TD12: Iterative Rounding

Exercise 1. Suppose that $\mathcal{L}$ is a laminar family of sets on the ground set V. Let $S, T \subseteq V$ be two sets such that $S \notin \mathcal{L}$ and $T \in \mathcal{L}$. Suppose that S intersects r sets in $\mathcal{L}$, one of which is T. Show that $S \cap T$ and $S \cup T$ each intersect fewer than r sets in $\mathcal{L}$.

Exercise 2. In the *generalized Steiner network* problem, we are given a graph $G = (V, E)$ where every edge e has a non negative cost c_e and a set of (integer) request values $\{r(u,v) : \{u,v\} \in \binom{V}{2}\}$. The goal is to find a subset $H \subseteq E$ with minimum total cost such that $G' = (V, H)$ satisfies: for every u, v, there are at least $r(u,v)$ edge disjoint uv-paths.

a. Show how the generalized Steiner network problem models some classical problems.

b. Deduce that the generalized Steiner network problem is NP-hard.

c. Recall the linear program P_{GSN} from Lecture 13. Show that there is a polynomial-time separation oracle for P_{GSN}.

d. A function g on 2^V is *weakly supermodular* if $g(\emptyset) = g(V) = 0$ and for all $A, B \subseteq V$, at least one of the following holds:

$$g(A) + g(B) \leq g(A \cup B) + g(A \cap B)$$

$$g(A) + g(B) \leq g(A \setminus B) + g(B \setminus A)$$

In lecture, we defined the function $f(\cdot)$ as follows: $f(S) = \max_{u \in S, v \notin S} r(u,v)$. Show that f is weakly supermodular.

e. Let $H \subseteq E$ and $\delta_H(S)$ be the border function associated to H, i.e. $\delta_H(S) = \delta(S) \cap H$. Show that the function $z(X) = |\delta_H(X)|$ for all $X \subseteq V$ satisfies both of:

$$z(A) + z(B) \geq z(A \cup B) + z(A \cap B)$$

$$z(A) + z(B) \geq z(A \setminus B) + z(B \setminus A)$$

f. Deduce that $f' := f - z$ defined by $f'(X) = f(X) - z(X)$ is weakly supermodular.

g. Show that if $f(\cdot)$ is weakly supermodular, then we can find a maximal laminar family of tight sets corresponding to an extreme point for P_{GSN}. (Show that the constraints tight for $\mathbf{x}$ can be uncrossed.)

h. Use Lemma 11 from Lecture 13 to show that there is a 2-approximation for the generalized Steiner network problem.

Exercise 3. Consider an instance of the 2-edge connected subgraph problem introduced in Lecture 13. From the previous exercise, we may be tempted to think that for *all* nonzero entries x_e in an extreme point $\mathbf{x}$ of (P_{2EC}), it is the case that $\frac{1}{2} \leq x_e \leq 1$. Is this true? Prove or find a counterexample.

Exercise 4. In class, we saw the following linear program for the minimum spanning tree problem.

$$\begin{aligned} \min \sum_{e \in E} c_e x_e & \\ \text{subject to: } \sum_{e \in E(S)} x_e \le |S| - 1, & \text{ for all } S \subset V,\ |S| \ge 2, \\ \sum_{e \in E} x_e = |V| - 1, & \\ x_e \ge 0. & \end{aligned} \tag{P_{MST}}$$

Give a polynomial-time separation oracle for (P_{MST}).

Exercise 5. Here is another candidate linear program for the minimum spanning tree problem.

$$\begin{aligned} \min \sum_{e \in E} c_e x_e & \\ \text{subject to: } \sum_{e \in \delta(S)} x_e \ge 1, & \text{ for all } S \subset V,\ |S| \ne \emptyset, \\ \sum_{e \in E} x_e = |V| - 1, & \\ x_e \ge 0. & \end{aligned} \tag{P_{cut}}$$

a. Give a polynomial-time separation oracle for (P_{cut}).

b. Show that $(P_{MST}) \subseteq (P_{cut})$, i.e. every point $\mathbf{x} \in (P_{MST})$ belongs to (P_{cut}).

c. Is it the case that $(P_{cut}) \subseteq (P_{MST})$?

d. Is (P_{cut}) an integer polytope?

e. What is the integrality gap of (P_{cut})?