TD11: Algorithmic Applications of Extreme Point Structure

Exercise 1. Back to the Simplex Algorithm
Consider the following linear program.

$$\begin{aligned} \max \quad & 6x_1 + 8x_2 + 5x_3 + 9x_4 \\ \text{subject to: } & 2x_1 + x_2 + x_3 + 3x_4 \leq 5 \\ & x_1 + 3x_2 + x_3 + 2x_4 \leq 3 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned} \tag{P}$$

a. Suppose you run the simplex algorithm on (P) and obtain an optimal solution $\mathbf{x}^*$. How many nonzero values does $\mathbf{x}^*$ contain?

b. If we add the constraint $x_1 \leq 1$, how many nonzero values can an optimal solution $\mathbf{x}^*$ for the simplex algorithm have?

Exercise 2. Cycle Covers via Iterative Rounding
Let $G = (V, E)$ be a bipartite graph where the vertex set V can be paritioned into two sets A and B such that all edges go between A and B. Our goal is to find a set of cycles such that each vertex is included in exactly one cycle. Consider the following linear program:

$$\begin{aligned} \sum_{j:ij \in E} x_{ij} &= 2 \text{ for all } i \in A, \\ \sum_{i:ij \in E} x_{ij} &= 2 \text{ for all } j \in B, \\ 0 \leq x_{ij} &\leq 1 \text{ for all } ij \in E. \end{aligned} \tag{P$_{cover}$}$$

The above linear program is actually totally unimodular. Suppose, however, that we do not know this (but we do know that the extreme points of the matching polytope on a bipartite graph are integral). We would like to nevertheless use the relaxation (P$_{cover}$) to find a cycle cover. In this exercise, we will use a technique known as *iterative rounding*, in which we find a variable x_{ij} with high value, include it in the solution, and iterate.

Suppose that the given graph G does have a cycle cover, i.e. the above linear program is feasible for G. Let $\mathbf{x}^*$ denote an extreme point of (P$_{cover}$).

a. How many fractional entries does $\mathbf{x}^*$ have?

b. Show that $\mathbf{x}^*$ must contain 1-edge, i.e. an edge ij such that $x^*_{ij} = 1$.

c. Show that if there is a 1-edge, we can add it so a (initially emtpy) solution set S and modify the linear program so that the set S plus an integral solution for this modified linear program is a cycle cover.

d. Show that as long as the linear program has a constraint where the right-hand-side value of a constraint is 2, an extreme point solution contains a 1-edge.

e. Conclude that we can find an integer solution for the cycle cover problem. (Hint: Use the fact that an extreme point for the matching polytope in bipartite graphs is integral.)

Exercise 3. Bin Packing

Given and instance I consisting of n items with sizes $s_1 \leq s_2 \leq \cdots \leq s_n \leq 1$, the *bin packing* problem is to find the minimum number of unit capacity bins required to pack all of the items. Consider the following linear programming relaxation:

$$
\begin{aligned}
\sum_{i=1}^{n} s_i \cdot x_{ij} &\leq 1 \text{ for all } j : 1 \leq j \leq B, \\
\sum_{j=1}^{B} x_{ij} &= 1 \text{ for all } i : 1 \leq i \leq n, \\
0 \leq x_{ij} &\leq 1.
\end{aligned} \tag{Q}
$$

a. Show that the linear program (Q) is feasible if the input instance has a packing using at most B bins.

b. Argue that you can find the minimum value of B for which (Q) is feasible. Call this value B^*.

c. Show that $B^* = \lceil SIZE(I) \rceil$. (Recall that $SIZE(I) = \sum_{i=1}^{n} s_i$, where instance $I = \{s_1, s_2, \ldots, s_n\}$.)

d. What is the integrality gap of (Q)? (Find an instance with a large gap between B^* and the optimal number of bins.)

e. What is the maximum number of items that can be packed in B^* bins? Give a lower bound in terms of n and B^*.

f. Use an optimal solution to (Q) to obtain a c-approximation algorithm for c small as possible.

Exercise 4. Bin Packing with Few Items Sizes

Suppose we are given an instance of bin packing with n items such that each item has a size from the set set $\{s_1, s_2, \ldots, s_{10}\}$, i.e. there are only ten item sizes. Give an algorithm to find a bin packing that uses at most $OPT + C \cdot 10$ for some constant $C > 0$.