- Exercise 1 - Private Vectors
Alice has two unit vectors, \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \). Bob wants to know the value \( \mathbf{x} \cdot \mathbf{y} \). He is allowed to choose any vector \( \mathbf{v} \in \mathbb{R}^n \), send \( \mathbf{v} \) to Alice and she will send him \( \mathbf{v} \cdot \mathbf{x} \) and \( \mathbf{v} \cdot \mathbf{y} \).

a. Can Bob compute \( \mathbf{x} \cdot \mathbf{y} \) using fewer than \( n \) queries?

b. Suppose Bob wants to estimate\(^2\) the dot product \( \mathbf{x} \cdot \mathbf{y} \). Design an algorithm for Bob that uses the fewest possible queries to Alice to estimate \( \mathbf{x} \cdot \mathbf{y} \).

- Exercise 2 - Jaccard Distance
Let \( U \) denote a universe of \( N \) items. Let \( A \subseteq U \) and \( B \subseteq U \) denote two sets, each containing \( n \) unique items, where \( n < N \). The \textit{Jaccard distance} between \( A \) and \( B \) is defined as
\[
\frac{|A \Delta B|}{|A \cup B|},
\]
where \( A \Delta B \) is the symmetric difference of the sets \( A \) and \( B \) (i.e., the number of elements in \( A \setminus B \) plus the number of elements in \( B \setminus A \)).

1. Suppose that \( A \) and \( B \) are each constructed by (independently) choosing an \( n \)-item subset of \( U \) uniformly at random among all such subsets. What is the expected value of the Jaccard distance between \( A \) and \( B \)?

2. \textit{Prove or disprove}: The Jaccard distance obeys the triangle inequality. (The \textit{triangle inequality} holds if \( d(A, B) \leq d(A, C) + d(B, C) \) for any three sets \( A, B, C \subseteq U \).)

- Exercise 3 - Small-Set Expansion on the Hypercube
A graph \( G = (V, E) \) is a \((\delta, \epsilon)\)-\textit{small-set expander} if for all subsets of vertices \( S \subseteq V \) such that \( |S| = \delta |V| \),\(^3\)
\[
\frac{E(S, \bar{S})}{\sum_{v \in S} \deg(v)} \geq 1 - \epsilon.
\]
A \textit{hypercube} is a graph on \( 2^n \) vertices, where each vertex corresponds to a (distinct) length-\( n \) binary string. Two vertices are connected by an edge if their respective binary strings differ in exactly one bit. For what values of \( \epsilon, \delta \) is the hypercube a \((\delta, \epsilon)\)-small-set expander?

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\(^1\)Homework submitted by email must be written using LaTeX and attached in .pdf format.
\(^2\)By \textit{estimate}, we mean compute a multiplicative \( 1 \pm \epsilon \) or an additive \( \pm \epsilon \) approximation, whichever is larger.
\(^3\)If you prefer to use the constraint \( |S| \leq \delta |V| \) rather than \( |S| = \delta |V| \), you are free to do so.
- **Exercise 4 - Unique Games and Cycle Covers**

Suppose we are given a cubic graph $G = (V, E)$ and a constraint $x_i - x_j \equiv c_{ij} \mod p$ for each edge $ij \in E$. Our objective is to find an assignment from $[0, p)$ to the variables $\{x_i\}$ that maximizes the number of satisfied constraints.

a. Give a $\frac{2}{3}$-approximation algorithm for this problem.

b. Suppose that $G$ has a cycle cover containing at least $\frac{n}{10}$ cycles. Give an approximation algorithm for this problem with the best approximation factor you can find. (A cycle cover is a subgraph of $G$ with vertex set $V$ and edgeset $F \subset E$ such that each vertex $v \in V$ has degree exactly two.)

- **Exercise 5 - Spectral Partitioning**

a. Give an example of a $d$-regular graph $G$ in which the spectral partitioning algorithm (Lecture 8) has an output close to its upper bound.

b. Give an example of a $d$-regular graph $G$ in which the spectral partitioning algorithm has a output close to the lower bound (i.e., close to $\lambda_2(G)/2$).

- **Exercise 6 - Local Connectivity**

Assume that $G = (V, A)$ is an directed, unweighted graph such that the solution for the linear programming relaxation for ATSP on $G$ has value $|V|$. Give an algorithm for Local-Connectivity ATSP that is 2-light.