Optimization and Approximation	ENS Lyon
Lecture 13	
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1 Extreme Point Structure using the Rank Lemma

In this lecture, we further explore how extreme point structure can be used to design *deterministic rounding* procedures for linear programs. Let $P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \geq \mathbf{b},\ \mathbf{x} \geq 0\}$ be a polyhedron in $\mathbb{R}^n$. Recall the definition of a basic feasible solution.

Definition 1. *A point $\mathbf{x}^* \in P$ is a* basic feasible solution *of P if among all tight constraints, n of them are linearly independent.*

Last time, we saw applications of the fact that a basic feasible solution has at least n tight constraints. Now we will see how to use the fact that it has n tight *linearly independent* constraints. We have the following key lemma.

Lemma 2. (Rank Lemma) *Let $\mathbf{x}^*$ be a basic feasible solution of P. Let r denote the cardinality of a maximal set of linear independent tight constraints of the form $\mathbf{A}_j\mathbf{x}^* = b_j$ for some row j of $\mathbf{A}$. Then $\mathbf{x}^*$ has r nonzero entries.*

Proof. Let $\mathbf{B}$ denote the submatrix of $\mathbf{A}$ restricted to the rows representing tight constraints and the columns representing nonzero variables. We know that the columns of $\mathbf{B}$ are linearly independent (using fact that $\mathbf{x}^*$ is a vertex iff the columns in $\mathbf{B}$ are linearly independent). This implies that $\mathbf{B}$ has full column rank, which is equal to the number of nonzero variables. Since the row rank of $\mathbf{B}$ equals the column rank, the row rank of $\mathbf{B}$ also equals the number of nonzero variables. The row rank of $\mathbf{B}$ corresponds to a maximal number of linearly independent rows of $\mathbf{B}$, which equals the cardinality of a maximal set of of linearly independent constraints. So we can conclude that the cardinality of a maximal set of linearly independent constraints equals the number of nonzero variables. □

2 Minimum Spanning Tree

Given a graph $G = (V, E)$ with edge costs $c : E \to \mathbb{R}$, the goal is to find a minimum cost spanning tree. Well-known algorithms for this problem include Kruskal's algorithm (sort the edges in nondecreasing order and add edge to current solution set if it does not create a cycle) and Prim's algorithm. Here, we would like to see how to solve this problem using a linear program. The techniques we introduce here will be applicable to other problems.

How do we write a linear program? A first attempt is:

$$\min \sum_{e \in E} c_e x_e$$
$$\sum_{e \in \delta(S)} x_e \geq 1, \text{ for all } S \subset V, S \neq \emptyset,$$
$$x_e \geq 0.$$

However, the integrality gap of this linear program is 2 (take cycle on k vertices and for each edge, set $x_e = \frac{1}{2}$), and we would like to find an exact linear programming relaxation, i.e. with integrality gap 1.

2.1 Linear Program for MST

Here is another linear program for the minimum spanning tree problem. Note that the bad example on the k-cycle is no longer a feasible solution. If we restrict each edge to have value $x_e \in \{0,1\}$, then $\mathbf{x}$ is a feasible solution for (P_{MST}) iff it is a spanning tree.

$$\begin{aligned}
& \min \sum_{e \in E} c_e x_e \\
\text{subject to:} \quad & \sum_{e \in E(S)} x_e \leq |S| - 1, \text{ for all } S \subset V,\ |S| \geq 2, \\
& \sum_{e \in E} x_e = |V| - 1, \\
& x_e \geq 0. \qquad\qquad (P_{MST})
\end{aligned}$$

We use $E(S)$ to denote the set of edges with both endpoints in S. The linear program (P_{MST}) has an exponential number of constraints. So to use the ellipsoid algorithm, we need to find an efficient separation oracle, which we leave as an exercise.

Suppose that $\mathbf{x}$ is an extreme point solution for (P_{MST}). How many nonzero entries can $\mathbf{x}$ have? If there were only $n-1$ nonzero entries in $\mathbf{x}$, then $\mathbf{x}$ would correspond to an integer solution. It is not the case for (P_{MST}) that there are few constraints, so the techniques from last lecture are insufficient to show that $\mathbf{x}$ is sparse. Instead, we will find a maximal set of linearly independent constraints, show that this set is not too large and then apply the Rank Lemma.

2.2 Uncrossing Techniques

Let $\mathbf{x}$ be an extreme point of (P_{MST}) and let $x(E(S)) = \sum_{e \in E(S)} x_e$. We use $\mathcal{F} = \{S \mid x(E(S)) = |S| - 1\}$ to denote the set of tight constraints for $\mathbf{x}$. We can show that $\mathcal{F}$ is closed under union and intersection. This will allow us to *uncross* two tight intersecting sets and replace them with their union and intersection.

Claim 3. *For $S, T \subset V$, $E(S) + E(T) \subseteq E(S \cup T) + E(S \cap T)$. Equality holds iff the set of edges $E(S \setminus T, T \setminus S) = \emptyset$.*

Proof. By inspection, we observe that:

$$E(S) + E(T) + E(S \setminus T, T \setminus S) \quad = \quad E(S \cup T) + E(S \cap T).$$

□

Now we show that if S and T are tight sets, then both the intersection $S \cap T$ and the union $S \cup T$ are also tight sets.

Lemma 4. *If $S, T \in \mathcal{F}$ and $S \cap T \neq \emptyset$, then both $S \cap T$ and $S \cup T$ are in $\mathcal{F}$. Moreover, $E(S) + E(T) = E(S \cap T) + E(S \cup T)$.*

Proof. Since both S and T belong to $\mathcal{F}$, they are both tight constraints. So:

$$\begin{aligned}
|S| - 1 + |T| - 1 &= x(E(S)) + x(E(T)) \\
&\leq x(E(S \cap T)) + x(E(S \cup T)) \qquad (1) \\
&\leq |S \cap T| - 1 + |S \cup T| - 1 \qquad (2) \\
&= |S| - 1 + |T| - 1. \qquad (3)
\end{aligned}$$

The first inequality (1) follows from Claim 3. The second inequality (2) follows from the fact that $\mathbf{x}$ is a feasible solution for (P_{MST}). The last equality (7) is because $|S| + |T| = |S \cap T| + |S \cup T|$ for any two subsets $S, T \subset V$.

All the inequalities must hold with equality. We can conclude that both $S \cap T$ and $S \cup T$ are tight sets. This implies that both $S \cap T$ and $S \cup T$ belong to $\mathcal{F}$. Moreover, due to the tight inequality (1), we can conclude that $E(S) + E(T) = E(S \cap T) + E(S \cup T)$. □

2.3 Laminar Family

We say that two sets S and T *intersect* if $S \cap T$, $S \setminus T$ and $T \setminus S$ are nonempty. A family of sets is *laminar* if no pair of sets intersect.

Definition 5. *Given a ground set V, a collection of subset $\mathcal{L} \subseteq 2^V$ is called* laminar *if for each $S, T \in \mathcal{L}$, either $S \cap T = \emptyset$ or $S \subseteq T$ or $T \subseteq S$.*

By Lemma 4, we can *uncross* two intersecting sets S and T: replace S and T with the set $S \cap T$ and $S \cup T$. Doing this until no intersecting sets remain results in a *laminar family* of sets.

Lemma 6. *A laminar family $\mathcal{L}$ over the ground set V without singleton sets has at most $|V| - 1$ distinct members.*

Proof. The proof is by induction on the size of the ground set. If $|V| = 2$, then the lemma follows. Let $n = |V|$ and assume that the lemma holds for all sets of size strictly smaller than n. Let $S \subset V$ be a maximal set in the laminar family. Each set in $\mathcal{L}$, except for V, is either contained in S or does not intersect S. The number of sets in $\mathcal{L}$ contained in S (including S itself) is at most $|S| - 1$ by the induction hypothesis. The sets in $\mathcal{L}$ that do not intersect S form a laminar family over the ground set $V \setminus S$ and there are therefore at most $|V| - |S| - 1$ such sets. Along with V, this gives a total of at most $(|S| - 1) + (|V| - |S| - 1) + 1 = |V| - 1$ sets in $\mathcal{L}$. □

Now let *span*$(\mathcal{F})$ denote the vector space generated by the set of vectors $\{\chi(E(S)) \mid S \in \mathcal{F}\}$, where $\chi(E(S))$ is the indicator vector in $\{0,1\}^{|E|}$ of the edge set $E(S)$. We now show that if $\mathcal{L}$ is a maximal laminar family, then *span*$(\mathcal{L})$ = *span*$(\mathcal{F})$.

Lemma 7. *If $\mathcal{L}$ is a maximal laminar family, then span$(\mathcal{L})$ = span$(\mathcal{F})$.*

Proof. Proof by contradiction. Suppose that $\mathcal{L}$ is a maximal laminar subfamily of $\mathcal{F}$, but *span*$(\mathcal{L}) \neq$ *span*$(\mathcal{F})$. Then there is some $S \in \mathcal{F}$ such that $\chi(E(S)) \notin$ *span*$(\mathcal{L})$. Choose such a set that intersects the minimum number of sets in $\mathcal{L}$. S must intersect at least one set in $\mathcal{L}$, otherwise $\mathcal{L}$ is not a maximal laminar family.

Let T be a set which intersects S. Since $S, T \in \mathcal{F}$, by Lemma 4, both $S \cap T$ and $S \cup T$ are also in $\mathcal{F}$. We can see that $S \cap T$ and $S \cup T$ each intersect fewer sets in $\mathcal{L}$ than S does. Therefore, both $S \cap T$ and $S \cup T$ must be in *span*$(\mathcal{L})$. By Lemma 4, $\chi(E(S)) + \chi(E(T)) = \chi(E(S \cap T)) + \chi(E(S \cup T))$. Since $T \in \mathcal{L}$ and $\chi(E(S \cap T))$ and $\chi(E(S \cup T))$ are in *span*$(\mathcal{L})$, this implies that $S \in$ *span*$(\mathcal{L})$, which is a contradiction. □

Now we can prove that an extreme point solution $\mathbf{x}$ to (P_{MST}) is integral.

Lemma 8. *Let $\mathbf{x}$ be an extreme point for (P_{MST}). Then $\mathbf{x}$ has integer entries.*

Proof. By the Rank Lemma, it is sufficient to find a maximal set of linearly independent constraints for $\mathbf{x}$. By Lemma 7, we can assume such a set of constraints corresponds to a laminar family. By Lemma 6, we conclude that such a family contains at most $|V| - 1$ tight constraints. Thus, $\mathbf{x}$ can have at most $|V| - 1$ nonzero entries. □

3 2-Edge Connectivity

Now let us look at a related NP-hard network design problem called *2-edge connectivity*. Given a 2-edge connected graph $G = (V, E)$, the goal is to find a subset $H \subseteq E$ of edges with minimum cardinality $|H|$ such that H is a 2-edge connected graph. Here is a linear programming relaxation for this problem.

$$\begin{aligned} &\min \sum_{e \in E} x_e \\ \text{subject to:} \quad &\sum_{e \in \delta(S)} x_e \geq 2, \text{ for all } S \subset V,\ S \neq \emptyset, \\ &x_e \geq 0. \end{aligned} \tag{P_{2EC}}$$

We use $\delta(S)$ to denote all edges with exactly one endpoint in S. Given an extreme point $\mathbf{x}$ of (P_{2EC}), we have the following lemma that allow us to uncross tight sets.

Claim 9. *For $S, T \subset V$, $\delta(S \cap T) + \delta(S \cup T) \subseteq \delta(S) + \delta(T)$.*

Proof. By inspection. □

Now let $\mathbf{x}$ be an extreme point of (P_{2EC}). Define $\mathcal{F} = \{S \mid x(\delta(S)) = 2\}$, i.e. $\mathcal{F}$ is the set of all tight constraints in (P_{2EC}) for $\mathbf{x}$. As before, we can show that $\mathcal{F}$ is closed under both union and intersection, allowing us to uncross the constraints and obtain a laminar family.

Lemma 10. *If $S, T \in \mathcal{F}$ and $S \cap T \neq \emptyset$, then both $S \cap T$ and $S \cup T$ are in $\mathcal{F}$.*

Proof. Since S and T belong to $\mathcal{F}$, they are both tight constraints. So:

$$\begin{aligned} 2 + 2 &= x(\delta(S)) + x(\delta(T)) \\ &\geq x(E(S \cap T)) + x(E(S \cup T)) \quad (4) \\ &\geq 2 + 2. \quad (5) \end{aligned}$$

The first inequality (4) follows from Claim 9. The second inequality (5) follows from the fact that $\mathbf{x}$ is a feasible solution for (P_{2EC}). All the inequalities must hold with equality. We can conclude that both $S \cap T$ and $S \cup T$ are tight sets. This implies that both $S \cap T$ and $S \cup T$ belong to $\mathcal{F}$. □

Now our laminar family may contain singleton sets, of which there are at most $|V|$. Therefore, there are at most $2|V| - 1$ sets in $\mathcal{L}$. So an extreme point contains at most $2|V| - 1$ nonzero values. If $x_e < 2$ for all edges, then the subgraph corresponding to the support of $\mathbf{x}$ is 2-edge connected. In this case, we can include all nonzero edges in the solution set for a total of at most $2|V| - 1$ edges. If there exists an edge with $x_e = 2$, we add a doubled edge e to solution set and use induction on the connected components. Since a connected subgraph contains at least $|V| - 1$ edges, this procedure results in a solution set with at most twice as many edges as in an optimal solution.

4 Generalized Steiner Network

Now we sketch an algorithm for the *generalized Steiner network problem* and show how a laminar family of tight sets can be used to design an iterative rounding procedure. Given a graph $G = (V, E)$, a cost function on the edges $c : E \to \mathbb{R}$ and a connectivity requirement $r(u, v)$ (a nonnegative integer) for each $u, v \in V$, the goal is to find a minimum cost subset of edges that contains $r(u, v)$ paths from u to v.

For a subset $S \subset V$, let $f(S) = \max_{u \in S, v \notin S} r(u,v)$. Then we have the following linear programming relaxation for the generalized Steiner network problem.

$$\min \sum_{e \in E} c_e x_e$$
$$\text{subject to: } x(\delta(S)) \geq f(S), \text{ for all } S \subset V,$$
$$0 \leq x_e \leq 1. \qquad (P_{GSN})$$

We say a function g on the ground set V is *weakly supermodular* if $g(V) = g(\emptyset) = 0$ and for all $S, T \subset V$, one of the following holds:

$$g(S) + g(T) \leq g(S \cap T) + g(S \cup T), \text{ or}$$
$$g(S) + g(T) \leq g(S \setminus T) + g(T \setminus S).$$

It can be shown that f is weakly supermodular. Moreover, suppose we have a partial solution set H. We update the graph to obtain $G' = (V, E \setminus H)$ and we update the requirement function to obtain f':

$$f'(S) = f(S) - |\delta_H(S)|.$$

It can also be shown that the function f' is weakly submodular. The fact that $f(S)$ is weakly submodular can be used to uncross the tight sets corresponding to an extreme point resulting in a maximal laminar family. This fact is essential to the proof of of the next lemma, which immediately results in a 2-approximation for the generalized Steiner network problem. (We leave this claim as an exercise.)

Lemma 11. *Let* $\mathbf{x}$ *be an extreme point of* (P_{GSN})*, where* f *is (re-)defined to be any weakly supermodular function. Then there exists and edge* $e \in E$ *such that* $x_e \geq \frac{1}{2}$.

Proof. We can assume that for all edges, we have: $0 < x_e < \frac{1}{2}$. (If $x_e = 0$, we can discard the edge. If $x_e \geq \frac{1}{2}$, then the lemma holds.)

Let $\mathcal{L}$ denote a maximal laminar family that spans the set of constraints that are tight for the extreme point $\mathbf{x}$. Then we have the following:

1. $\mathbf{x}$ is the unique solution for the system of equations: $\{x(\delta(S)) = f(S), \forall S \in \mathcal{L}\}$.
2. The vectors $\chi(\delta(S))$ are linearly independent for $S \in \mathcal{L}$.
3. $|E| = |\mathcal{L}|$.

This proof uses a token argument. Each edge will have exactly one token to distribute among sets. In total, there are therefore $|E|$ token. We will show that each set receives at least one token. This will show that $|\mathcal{L}| \leq |E|$. However, we will also show that there are token that were not assigned to any sets, which implies that $|\mathcal{L}| < |E|$, a contradiction.

Consider the following rules for assigning the token of edge (u,v) to sets in $\mathcal{L}$.

Rule 1. Let S be the smallest set in $\mathcal{L}$ containing u and let R be the smallest set in $\mathcal{L}$ containing v. Then S and R each get x_e tokens.

Rule 2. Let T be the smallest set in $\mathcal{L}$ containing both u and v. Then T gets $1 - 2x_e$ tokens.

Clearly, each edge distributes at most one token. Now we want to show that each set $S \in \mathcal{L}$ gets at least one token. Let $R_1, \ldots, R_k$ denote the maximal sets contained in S. Here, for the sake of clarity, we assume that S contains only two maximal sets, R_1 and R_2. The different type of edges with at least one endpoint in S are categorized in Figure 1. The amount of tokens that the set S gets is:

$$\sum_{e \in A} x_e + \sum_{e \in B} x_e + \sum_{e \in B}(1 - 2x_e) + \sum_{e \in C}(1 - 2x_e) = x(A) + x(B) + |B| - 2x(B) + |C| - 2x(C) \quad (6)$$
$$= x(A) - x(B) - 2x(C) + |B| + |C|. \quad (7)$$

Since all values on the left-hand side of (6) are positive, we conclude that the quantity in (7) is strictly positive. By inspection, we see that:

$$\begin{aligned} x(\delta(S)) - x(\delta(R_1)) - x(\delta(R_2)) &= x(A) + x(D) - x(B) - x(C) - x(C) - x(D) \\ &= x(A) - x(B) - 2x(C). \end{aligned}$$

Since all sets in $\mathcal{L}$ are tight, this implies:

$$f(S) - f(R_1) - f(R_2) = x(A) - x(B) - 2x(C). \quad (8)$$

Since all the constraints are linearly independent, the quantity in (8) is nonzero. Therefore, we can conclude that (7) is a positive integer. This implies that the number of tokens assigned to each set is at least one.

Now we argue that there are some unused tokens. Consider a maximal set R in $\mathcal{L}$. We see that $R \neq V$, since $f(V) = 0$ and therefore $V \notin \mathcal{L}$. Then there are edges in $\delta(R)$ that are not contained in any sets in $\mathcal{L}$. Thus, the tokens dispensed via Rule 2 are unused. So we can conclude that $|\mathcal{L}| < |E|$, which is a contradiction. □

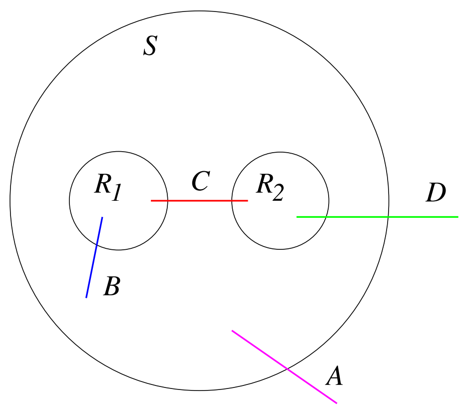

Figure 1: The different types of edges contained in the sets S and $\delta(S)$.

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The proofs in Section 2 are from Mohit Singh's thesis [Sin08]. Bounds on the size of the support of an extreme point of (P_{2EC}) were given by Cornuejols, Fonlupt and Naddef [CFN85]. The algorithm in Section 4 is due to Jain [Jai01] with a simplified analysis due to Nagarajan, Ravi and Singh [NRS10].