TD2: Geometry of LP and the Simplex Algorithm

- Exercise 1 - Lines and Extreme Points.

We say that a polyhedron P *contains a line* if there exists a point $\mathbf{x} \in P$ and a nonzero vector $\mathbf{d} \in \mathbb{R}^n$ such that $\mathbf{x} + \lambda\mathbf{d} \in P$ for all scalars λ.

Prove the following: Let P be a non-empty polyhedron. Show that if P does not contain a line, then P contains an extreme point.

- Exercise 2 - Equivalence of Vertices, Extreme Points and Basic Feasible Solutions.

Let P be a non-empty polyhedron and let $\mathbf{x}^* \in P$. Prove that the following statements are equivalent.

(i) $\mathbf{x}^*$ is a vertex;

(ii) $\mathbf{x}^*$ is an extreme point;

(iii) $\mathbf{x}^*$ is a basic feasible solution.

- Exercise 3 - Basic Solutions and Degeneracy

Consider the standard form polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$, and assume that the rows of the matrix $\mathbf{A}$ are linearly independent. Recall that a point $\mathbf{x} \in \mathbb{R}^n$ is *degenerate* if it is tight for more than n constraints in P.

1. Suppose that two different bases lead to the same basic solution. Show that this basic solution is degenerate.
2. Consider a degenerate basic solution. Is it true that it corresponds to two or more distinct bases? Prove or give a counterexample.
3. Suppose that a basic solution is degenerate. Is it true that there exists an adjacent basic solution which is degenerate? Prove or give a counterexample.

- Exercise 4 - Simple Linear Programs.

Solve the following linear programs using the simplex algorithm:

$$\begin{array}{rcl}
 & \max & 6x_1 + 8x_2 + 5x_3 + 9x_4 \\
\text{subject to: } 2x_1 + x_2 + x_3 + 3x_4 & \leq & 5 \\
x_1 + 3x_2 + x_3 + 2x_4 & \leq & 3 \\
x_1, x_2, x_3, x_4 & \geq & 0.
\end{array}$$

$$\begin{array}{rcl}
 & \max & x_1 + 3x_2 \\
\text{subject to: } -x_1 - x_2 & \leq & -3 \\
-x_1 + x_2 & \leq & -1 \\
x_1 + 2x_2 & \leq & 2 \\
x_1, x_2 & \geq & 0.
\end{array}$$

- Exercise 5 - Batteries.

A battery factory wants to add two new products to its catalog: the Everlast III and the Xeros dry-cell. The Everlast III contains 2g of Cadmium and 4g of Nickel, whereas the Xeros needs 3g of Nickel and 4g of powder Zinc. The total quantity of metal available on the market is 1 ton of Cadmium, and 3 tons for Nickel. Zinc is unlimited and its sputtering is a mere formality. The production of 1000 Everlast III needs 2 hours on a Glunt II press, and the production of 1000 Xeros dry-cells needs 3 hours. The press is available 2400 hours per year. The company expects a profit of 1000 euros per thousand for the Everlast, and 1200 euros per thousand for the Xeros.

a. Translate this into a canonical linear program.

b. Solve the problem using a graphical method.

c. Maximize the gain over a year with the simplex method. Iterate for all possible entering variables for the first pivot.

d. Draw the graph of the evolution of the decision variables for each iteration of the simplex.

e. A study showing the high noxiousness of the Xeros pushes the company to increase the advertisement credits for this product. The profit for the Xeros is then impacted and reduced to 750 € per thousand Xeros. Recalculate the optimum.

- Exercise 6 - Nutritionist.

A nutritionist is in charge of creating a diet using the following ingredients: Eggs, Milk, Cheese and Bread. The compositions (in mg) of those products in Cadmium, Nickel and Zinc are respectively of: Eggs: 6,2,1. Milk: 8,1,3. Cheese: 5,1,1. Bread: 9,3,2. A recent study showed the high noxiousness of Nickel and Zinc, and it is assessed that the daily consumption should not exceed 15mg for the Nickel and 10mg for the Zinc. The study shows however that Cadmium is a trace element that is highly beneficial.

a. Use the simplex method to compute a diet that contains as much Cadmium as possible.

b. Show the unicity of the solution.

c. An error made its way into the report and swapped the roles of the Zinc and the Cadmium (Cadmium being highly toxic). It is also assessed that a diet has to contain at least one unit of bread and at most three of eggs. Recompute an optimal solution.

- Exercise 7 - Thief.

A thief, who has a backpack with a capacity of 60 liters, is confronted with the problem of choosing the objects to steal among seven possibilities. Respective volumes (in liters) and sell prices are given in the following table:

	object 1	object 2	object 3	object 4	object 5	object 6	object 7
volume	20	16	7	10	42	4	12
price	25	18	10	12	50	5	14

a. Solve the problem by hand. Try to show the optimality of your solution.

b. Model your problem with an integer linear program.

c. Solve the linear relaxation of this problem using a greedy algorithm.

d. Solve the linear relaxation using the simplex algorithm.

- Exercise 8 - Power Plants.

Consider three power plants with output 700, 400 and 500 megawatts, respectively. These power plants power two cities, each needing 800 megawatts. Each power plant can provide all or a percentage of its production output to each of the cities.

The costs (per megawatt) to route the power in the electric network are given in the following table:

	City 1	City 2
Power plant 1	20	25
Power plant 2	15	10
Power plant 3	10	15

The problem is to power each city minimizing the cost. Model this with a linear program. (We do not ask to solve the problem.)

- Exercise 9 - Fraction.

We want to maximize the following fraction:

$$\frac{3 + 2x_1 + 3x_2 + x_3}{1 + 3x_1 + x_2 + 4x_3}$$

subject to the constraints $5x_1 + x_2 + 6x_3 \leq 10$ and $x_1 + 2x_2 + x_3 \leq 2$ and non negative x_i.

Show that this problem can be modeled with the following linear program:

$$\begin{array}{lrcrcrcrcl}
\text{Maximize} & 3t & + & 2y_1 & + & 3y_2 & + & y_3 & & \\
\text{Subject to} & t & + & 3y_1 & + & y_2 & + & 4y_3 & = & 1 \\
 & -10t & + & 5y_1 & + & y_2 & + & 6y_3 & \leq & 0 \\
 & -2t & + & y_1 & + & 2y_2 & + & y_3 & \leq & 0 \\
 & & & & & t, y_1, y_2, y_3 & \geq & 0 & &
\end{array}$$

- Exercise 10 - Trucking company.

The daily needs of a trucking company consist of 13 truck drivers on Mondays, 18 on Tuesdays, 21 on Wednesdays, 16 on Thursdays, 12 on Fridays, 25 on Saturdays and 9 on Sundays. What is the smallest number of truck drivers that must be hired to ensure the company needs assuming that a driver works five days straight? Write the linear program.