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| **Optimization and Approximation** | ENS Lyon |
| Lecture 6 | |
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# 1 Integer Solutions

Suppose we have a linear program and we wish to find an optimal *integer* solution. In other words, suppose we have the following optimization problem:

$$\begin{array}{lrcl} \min & \mathbf{c}'\mathbf{x} & & \\ \text{subject to:} & \mathbf{A}\mathbf{x} & \geq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \text{ and integer.} \end{array}$$

If we could solve the above integer program, we could solve the following problem. Let $G = (V, E)$ and for each $v_i \in V$, let $x_i$ denote a variable with an integer value.

$$\begin{array}{rll} \min & \sum_{i \in V} x_i & \\ \text{subject to:} & x_i + x_j \geq 1, & \text{for } (i,j) \in E, \\ & x_1, \dots, x_n \geq 0, & \text{and integer.} \end{array}$$

The above integer program models the minimum vertex cover problem, which is NP-hard. Thus, it is NP-hard to solve the first (more generally phrased) integer program. In fact, the problem of deciding if a polyhedron contains an integer point is NP-complete. So, we cannot expect to find a general method to solve an integer program. However, sometimes finding integer solutions using a linear program is possible, e.g. when we can efficiently describe an integer polytope for a problem. An integer polytope is defined to be a polytope for which all of its extreme points are integral. An example of such a problem is finding a maximum matching in bipartite graphs.

## 1.1 Extreme Points of Integer Polytopes

We have defined an extreme point of a polytope $P$ to be a point $\mathbf{x} \in P$ that cannot be written as a convex combination of two other points in $P$. If $\mathbf{x}$ is *not* an extreme point in $P$, then $\mathbf{x}$ *can* be written as a convex combination of other points in $P$. In particular, since $P$ is the convex hull of its vertices, $\mathbf{x}$ can be written as a convex combination of the extreme points of $P$. More precisely, suppose $\mathbf{x}_1^*, \mathbf{x}_2^*, \dots, \mathbf{x}_d^*$ are the extreme points of $P$. Then there exists $\lambda_i \in [0, 1)$ for $i \in \{1, 2, \dots, d\}$ such that:

$$\sum_{i=1}^{d} \lambda_i = 1, \quad \mathbf{x} = \sum_{i=1}^{d} \lambda_i \mathbf{x}_i^*.$$

Carathéodory's Theorem states that $d \leq n + 1$ when $P \in \mathbb{R}^n$.

An integer polytope is a polytope that has integer (and only integer) extreme points. If $P$ is an integer polytope, then any point in $P$ can be decomposed into a convex combination of integer solutions. This seemingly simple statement has powerful consequences. Suppose we would like to prove that a graph $G$ has a certain combinatorial property (e.g. $G$ has a matching with at least $k$ edges; $G$ has a minimum cut containing at most $k$ edges). If we can specify a set of inequalities that describe the convex hull of corresponding integer solutions, then it suffices to show that the polytope contains a point—not necessarily integral—for which the property holds fractionally to prove that $G$ has the (integral) property.

## 2 Matching Polytope in Bipartite Graphs

Given a bipartite graph $G = (V, E)$ with edge weights $w_e$, let us consider the problem of finding a maximum weight *matching*. A matching is set of edges $M \subseteq E$ such that each vertex $V$ is adjacent to at most one edge in $M$. If each vertex in $V$ is adjacent to exactly one edge in $M$, then $M$ is a *perfect matching*. For each matching $M$ in $G$, let $\chi^M \in \{0,1\}^{|E|}$ be the indicator vector of the matching. Let $\mathcal{M}(G)$ denote the convex hull of all such indicator vectors of matchings in $G$.

$$\mathcal{M}(G) \quad = \quad \text{convex-hull}\{\chi^M \mid M \text{ is a matching}\}.$$

Consider the following polyhedron consisting of valid matching constraints:

$$Q^{\mathrm{M}}_{\mathrm{BIP}}(G) \quad = \quad \{\mathbf{x} \mid \sum_{e \in \delta(v)} x_e \leq 1, \forall v \in V;\ x_e \geq 0, \forall e \in E\}.$$

We observe that for any graph $G$ containing at least one edge, $Q^{\mathrm{M}}_{\mathrm{BIP}}(G)$ is a *bounded* polyhedron (since it does not contain a line) and is therefore a polytope.

**Lemma 1.** $\mathcal{M}(G) \subseteq Q^{M}_{\mathrm{BIP}}(G)$.

*Proof.* Each indicator vector of a matching obeys the constraints in $Q^{\mathrm{M}}_{\mathrm{BIP}}(G)$, since the constraints in $Q^{\mathrm{M}}_{\mathrm{BIP}}(G)$ are valid for any matching in $G$. □

**Lemma 2.** *If $G$ is bipartite, then $Q^{M}_{\mathrm{BIP}}(G) \subseteq \mathcal{M}(G)$.*

*Proof.* Consider the smallest graph $G$ (in terms of $|V| + |E|$) such that $Q^{\mathrm{M}}_{\mathrm{BIP}}(G)$ has a vertex that is not a matching, i.e. is not integral. Consider such a vertex $\mathbf{x}^\star \in Q^{\mathrm{M}}_{\mathrm{BIP}}(G)$. Note that $\mathbf{x}^\star$ has no integer entries, as that would allow us to delete the corresponding edges from $G$ and consider an even smaller graph. Now let us consider two cases:

(i) Suppose that $G$ contains a cycle $C$. Since $G$ is bipartite, $C$ is an even cycle. Define:

1. $\alpha = \min_{e \in C} x^\star_e$,
2. $\beta = \max_{e \in C} x^\star_e$.

Let $\epsilon = \min\{\alpha, 1 - \beta\}$ and let $\mathbf{z} = \{1, -1, 1, \ldots, -1\}$, where $\mathbf{z}$ has length $|C|$. Then we obtain the following two vectors:

$$\mathbf{x}' = \mathbf{x}^\star + \epsilon\mathbf{z}, \qquad \mathbf{x}'' = \mathbf{x}^\star - \epsilon\mathbf{z}.$$

Both $\mathbf{x}'$ and $\mathbf{x}''$ satisfy the constraints in $Q^{\mathrm{M}}_{\mathrm{BIP}}(G)$, so $\mathbf{x}', \mathbf{x}'' \in Q^{\mathrm{M}}_{\mathrm{BIP}}(G)$. Since $\mathbf{x}^\star = (\mathbf{x}' + \mathbf{x}'')/2$, it follows that $\mathbf{x}^\star$ is not an extreme point of $Q^{\mathrm{M}}_{\mathrm{BIP}}(G)$, which is a contradiction.

(ii) Suppose $G$ does not contain a cycle. Then it contains a path $P$. Define:

1. $\alpha = \min_{e \in S} x^\star_e$,
2. $\beta = \max_{e \in S} x^\star_e$.

Let $\epsilon = \min\{\alpha, 1 - \beta\}$ and let $\mathbf{z} = \{1, -1, 1, \ldots, -1\}$, where $\mathbf{z}$ has length $|P|$. As in case (i), we obtain the following two vectors:

$$\mathbf{x}' = \mathbf{x}^\star + \epsilon\mathbf{z}, \qquad \mathbf{x}'' = \mathbf{x}^\star - \epsilon\mathbf{z}.$$

Both $\mathbf{x}'$ and $\mathbf{x}''$ satisfy the constraints in $Q^{\mathrm{M}}_{\mathrm{BIP}}(G)$, so $\mathbf{x}', \mathbf{x}'' \in Q^{\mathrm{M}}_{\mathrm{BIP}}(G)$. Since $\mathbf{x}^\star = (\mathbf{x}' + \mathbf{x}'')/2$, it follows that $\mathbf{x}^\star$ is not an extreme point of $Q^{\mathrm{M}}_{\mathrm{BIP}}(G)$, which is a contradiction. □

Theorem 3 follows from Lemmas 1 and 2.

**Theorem 3.** *If $G$ is bipartite, then $Q^{M}_{\mathrm{BIP}}(G) = \mathcal{M}(G)$.*

### 2.1 Perfect Matchings

Analogously, we can define the convex hull of perfect matchings. For each perfect matching $M$ in $G$, let $\chi^M \in \{0,1\}^{|E|}$ be the indicator vector of the matching. Let $\mathcal{PM}(G)$ denote the convex hull of all such indicator vectors of matchings in $G$.

$$\mathcal{PM}(G) \quad = \quad \text{convex-hull}\{\chi^M \mid M \text{ is a perfect matching}\}.$$

We can also define a polyhedron consisting of constraints valid for perfect matchings:

$$Q_{\text{BIP}}^{\text{PM}}(G) \quad = \quad \{\mathbf{x} \mid \sum_{e \in \delta(v)} x_e = 1, \forall v \in V;\ x_e \geq 0, \forall e \in E\}.$$

**Theorem 4.** *If $G$ is bipartite, then $Q_{\text{BIP}}^{PM}(G) = \mathcal{PM}(G)$.*

*Proof.* $\mathcal{PM}(G) \subseteq Q_{\text{BIP}}^{\text{PM}}(G)$, since each indicator vector of a perfect matching obeys the constraints in $Q_{\text{BIP}}^{\text{PM}}(G)$, since the constraints in $Q_{\text{BIP}}^{\text{PM}}(G)$ are valid for any perfect matching in $G$. To show that $Q_{\text{BIP}}^{\text{PM}}(G) \subseteq \mathcal{PM}(G)$, it is sufficient to consider case (i) in the proof of Lemma 2. □

Theorems 3 and 4 do not hold when $G$ is not bipartite. Let $G$ be an odd cycle and define $x_e = 1/2$ for each edge in the cycle. Then $\mathbf{x}$ is in $Q_{\text{BIP}}^{\text{M}}(G)$, but $\mathbf{x}$ is not a convex combination of matchings. In other words, if $G$ is an odd cycle, then $Q_{\text{BIP}}^{\text{M}}(G)$ is not an integer polytope.

### 2.2 Applications: Perfect Matchings in $d$-Regular Graphs

Let us look at some applications of Theorem 4.

**Theorem 5.** *Let $G = (V, E)$ be a bipartite $d$-regular graph. Then $G$ has a perfect matching.*

*Proof.* Define the vector $\mathbf{y} \in \mathbb{R}^{|E|}$ as follows. Let $y_e = 1/d$ for each edge $e$ in $E$. Then $\mathbf{y} \in Q_{\text{BIP}}^{\text{PM}}(G)$. Thus, the polytope $Q_{\text{BIP}}^{\text{PM}}(G)$ is non-empty and therefore contains an extreme point (see Exercise 1 from TD2), which corresponds to a perfect matching in $G$. □

Note that to prove Theorem 5, we do not even need to use the fact that any (non-extreme) point can be decomposed into a convex combination of extreme points (á la Carathéodory). Since all extreme points of $Q_{\text{BIP}}^{\text{PM}}(G)$ are *perfect* matchings, we simply need to show that the polytope $Q_{\text{BIP}}^{\text{PM}}(G)$ is non-empty.

Theorem 4 can also be used to prove the following theorem, which we leave as an exercise.

**Theorem 6.** *Let $G = (V, E)$ be a bipartite $d$-regular graph. Then $G$ has $d$ perfect matchings.*

## 3 Totally Unimodular Matrices and Integer Polytopes

A square matrix is called *unimodular* if it has determinant $\pm 1$. Equivalently, a matrix is unimodular if it has integer entries and its inverse also has integer entries. A matrix $\mathbf{A}$ is *totally unimodular* (TU) if the determinant of every square submatrix of $\mathbf{A}$ has value 0 or $\pm 1$. (Note: All entries must be 0 or $\pm 1$.)

**Theorem 7.** *If $\mathbf{A}$ is totally unimodular, and $\mathbf{b}$ is an integer vector, all the vertices of the polyhedron $P = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq 0\}$ have integer values. In other words, $P$ is an integer polytope.*

*Proof.* If $\mathbf{y}$ is a vertex, it is determined by an $m \times m$ submatrix $A_B$ for some basis $B$, where $A_B$ consists of $m$ columns of $A$ and $A_B$ has rank $m$.

The determinant of $A_B$ has value $\pm 1$, since $A_B$ is a square submatrix of $A$ and it has full rank (i.e. $A_B$ is nonsingular). Thus, $A_B$ is unimodular implying that $A_B^{-1}$ has integer entries. We have:

$$\mathbf{y} \quad = \quad A_B^{-1}\mathbf{b}.$$

Since both $A_B^{-1}$ and $\mathbf{b}$ have integer entries, each entry in $\mathbf{y}$ is also an integer. □

**Corollary 8.** *When the constraint matrix* $\mathbf{A}$ *of a linear program is TU and the vector* $\mathbf{b}$ *is integer, the solution vector is also integer.*

## 3.1 Bipartite Matching

Now we can show that the constraint matrix $\mathbf{A}$ for $Q^{\mathrm{M}}_{\mathrm{BIP}}(G)$ is totally unimodular. This provides another proof that $Q^{\mathrm{M}}_{\mathrm{BIP}}(G)$ is an integer polytope. Let us consider the incidence matrix $\mathbf{A}$ of $G$, which has $|V|$ rows and $|E|$ columns. Each row of $\mathbf{A}$ corresponds to a vertex $v \in V$, and each column of $\mathbf{A}$ corresponds to an edge $e$. An entry $a_{ve} = 1$ if $e \in \delta(v)$ and $a_{ve} = 0$ if $e \notin \delta(v)$.

**Lemma 9.** *The incidence matrix* $\mathbf{A}$ *of* $G$ *is TU iff* $G$ *is bipartite.*

*Proof.* ($\Leftarrow$ If $G$ is bipartite, then $\mathbf{A}$ is TU.)
Consider any $k \times k$ submatrix $\mathbf{A}'$ of $\mathbf{A}$. We will prove ($\Leftarrow$) by induction on $k$. For $k = 1$, this is true, because each element in $\mathbf{A}$ is in $\{0, \pm 1\}$ by construction. Now assume ($\Leftarrow$) for all $(k-1) \times (k-1)$ submatrices. We will prove ($\Leftarrow$) for a $k \times k$ submatrix $\mathbf{A}'$.

1. If any column in $\mathbf{A}'$ is all zeros, then $\det(\mathbf{A}') = 0$.
2. If any column in $\mathbf{A}'$ has only a single nonzero entry, then we can use the induction hypothesis to determine that $\det(\mathbf{A}')$ is 0 or $\pm 1$.
3. If all columns of $\mathbf{A}'$ have at least two nonzero entries, then separate the rows into those belonging to $V_1$ and those belonging to $V_2$ (via the bipartition of $V$). Each edge belongs to exactly two rows. Since we are not in an above case, for each edge $e = (i, j)$, both rows corresponding to $i$ and $j$ are included in $\mathbf{A}'$. This implies that the sum of all rows in $V_1 \cap \mathbf{A}'$ equals the sum of all rows in $V_2 \cap \mathbf{A}'$, which implies that the rows of $\mathbf{A}'$ are not linearly independent. Thus, $\det(\mathbf{A}') = 0$.

($\Rightarrow$ If $\mathbf{A}$ is TU, then $G$ is bipartite.)
If $G$ is not bipartite, then $G$ contains an odd cycle $C$. Consider the submatrix corresponding to the vertices and edges in $C$. One can verify that this submatrix has determinant with value 2. □

Note that Lemma 9 is more powerful than showing that the matching polytope is integral, because $\mathbf{b}$ can be a vector with positive integers. The solution to this more general problem is called a $\mathbf{b}$-*matching*.

## 3.2 Maximum Flow

There are other fundamental optimization problems whose associated constraint matrices are totally unimodular. One example is the maximum flow problem. Recall the linear program from previous lectures:

$$\begin{aligned} &\max \sum_{su \in A} x_{su} \\ \text{subject to: } &\sum_{uv \in A} x_{uv} - \sum_{vw \in A} x_{vw} = 0, \quad \forall v \neq s, t, \\ &x_{uv} \leq c_{uv}, \quad \forall uv \in A, \\ &x_{uv} \geq 0. \quad \forall uv \in A. \end{aligned} \tag{$\mathcal{P}_{max-flow}$}$$

Thus, if the edge capacities are integer, then the optimal flow value is integer.

### 3.3 Transportation Problem

Another basic optimization problem is known as the *transportation problem.* Consider the following single-commodity setting. There are $m$ suppliers and $n$ consumers. The $i^{th}$ supplier can provide $s_i$ units of the product, and the $j^{th}$ customer has a demand for $d_j$ units of the product. The total supply equals the total demand:

$$\sum_{i=1}^{m} s_i \quad = \quad \sum_{j=1}^{n} d_j.$$

For each pair of supplier $i$ and consumer $j$, our goal is to find $f_{ij}$, which is the amount of the product supplier $i$ ships to consumer $j$. Each such shipment has cost $c_{ij}$. This leads to the following problem:

$$\begin{aligned} \min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} f_{ij} \\ \text{subject to:} \quad & \sum_{i=1}^{m} f_{ij} = d_j, \quad j = 1, \ldots, n, \\ & \sum_{j=1}^{n} f_{ij} = s_i, \quad i = 1, \ldots, m, \\ & f_{ij} \geq 0, \quad \forall i, j. \end{aligned} \tag{$\mathcal{P}_{transport}$}$$

One can prove that the constraint matrix corresponding to $(\mathcal{P}_{transport})$ is totally unimodular, but we leave this as an exercise.

# 4 Matching Polytope for General Graphs

As mentioned earlier, if $G$ is not a bipartite graph, the polytope $Q^{\mathrm{M}}_{\mathrm{BIP}}(G)$ is not an integer polytope, i.e. it has vertices that do not correspond to matchings. Nevertheless, we can describe additional linear inequalities such that the resulting polytope is in fact equal to $\mathcal{PM}(G)$! This is a famous result due to Jack Edmonds [Edm65b, Edm65a].

Let $G$ be a 3-cycle (triangle) and assign $y_e = 1/2$ for each of the three edges. Then $\mathbf{y} \in Q^{\mathrm{PM}}_{\mathrm{BIP}}(G)$, but $G$ does not contain a perfect matching. The following constraint addresses this issue. It essentially says that in the subgraph induced by an odd-cardinality subset of vertices, there is at least one unmatched vertex.

$(\star\star\star)$ $\sum_{e \in U} x_e \leq \frac{|U|-1}{2}$, for each $U \subset V$, $|U|$ odd, $|U| \geq 3$.

Note that if $G$ is a 3-cycle, the point $\mathbf{y} = (1/2, 1/2, 1/2)$ does not satisfy this constraint. Furthermore, if we modify $\mathbf{y}$ to satisfy constraint $(\star\star\star)$, then $\mathbf{y}$ no longer belongs to $Q^{\mathrm{PM}}_{\mathrm{BIP}}(G)$.

We now show how to use the constraint $(\star\star\star)$ to construct a polyhedral description of $\mathcal{PM}(G)$. We therefore only consider graphs $G = (V, E)$ for which $|V|$ is even (otherwise $G$ does not have a perfect matching). If $|V|$ is even, then for $U \subset V$, if $|U|$ is odd, $|\bar{U}|$ is also odd. Adding constraint $(\star\star\star)$ for $U$ and $\bar{U}$, and using the fact that a perfect matching contains exactly $n/2$ edges, we have:

$$\sum_{e \notin E(U,\bar{U})} x_e \leq \frac{|U| + |\bar{U}|}{2} - 1 \;=\; \frac{n}{2} - 1 \quad \Rightarrow \quad \sum_{e \in E(U,\bar{U})} x_e \geq 1.$$

Let us consider the following constraints:

(*) $x_e \geq 0$, for each $e \in E$,

(**) $\sum_{e\in\delta(v)} x_e = 1$, for each $v \in V$,

(***) $\mathbf{x}(\delta(U)) \geq 1$, for each $U \subset V$, $|U|$ odd, $|U| \geq 3$.

We now define the polytope $Q^{\rm PM}(G)$ as follows:

$$Q^{\rm PM}(G) \quad = \quad \{\mathbf{x} \mid \mathbf{x} \text{ obeys constraints (*), (**) and (***)}\}.$$

The following theorem is due to Edmonds [Edm65a, Sch03]:

**Theorem 10.** *Let $G = (V, E)$ be a graph such that $|V|$ is even. Then $Q^{PM}(G) = \mathcal{PM}(G)$.*

*Proof.* It is clear that $\mathcal{PM}(G) \subseteq Q^{\rm PM}(G)$. Suppose that the converse does not hold: let $\mathbf{z}$ be a vertex of $Q^{\rm PM}(G)$ such that $\mathbf{z} \notin \mathcal{PM}(G)$, i.e. $\mathbf{z}$ is not integral.

We can assume we choose this counterexample graph $G$ as small as possible (i.e. $|V|+|E|$ minimal). Otherwise, we can delete edges for which $z_e$ are 0 or 1. Therefore, $0 < z_e < 1$ for all $e \in E$, and each vertex in $G$ has degree at least 2, so $|E| \geq |V|$.

If all vertices have degree exactly 2, then either $G$ contains an even cycle and we are done (Case (i), Theorem 3), or $G$ contains an odd cycle. But the only feasible solution for an odd cycle has 1/2 on each edge, which violates constraint (***).

Thus, it must be the case that $|E| > |V|$, i.e. there is at least one vertex with degree at least 3. Since $\mathbf{z}$ is an extreme point and there are $|E|$ non-zero variables, there must be $|E|$ tight constraints. At most $|V|$ of them come from the tight degree constraints (**). Therefore, we must have one tight odd-cut constraint (***) for some odd-cardinality set $U \subset V$, where $|U| \geq 3$.

Let consider two subgraphs: First shrink $U$ to a single point and then shrink $\bar{U}$ to a single point. Call the graphs $G_1$ and $G_2$, respectively. These graphs are smaller than $G$ and they each have a vertex set with even cardinality. Therefore, both $Q^{\rm PM}(G_1)$ and $Q^{\rm PM}(G_2)$ are integer polytopes. Let $\mathbf{z}'$ and $\mathbf{z}''$ be the projection of $\mathbf{z}$ onto the edge sets of these two graphs. Since $\mathbf{z}' \in Q^{\rm PM}(G_1)$, it can be written as a convex combination of perfect matchings in $G_1$. (Since $\mathbf{z}$ is rational as it is a vertex of a rational polyhedron, the convex combination has rational coefficients.) The same holds for $\mathbf{z}''$ and $G_2$. Denote this set of perfect matchings in $G_1$ by $\mathcal{M}_1$ and this set of perfect matchings in $G_2$: $\mathcal{M}_2$. Now we make $m_2$ copies of each matching in $\mathcal{M}_1$ and $m_1$ copies of each matching in $\mathcal{M}_2$ to obtain the sets $\mathcal{M}'$ and $\mathcal{M}''$, respectively. Let $k = m_1 m_2$ and note that $|\mathcal{M}'| = |\mathcal{M}''| = k$.

Then $G_1$ has perfect matchings $M'_1, M'_2, \ldots, M'_k$ and $G_2$ has perfect matchings $M''_1, M''_2, \ldots, M''_k$ such that:

$$\mathbf{z}' = \frac{1}{k}\sum_{i=1}^{k} \chi^{M'_i} \quad \text{and} \quad \mathbf{z}'' = \frac{1}{k}\sum_{i=1}^{k} \chi^{M''_i}.$$

For each edge $e \in E(U, \bar{U})$, $z_e = z'_e = z''_e$. Thus, edge $e$ appears in a $z_e$-fraction of the perfect matchings in each of the sets $\mathcal{M}'$ and $\mathcal{M}''$. Since both sets have cardinality $k$, edge $e$ appears $kz_e$ times in $\mathcal{M}'$ and in $\mathcal{M}''$. So let $M_i = M'_i \cup M''_i$. Then $M_i$ is a perfect matching of $G$. Thus,

$$\mathbf{z} \quad = \quad \frac{1}{k}\sum_{i=1}^{k} \chi^{M_i},$$

and $\mathbf{z}$ can be written as a convex combination of matchings, or points that belong to $Q^{\rm PM}(G)$, which is a contradiction. $\square$

For a graph $G$, let us define the following polytope:

$$Q^{\rm M}(G) \quad = \quad \{\mathbf{x} \mid \mathbf{x} \text{ obeys constraints (*), (**) and } (\star\star\star)\}.$$

Edmonds showed the following theorem.

**Theorem 11.** $Q^M(G) = \mathcal{M}(G)$.

(See Corollary 25.1a in [Sch03] for a proof.)

### 4.1 Application: Petersen's Theorem

In 1891, Petersen proved the following theorem [Pet91]. A bridgeless graph is a graph that is 2-edge connected (i.e. the minimum cut consists of at least two edges).

**Theorem 12.** *Every bridgeless, cubic graph has a perfect matching.*

*Proof.* Let $G = (V, E)$ be a bridgeless, cubic graph. To prove Petersen's Theorem, we can construct a vector $\mathbf{y} \in \mathbb{R}^{|E|}$ such that $y_e = 1/3$ for all $e \in E$. Clearly $\mathbf{y}$ satisfies constraints (*) and (**). Also, note that $\mathbf{y}$ satisfies (***) if $|\delta(U)| \geq 3$. Thus, we only need to consider sets $U \subset V$ such that $|\delta(U)| = 2$.

But we claim that if $|\delta(U)| = 2$, then $|U|$ cannot be odd. Thus, $\mathbf{y}$ belongs to $Q^{\mathrm{PM}}(G)$. This implies that $Q^{\mathrm{PM}}(G)$ is non-empty, and therefore $G$ has a perfect matching. $\square$

## References


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These lecture notes are partly based on the following sources: lecture notes by Stéphan Thomassé from previous versions of the same course and Chapters 18 and 25 in [Sch03].