Homework 1
Due December 3 in class or by email at 15h00 [1]

- Exercise 1 - Triangle Transversals

Given an undirected graph $G = (V, E)$, a *triangle transversal* is a subset of edges, $F \subset E$, such that $G' = (V, F')$ contains no triangles. In Lecture 3, we presented a 2-approximation for the problem of finding a minimum cardinality triangle transversal. In class, someone suggested the following algorithm:

REMOVE-EDGE-FROM-TRIANGLE(G)

1. $F \leftarrow \emptyset$.
2. Repeat until G contains no triangles:
   (a) Find any triangle t in G.
   (b) Let e be any edge in t.
      i. Add e to solution set F.
      ii. Remove e from E (i.e., from G).
3. Return F.

Prove or disprove: The algorithm REMOVE-EDGE-FROM-TRIANGLE is a 3-approximation algorithm for the minimum triangle transversal problem.

- Exercise 2 - Tournaments

A *tournament* $T = (V, A)$ is an oriented complete graph: for each pair of distinct vertices i and j in V, there is either an arc ij or ji (but not both). A tournament is *s*trongly connected if for each pair of distinct vertices i and j in V, there is a directed path from i to j and a directed path from j to i.

a. Prove or disprove: A strongly connected tournament on at least three vertices contains a directed triangle. (A directed triangle consists of the arcs ij, jk and ki where i, j and k are three distinct vertices.)

b. Prove or disprove: A strongly connected tournament on at least three vertices contains a directed Hamilton cycle. (A Hamilton cycle is a cycle that visits each vertex exactly once.)

- Exercise 3 - Semi-Kernels in Digraphs

A *s*emi-kernel in a digraph $D = (V, A)$ is a subset of vertices, $S \subseteq V$, such that S is a stable set and for every vertex $v \in V$, either $v \in N^+[S]$ or there is some vertex w such that $v \in N^+(w)$ and arc $(u, w) \in A$ for some $u \in S$. In other words, $N^{++}[S] = V$.

Prove or disprove: every digraph has a semi-kernel.

[1] Homework submitted by email must be written using LaTex and attached in .pdf format.

- Exercise 4 - Duality

Let $G = (V, E)$ be an undirected graph and let $\mathcal{S}$ be the set of all stable sets. The indicator vector x consists of the entries x_S for each stable set $S \in \mathcal{S}$. In Lecture 4, we gave the following linear programming relaxation for the graph coloring problem.

$$\begin{aligned} \min \sum_{S \in \mathcal{S}} x_S & \\ \text{subject to:} \quad \sum_{S: v \in S} x_S \geq 1, & \quad \text{for all } v \in V, \\ x_S \geq 0. & \end{aligned} \qquad (P_{frac\text{-}color})$$

a. Write the dual linear program for $(P_{frac\text{-}color})$.

b. For which discrete optimization problem is the dual a relaxation?

c. What is the integrality gap of this relaxation?

- Exercise 5 - Red and Blue Tournaments

Let $T = (V, A)$ be a tournament (i.e., an oriented complete graph K_n on n vertices). Suppose that each arc is colored either red or blue. For two vertices u and v, we say that *there is a red path from u to v* if there exists a directed path from u to v consisting of only red arcs. We define a *blue path from u to v* analogously. (Note that if $u = v$, then there is trivially a red and a blue path from u to v.

Prove or disprove: There is a vertex u in V such that for each vertex v in V, there is either a red path or a blue path from u to v.