Advanced Heuristic and Approximation Algorithms	Grenoble-INP (ORCO)
Lecture 6	
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1 Asymmetric TSP

Let $G = (V, A)$ be a directed graph with edge weights obeying triangle inequality, i.e. $w_{ik} \leq w_{ij} + w_{jk}$. These edge weights might be *asymmetric*, i.e. $w_{ij} \neq w_{ji}$. Let G_{met} denote the metric completion of G, which is a complete, weighted graph. The *asymmetric traveling salesman problem* (ATSP) is to find a Hamilton cycle in G_{met} of minimum weight.

1.1 LP Relaxation

For a nonempty subset $S \subset V$, let $\delta^+(S) \subset A$ denote all edges $(i, j) \in A$ with $i \in S$ and $j \notin S$. Similarly, let $\delta^-(S) \subset E$ denote all edges $(i, j) \in A$ with $j \in S$ and $i \notin S$. Here is a linear programming relaxation for ATSP.

$$\begin{aligned}
\min \sum_{i,j \in V \times V} & w_{ij} \cdot x_{ij} \\
x(\delta^+(i)) = 1, &\quad \text{for all } i \in V, \\
x(\delta^-(i)) = 1, &\quad \text{for all } i \in V, \\
x(\delta(S)) \geq 1, &\quad \text{for all } S \subset V \text{ such that } S \neq \emptyset, \\
x_{ij} \geq 0, &\quad \text{for all } i, j \in V \times V. \qquad (P^{met}_{ATSP})
\end{aligned}$$

Here is a equivalent linear programming relaxation, which only uses variables corresponding to edges in G.

$$\begin{aligned}
\min \sum_{ij \in A} & w_{ij} \cdot x_{ij} \\
x(\delta^+(i)) = x(\delta^-(i)), &\quad \text{for all } i \in V, \qquad (1) \\
x(\delta(S)) \geq 1, &\quad \text{for all } S \subset V \text{ such that } S \neq \emptyset, \\
x_{ij} \geq 0, &\quad \text{for each } ij \in A. \qquad (P_{ATSP})
\end{aligned}$$

Both linear programs can be solved in polynomial time using the following separation oracle. We fix some vertex $s \in V$. Then for all $t \neq s$, we compute the minimum s-t-cut using x_{ij} as edge weights. If there exists and s-t-cut where $s \in S$ and $x(\delta^+(S)) < 1$, then we have found a violated constraint. (Note that if Constraint (1) holds, then we can set $y_{ij} = x_{ij} + x_{ji}$ and check if the global minimum cut is at least 2 on the graph G with edge weights y_{ij}.)

1.2 LP Integrality Gap

In Figure 1, a directed graph is shown for which the integrality gap is $\frac{3}{2}$. Each directed edge has LP value $\frac{1}{2}$. The total LP value is therefore $2 \cdot 2k \cdot \frac{1}{2}$ for the black edges plus $2k$ for the red edges for a total LP value of $4k$. Meanwhile, the optimal integer solution is $\approx 6k$. For more on the integrality gap see [CGK06], where an example with integrality gap arbitrarily close to 2 is presented. It has been conjectured that the integrality gap of both P_{ATSP}^{met} and P_{ATSP} is at most 2.

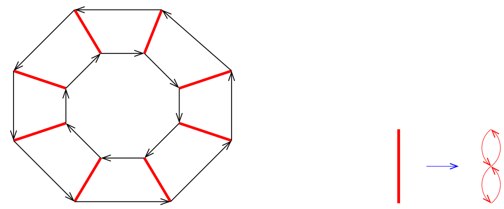

Figure 1: Each inter and outer directed cycle consists of k black edges. Each bold red edge denotes a double directed path of length two. Black edges have weight 2 and red edges have weight 1.

2 A $\log n$-Approximation Algorithm for ATSP

For this algorithm, we will consider the metric completion of the input graph $G = (V, A)$, which we refer to as G_{met}, and the linear program P_{ATSP}^{met}. The first step in the algorithm is to set up a perfect matching problem in a bipartite graph. For each vertex $i \in V$, we make two copies of vertex i, which we refer to as i' and i''. For each edge $ij \in G_{met}$, we include edge $i' \to j''$ and edge $j' \to i''$. The former edge is assigned weight w_{ij} and the latter, w_{ji}. Note that a feasible solution for P_{ATSP}^{met} corresponds to a feasible solution for the bipartite matching linear program (in which each vertex has fractional degree exactly 1). So we can find a bipartite matching in this new graph with weight at most that of an optimal solution for P_{ATSP}^{met}. Furthermore, observe that a perfect matching in this new graph corresponds to a cycle cover in G_{met}. This leads to the following algorithm.

Contract-Cycle-Cover(G)

1. If $|V| > 1$, find a minimum weight cycle cover C in G_{met}.
2. Contract cycle cover to obtain a new graph G' on at most $|V|/2$ vertices.
3. Return $C \cup$ Contract-Cycle-Cover(G').

We can observe that Contract-Cycle-Cover algorithm will be executed at most $\log n$ times on an input graph $G = (V, A)$, where $n = |V|$. Since at each round, the weight of the cycle cover is at most that of an optimal tour, the final solution has weight at most $\log n \cdot OPT$.

3 Hoffman Circulation Theorem

Let $G = (V, A)$ be a directed graph. A function $f : A \to \mathbb{R}_{\geq 0}$ is a *circulation* if

$$f(\delta^-(v)) = f(\delta^+(v)) \text{ for all } v \in V.$$

Theorem 1. [Hoffman Circulation Theorem] *For a directed graph $G = (V, A)$, suppose we are given functions:*

$\ell : A \to \mathbb{Z}_{\geq 0}$ *(lower bound),*

$u : A \to \mathbb{Z}_{\geq 0}$ *(upper bound), and*

$f : A \to \mathbb{R}_{\geq 0}$ *such that* $f(\delta^-(v)) = f(\delta^+(v))$ *for all* $v \in V$ *and* $\ell(e) \leq f(e) \leq u(e)$ *for all* $e \in A$.

Then there exists an integer circulation $f' : A \to \mathbb{Z}_{\geq 0}$ *such that*

$\ell(e) \leq f'(e) \leq u(e)$ *and* $\sum_{e \in A} f'(e) \leq \sum_{e \in A} f(e)$.

Moreover, if G *is a weighted graph with* $w : A \to \mathbb{R}^+$, *then* $\sum_{e \in A} w(e) \cdot f'(e) \leq \sum_{e \in A} w(e) \cdot f(e)$.

3.1 Application 1: Cycle Covers

Consider the metric completion of a directed graph $G = (V, A)$. Suppose we have a solution for the linear program P_{ATSP}^{met} on this graph. Let $f(ij) = x_{ij}$. For each vertex $v \in V$, apply the reduction shown in Figure 2 and for the new edge e, let $\ell(e) = 1$ and $u(e) = \infty$. For each original edge e in G_{met}, we set $\ell(e) = 0$ and $u(e) = 1$. Then we can apply Theorem 1 to obtain an integral circulation, which corresponds to a cycle cover in G_{met}.

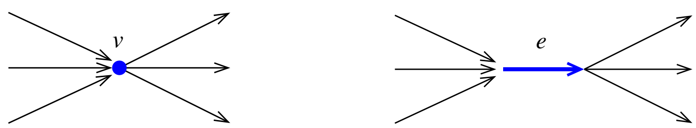

Figure 2: Each vertex corresponds to an edge with unit demand (i.e. lower bound 1).

3.2 Application 2: 2-Edge-Connected Subgraphs in 3-EC Cubic Graphs

We now show how Theorem 1 can be applied to obtain an algorithm for the 2-edge-connected subgraph problem. Given an unweighted, undirected graph $G = (V, E)$, the *2-edge-connected subgraph problem* is to find a subset of edges, $F \subseteq E$, such that $|F|$ is minimum and F is a 2-edge-connected graph. We consider this problem for the following restricted class of graphs. Let $G = (V, E)$ be an unweighted, undirected, cubic graph (i.e. each vertex has degree three) that is additionally 3-edge-connected. Now we describe an algorithm which produces a set F such that $|F| \leq \frac{5n}{4}$.

Augment-DFS(G)

$F \leftarrow \emptyset$.
For all $e \in E$, set $f(e) = 0$.

1. Find a depth-first-search tree of G, called T with root r.
2. For each $e \in T$, direct edge away from r and set:

   $w(e) = 0,\ \ell(e) = 1$ and $u(e) = \infty$.

3. For each $e \in E \setminus T$, direct edge towards r and set:

   $w(e) = 1,\ \ell(e) = 0$ and $u(e) = 1$.

4. For each $e \in E \setminus T$:

   Let C denote the unique cycle formed by e and edges in T.
   For each edge $e' \in C$, set $f(e') := f(e') + \frac{1}{2}$.

5. Find an integer circulation, f', such that $\sum_{e \in E} w(e) \cdot f'(e) \leq \sum_{e \in E} w(e) \cdot f(e)$.
6. For each edge $e \in E \setminus T$, add e to F if $f'(e) > 0$.
7. Output $F := T \cup F$.

Theorem 2. *The set F output by* Augment-DFS *has cardinality* $|F| \leq \frac{5n}{4}$.

Proof. The function f defined in the algorithm is a circulation, because it is the union of (fractionally weighted) directed cycles. Moreover, since G is 3-edge-connected, each edge e in T belongs to at least two of these cycles, and therefore has $f(e) \geq 1 = \ell(e)$. So f is a circulation that conforms to the specified upper and lower bounds.

Applying Theorem 1, we conclude that there is an integer circulation function f' also obeying these upper and lower bounds such that:

$$\sum_{e \in E} w(e) \cdot f'(e) \quad \leq \quad \sum_{e \in E} w(e) \cdot f(e).$$

To analyze the cardinality of F, we have:

$$\begin{aligned} |F| &= |T| + |\{e \in E \setminus T : f'(e) = 1\}| \\ &= (n-1) + \sum_{e \in E \setminus T} w(e) \cdot f'(e). \end{aligned}$$

Since each edge $e \in E \setminus T$ has $f(e) = \frac{1}{2}$, we have:

$$\begin{aligned} \sum_{e \in E \setminus T} w(e) \cdot f'(e) &\leq \sum_{e \in E} w(e) \cdot f'(e) \\ &\leq \sum_{e \in E} w(e) \cdot f(e) \\ &= \sum_{e \in E \setminus T} f(e) = \sum_{e \in E \setminus T} \frac{1}{2} = (\frac{n}{2} + 1) \cdot \frac{1}{2} = \frac{n}{4} + \frac{1}{2}. \end{aligned}$$

We conclude:

$$\begin{aligned} |F| &\leq n + \frac{n}{4} - \frac{1}{2} \\ &< \frac{5n}{4}. \end{aligned}$$

□

References

[BFS16] Sylvia Boyd, Yao Fu, and Yu Sun. A $\frac{5}{4}$-approximation for subcubic 2EC using circulations and obliged edges. *Discrete Applied Mathematics*, 209:48–58, 2016.

[CGK06] Moses Charikar, Michel X. Goemans, and Howard Karloff. On the integrality ratio for the asymmetric traveling salesman problem. *Mathematics of Operations Research*, 31(2):245–252, 2006.

[FGM82] Alan M. Frieze, Giulia Galbiati, and Francesco Maffioli. On the worst-case performance of some algorithms for the asymmetric traveling salesman problem. *Networks*, 12(1):23–39, 1982.

The integrality example in Section 1.2 is due to Michel Goemans[1]. The algorithm in Section 2 for ATSP is due to Frieze, Galbiati and Maffioli [FGM82]. The algorithm in Section 3.2 for the 2-edge-connected subgraph problem on 3-edge-connected, cubic graphs is due to Boyd, Fu and Sun [BFS16].

[1] To the best of my recollection, this is the example that was presented in an informal seminar in 2004.