Exercise 1. Suppose that $\mathcal{L}$ is a laminar family of sets on the ground set $V$. Let $S, T \subseteq V$ be two sets such that $S \not\in \mathcal{L}$ and $T \in \mathcal{L}$. Suppose that $S$ intersects $r$ sets in $\mathcal{L}$, one of which is $T$. Show that $S \cap T$ and $S \cup T$ each intersect fewer than $r$ sets in $\mathcal{L}$.

Exercise 2. In the generalized Steiner network problem, we are given a graph $G = (V,E)$ where every edge $e$ has a non-negative cost $c_e$ and a set of (integer) request values $\{r(u,v) : \{u,v\} \in (V)^2\}$. The goal is to find a subset $H \subseteq E$ with minimum total cost such that $G' = (V,H)$ satisfies: for every $u,v$, there are at least $r(u,v)$ edge disjoint $uv$-paths.

a. Show how the generalized Steiner network problem models some classical problems.

b. Deduce that the generalized Steiner network problem is NP-hard.

c. Recall the linear program $P_{GSN}$ from Lecture 13. Show that there is a polynomial-time separation oracle for $P_{GSN}$.

d. A function $g$ on $2^V$ is weakly supermodular if $g(\emptyset) = g(V) = 0$ and for all $A, B \subseteq V$, at least one of the following holds:

$$g(A) + g(B) \leq g(A \cup B) + g(A \cap B)$$

$$g(A) + g(B) \leq g(A \setminus B) + g(B \setminus A)$$

In lecture, we defined the function $f(\cdot)$ as follows: $f(S) = \max_{u \in S, v \notin S} r(u,v)$. Show that $f$ is weakly supermodular.

e. Let $H \subseteq E$ and $\delta_H(S)$ be the border function associated to $H$, i.e. $\delta_H(S) = \delta(S) \cap H$. Show that the function $z(X) = |\delta_H(X)|$ for all $X \subseteq V$ satisfies both of:

$$z(A) + z(B) \geq z(A \cup B) + z(A \cap B)$$

$$z(A) + z(B) \geq z(A \setminus B) + z(B \setminus A)$$

f. Deduce that $f' := f - z$ defined by $f'(X) = f(X) - z(X)$ is weakly supermodular.

g. Show that if $f(\cdot)$ is weakly supermodular, then we can find a maximal laminar family of tight sets corresponding to an extreme point for $P_{GSN}$. (Show that the constraints tight for $x$ can be uncrossed.)

h. Use Lemma 11 from Lecture 13 to show that there is a 2-approximation for the generalized Steiner network problem.

Exercise 3. Consider an instance of the 2-edge connected subgraph problem introduced in Lecture 13. From the previous exercise, we may be tempted to think that for all nonzero entries $x_e$ in an extreme point $x$ of $(P_{2EC})$, it is the case that $\frac{1}{2} \leq x_e \leq 1$. Is this true? Prove or find a counterexample.
Exercise 4. In class, we saw the following linear program for the minimum spanning tree problem.

\[
\begin{align*}
\text{min} & \quad \sum_{e \in E} c_e x_e \\
\text{subject to:} & \quad \sum_{e \in E(S)} x_e \leq |S| - 1, \text{ for all } S \subset V, |S| \geq 2, \\
& \quad \sum_{e \in E} x_e = |V| - 1, \\
& \quad x_e \geq 0.
\end{align*}
\]

\( (P_{\text{MST}}) \)

Give a polynomial-time separation oracle for \( (P_{\text{MST}}) \).

Exercise 5. Here is another candidate linear program for the minimum spanning tree problem.

\[
\begin{align*}
\text{min} & \quad \sum_{e \in E} c_e x_e \\
\text{subject to:} & \quad \sum_{e \in \delta(S)} x_e \geq 1, \text{ for all } S \subset V, |S| \neq \emptyset, \\
& \quad \sum_{e \in E} x_e = |V| - 1, \\
& \quad x_e \geq 0.
\end{align*}
\]

\( (P_{\text{cut}}) \)

a. Give a polynomial-time separation oracle for \( (P_{\text{cut}}) \).

b. Show that \( (P_{\text{MST}}) \subseteq (P_{\text{cut}}) \), i.e. every point \( x \in (P_{\text{MST}}) \) belongs to \( (P_{\text{cut}}) \).

c. Is it the case that \( (P_{\text{cut}}) \subseteq (P_{\text{MST}}) \)?

d. Is \( (P_{\text{cut}}) \) an integer polytope?

e. What is the integrality gap of \( (P_{\text{cut}}) \)?