TD13: Convex Combinations

Exercise 1. Consider the maximum *stable set* problem (also known as the maximum *independent set* problem). Given a graph $G = (V, E)$, find a subset of vertices $S \subseteq V$ such that $|S|$ is maximum and the graph induced on S contains no edges. Consider the following linear programming relaxation of the stable set problem.

$$\begin{aligned} \max \sum_{i \in V} x_i & \\ x_i + x_j \leq 1, & \text{ for all edges } ij \in E, \\ x_i \geq 0, & \text{ for all } i \in V. \end{aligned} \tag{P^0_{stab}}$$

To strengthen this relaxation, we can add the following constraint for every odd cycle:

$$\begin{aligned} \max \sum_{i \in V} x_i & \\ x_i + x_j \leq 1, & \text{ for all edges } ij \in E, \\ \sum_{i \in C} x_i \leq \frac{|C| - 1}{2}, & \text{ for all cycles } C \text{ where } |C| \text{ is odd.} \\ x_i \geq 0, & \text{ for all } i \in V. \end{aligned} \tag{P^1_{stab}}$$

1. What is the integrality gap of (P^0_{stab})?
2. A graph G is called *t-perfect* if the corresponding relaxation (P^1_{stab}) is an integer polytope. Show that if G is t-perfect, then there is a stable set that intersects all triangles in G.
3. An *odd hole* in a graph is subset of vertices $H \subseteq V$ such that $|H| \geq 5$, $|H|$ is odd and the induced subgraph on H is chordless. Suppose that G is t-perfect and G has no odd holes. Show that G is 3-colorable.
4. Here are two conjectures about coloring t-perfect graphs:

   Conjecture 1 *Every t-perfect graph is 4-colorable.*

   Conjecture 2 *Every triangle-free t-perfect graph is 3-colorable.*

   Show that Conjecture 2 implies Conjecture 1.