TD1: Modeling Problems with Linear Programming

- Exercise 1 - Lumberjack. A lumberjack has 100 hectares of hardwoods. Cutting an hectare of wood and letting the zone regenerate naturally cost 10k€ per hectare and the benefit is 50k€ . Alternatively, cutting an hectare of wood and re-seeding with pine-woods costs 50 k€ per hectare, and the benefit is 120 k€ . Assuming that the lumberjack has only 4000 k€ in funds in the beginning of the operation, find the best strategy to adopt and the benefit associated.

- Exercise 2 - Taxis. A taxi company has four free vehicles and has to transport four clients. The goal of the company is to assign one taxi per client while minimizing the sum of travelled distances. The respective distances (in kilometers) between taxis and clients are given in the following table:

distance	client 1	client 2	client 3	client 4
taxi 1	6	3	4	5
taxi 2	4	5	4	6
taxi 3	5	6	6	7
taxi 4	4	4	3	5

a. Model the problem with a canonical linear program with variables in $\mathbb{Z}$.

- Exercise 3 - Objective function. Suppose (P) is a program with variables x_1, x_2, x_3 and subject to linear constraints but with non linear objective function z. In the following cases, explain how to modify (P) to get a linear program (by possibly adding new variables and constraints).

a. When $z =$ Maximize (Minimum$(x_1 + x_2, x_2 + x_3)$).

b. When $z =$ Minimize $|x_1 - x_2 + x_3|$, where $|.|$ is the absolute value.

c. When $z =$ Minimize $|x_1| + |x_2| + |x_3|$.

d. When $z =$ Minimize $\text{pos}(x_1) + \text{pos}(x_2) + \text{pos}(x_3)$, where $\text{pos}(x)$ equals x if $x \geq 0$, 0 otherwise.

e. What is the complexity of maximizing a sum of absolute values?

- Exercise 4 - Boxes. A company with a stock of 10 000 m^2 of cardboard, builds and sells 2 kinds of cardboard boxes. Building a cardboard box of type 1 or 2 requires, respectively 1 and 2 m^2 of cardboard, and needs 2 and 3 minutes to be assembled. Only 200 hours of work are available next week. The boxes are stapled and the second kind of box needs four times as much staples. The staple stock allows to build at most 15 000 boxes of the first kind. Boxes are sold, respectively, 3€ and 5€ .

a. Write the problem of searching the production plan maximizing the company profit with a canonical linear program.

b. Determine an optimal production plan from the linear program written in a with a graphical method.

- Exercise 5 - Bus schedule. The following table contains the possible time slots for the bus drivers of a bus company. The company wants to determine the time slots to keep in order to ensure, with a minimal cost, that at any hour of the day (from 9:00 until 17:00) at least one driver is present.

Time slot	9 à 11h	9 à 13h	11 à 16h	12 à 15h	13 à 16h	14 à 17h	16 à 17h
Cost	18	30	38	14	22	16	9

Write a linear program with integers that can be used to solve the decision problem of the bus company.

- Exercise 6 - Bike and Run. In the bicycle problem, n people have to travel d km and can use only one bicycle (with at most one person on the bicycle). Walking speed w_j and bike speed b_j are given for any person j. The problem is to minimize the arrival time of the last person to arrive to its destination. The bicycle can be left for the next person.

a. Solve by hand the following case: $d = 100, n = 3, w_1 = 2, w_2 = w_3 = 1, b_1 = 8, b_2 = b_3 = 6$.

b. Write a linear program giving a lower bound on the optimum of this problem.

- Exercise 7 - Desks. A school director wants to change all desks in all classrooms. He managed to obtain tabletops at a good price and now has to buy metal rods to build the legs. The company offers him cheap rods of 2.10 meters.

Depending on the age of the kids (and especially on their height!), the height of the desks varies. The director wants to build 60 small desks (height: 50cm), 80 medium desks (height: 80cm) and 65 high desks (height: 1.10m). Hence, to build a small desk he has to cut 4 rods of 50cm.

The director wonders how to cut the rods of 2.10m to obtain the required number of legs minimizing the scrap metal. Translate this problem into a linear problem.

Three additional exercises (Taken from Vanderbei's book *Linear programming: Foundations and Extensions.*)

- Exercise 8 -

A steel company must decide how to allocate next week's time on a rolling mill, which is a machine that takes unfinished slabs of steel as input and can produce either of two semi-finished products: bands and coils. The mill's two products come off the rolling line at different rates:

Bands	200 tons/hr
Coils	140 tons/hr.

They also produce different profits:

Bands	\$ 25/ton
Coils	\$ 30/ton.

Based on currently booked orders, the following upper bounds are placed on the amount of each product to produce:

Bands	6000 tons
Coils	4000 tons

Given that there are 40 hours of production time available this week, the problem is to decide how many tons of bands and how many tons of coils should be produced to yield the greatest profit. Formulate this problem as a linear programming problem. Can you solve this problem by inspection?

- Exercise 9 -

A small airline, Ivy Air, flies between three cities: Ithaca, Newark and Boston. They offer several flights but, for this problem, let us focus on the Friday afternoon flight that departs from Ithaca, stops in Newark, and continues to Boston. There are three types of passengers:

a Those travelling from Ithaca to Newark.

b Those travelling from Newark to Boston.

c Those travelling from Ithaca to Boston.

The aircraft is a small commuter plane that seats 30 passengers. The airline offers three fare classes:

a Y class: full coach.

b B class: nonrefundable.

c M class: nonrefundable, 3-week advanced purchase.

Tickets prices, which are largely determined by external influences (i.e., competitors), have been set and advertised as follows:

	Ithaca-Newark	Newark-Boston	Ithaca-Boston
Y	300	160	360
B	220	130	280
M	100	80	140

Based on past experience, demand forecasters at Ivy Air have determined the following upper bounds on the number of potential customers in each of the 9 possible origin-destination/fare-class combinations:

	Ithaca-Newark	Newark-Boston	Ithaca-Boston
Y	4	8	3
B	8	13	10
M	22	20	18

The goal is to decide how many tickets from each of the 9 origin/destination/fare-class combinations to sell. The constraints are that the place cannot be overbooked on either of the two legs of the flight and that the number of tickets made available cannot exceed the forecasted maximum demand. The objective is to maximize the revenue. Formulate this problem as a linear programming problem.

- Exercise 10 -

Suppose that Y is a random variable taking on one of n known values:

$$a_1, a_2, \dots, a_n$$

Suppose we know that Y either has distribution p given by

$$\mathbb{P}(Y = a_j) = p_j$$

or it has distribution q given by

$$\mathbb{P}(Y = a_j) = q_j.$$

Of course, the numbers p_j, $j = 1, 2, \ldots, n$ are nonnegative and sum to one. The same is true for the q_j's. Based on a single observation of Y, we wish to guess whether it has distribution p or distribution q. That is, for each possible outcome a_j, we will assert with probability x_j that the distribution is p and with probability $1-x_j$ that the distribution is q. We wish to determine the probabilities x_j, $j = 1, 2, \ldots, n$ such that probability of saying the distribution is p when in fact it is q has probability no larger than β, where β is some small positive value (such as 0.05). Furthermore, given this constraint, we wish to maximize the probability that we say the distribution is p when in fact it is p. Formulate this maximization problem as a linear programming problem.