2 & \geq 5 \\ & 3x_1 & -2x_2 & = 4 \\ & x_1 & & \geq 0 \end{array}$$

Exercise 9.

We are given the following linear program:

$$\begin{array}{llll} \text{Maximise} & 120x_1 & +80x_2 & \\ \text{Subject to} & 2x_1 & +x_2 & \leq 6 \\ & 7x_1 & +8x_2 & \leq 28 \\ & x_1, & x_2 & \geq 0 \end{array}$$

a. Solve it using a graphical method.

b. Solve it using a graphical method with the additional assumption that x_1 and x_2 are integers.

Exercise 10.

Solve the following programs using the simplex method:

$$\begin{array}{lccccccc} \text{Maximise} & 3x_1 & + & 3x_2 & + & 4x_3 & & \\ \text{Subject to} & x_1 & + & x_2 & + & 2x_3 & \leq & 4 \\ & 2x_1 & & & + & 3x_3 & \leq & 5 \\ & 2x_1 & + & x_2 & + & 3x_3 & \leq & 7 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

$$\begin{array}{lccccccccc} \text{Maximise} & 5x_1 & + & 6x_2 & + & 9x_3 & + & 8x_4 & & \\ \text{Subject to} & x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & \leq & 5 \\ & x_1 & + & x_2 & + & 2x_3 & + & 3x_4 & \leq & 3 \\ & & & & & & & x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

$$\begin{array}{lrcrcl}
\text{Maximise} & 2x_1 & + & x_2 & & \\
\text{Subject to} & 2x_1 & + & 3x_2 & \leq & 3 \\
& x_1 & + & 5x_2 & \leq & 1 \\
& 2x_1 & + & x_2 & \leq & 4 \\
& 4x_1 & + & x_2 & \leq & 5 \\
& & & x_1, x_2 & \geq & 0
\end{array}$$

Exercise 11.
Is the polyhedron corresponding to the feasible region of the system

$$\begin{aligned}
-6x_1 + 3x_2 &\leq 7 \\
-3x_2 + 2x_3 &\leq 4 \\
3x_1 - x_3 &\leq 3 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}$$

bounded?

Exercise 12.
Use the simplex method to describe *all* solutions of the following linear program:

$$\begin{array}{lrcrcrcrcl}
\text{Maximise} & 2x_1 & + & 3x_2 & + & 5x_3 & + & 4x_4 & & \\
\text{Subject to} & x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & \leq & 5 \\
& x_1 & + & x_2 & + & 2x_3 & + & 3x_4 & \leq & 3 \\
& & & & & & & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}$$

Exercise 13.
Show that the following system cannot be a feasible dictionary when the variables x_3 and x_4 are the slack variables.

$$\begin{array}{rlll}
x_4 &= 1 & +x_3 & -x_2 \\
x_1 &= 3 & & -2x_2 \\
\hline
z &= 4 & -x_3 & -2x_2
\end{array}$$