TD10: Graph Coloring

Exercise 1. Max-Cut on 3-Vector Colorable Graphs

Let $G = (V, E)$ be a graph. We say G is *3-colorable* if there is a legal coloring of V with at most three colors. We say that G is *3-vector colorable* if there is a a feasible solution for the vector program P_{3color} from Lecture 11. Recall that the maximum cut of any graph has size at least $\frac{|E|}{2}$.

1. Assume that G is a 3-colorable graph. Give a lower bound on the size of the maximum cut of G that is strictly larger than half of the edges. What is the largest lower bound you can find?
2. Assume that G is a 3-vector colorable graph. Give a lower bound on the size of the maximum cut of G that is strictly larger than half of the edges. What is the largest lower bound you can find?

Exercise 2. Vector Chromatic Number

Let $G = (V, E)$ be a connected graph and $k > 1$ be an integer. We consider the vector program which assigns a unit vector v_i to every vertex i in V. The constraints are that $v_i \cdot v_j \leq \frac{-1}{k-1}$ for every edge $ij \in E$. The minimum integer $k > 1$ for which a solution exists is called the *vector chromatic number* of G. We denote it by $\chi_v(G)$.

a. Describe the graphs with $\chi_v(G) = 2$.

b. Show that $\chi_v(G)$ is lower bounded by the (NP-hard) clique number $\omega(G)$ of G.

c. Show that $\chi_v(G)$ is upper bounded by the usual (NP-hard) chromatic number $\chi(G)$ of G.

d. Recall that vector programs can be approximated to any fixed precision in polynomial time. Deduce that the chromatic number of a graph G satisfying $\omega(G) = \chi(G)$ can be computed in polynomial time.

e. Given a partial order P on n elements, we can construct an associated *comparability graph* as follows. For each element i in P, create a vertex x_i. Add an edge between vertices x_i and x_j iff the corresponding elements i and j are comparable in P.

Show that comparability graphs have the property $\omega(G) = \chi(G)$. Show also that complements of comparability graphs also have this property.

f. Provide examples showing that for general graphs $\chi(G)$ cannot be bounded in terms of $\chi_v(G)$.

Exercise 3. Correlation of Normally Distributed Random Vectors

We are told the following pieces of information:

- Vectors $v_i, v_j \in \mathbb{R}^n$ are unit vectors and $v_i \cdot v_j = .94$.
- Vector $r = (r_1, r_2, \ldots, r_n)$ is a random vector such that each $r_i \sim \mathcal{N}(0, 1)$.
- $r \cdot v_i = .98$.

a. What is the probability that $r \cdot v_j \geq .98$?

b. What is the probability that $r \cdot v_j \leq -.3$?

Exercise 4. Coloring Triangle-Free Graphs with Few Colors

Let $G = (V, E)$ be an undirected graph and let $n = |V|$. We say G is *triangle-free* if there is no set of three edges in E that forms a triangle (i.e. K_3). Give an algorithm to color a triangle-free graph with $O(\sqrt{n})$ colors.

Exercise 5. Cutting Cliques

For a graph $G = (V, E)$, let k be the vector chromatic number of G (as defined in the previous exercise). Show that there is partition of the vertices V into $(S, V \setminus S)$ such that neither S nor $V \setminus S$ contains a clique of size k.

Exercise 6. Integrality Gap

Find an example of a graph that is 3-vector colorable but not 3-colorable. (*Hard*)