- Exercise 1 - Triangle Transversals

Given an undirected graph $G = (V,E)$, a triangle transversal is a subset of edges, $F \subseteq E$, such that $G' = (V,F')$ contains no triangles. In Lecture 3, we presented a 2-approximation for the problem of finding a minimum cardinality triangle transversal. In class, someone suggested the following algorithm:

\begin{algorithm}
\textsc{Remove-Edge-From-Triangle}(G)
\begin{enumerate}
\item $F \leftarrow \emptyset$.
\item Repeat until $G$ contains no triangles:
\begin{enumerate}
\item Find any triangle $t$ in $G$.
\item Let $e$ be any edge in $t$.
\begin{enumerate}
\item Add $e$ to solution set $F$.
\item Remove $e$ from $E$ (i.e., from $G$).
\end{enumerate}
\end{enumerate}
\end{enumerate}
3. Return $F$.
\end{algorithm}

Prove or disprove: The algorithm \textsc{Remove-Edge-From-Triangle} is a 3-approximation algorithm for the minimum triangle transversal problem.

- Exercise 2 - Tournaments

A tournament $T = (V,A)$ is an oriented complete graph: for each pair of distinct vertices $i$ and $j$ in $V$, there is either an arc $ij$ or $ji$ (but not both). A tournament is strongly connected if for each pair of distinct vertices $i$ and $j$ in $V$, there is a directed path from $i$ to $j$ and a directed path from $j$ to $i$.

a. Prove or disprove: A strongly connected tournament on at least three vertices contains a directed triangle. (A directed triangle consists of the arcs $ij, jk$ and $ki$ where $i, j$ and $k$ are three distinct vertices.)

b. Prove or disprove: A strongly connected tournament on at least three vertices contains a directed Hamilton cycle. (A Hamilton cycle is a cycle that visits each vertex exactly once.)

- Exercise 3 - Semi-Kernels in Digraphs

A semi-kernel in a digraph $D = (V,A)$ is a subset of vertices, $S \subseteq V$, such that $S$ is a stable set and for every vertex $v \in V$, either $v \in N^+[S]$ or there is some vertex $w$ such that $v \in N^+(w)$ and arc $(u,w) \in A$ for some $u \in S$. In other words, $N^{++}[S] = V$.

Prove or disprove: every digraph has a semi-kernel.

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1Homework submitted by email must be written using \LaTeX{} and attached in .pdf format.
- Exercise 4 - Duality

Let $G = (V, E)$ be an undirected graph and let $\mathcal{S}$ be the set of all stable sets. The indicator vector $x$ consists of the entries $x_S$ for each stable set $S \in \mathcal{S}$. In Lecture 4, we gave the following linear programming relaxation for the graph coloring problem.

$$
\min \sum_{S \in \mathcal{S}} x_S \\
\text{subject to:} \sum_{S : v \in S} x_S \geq 1, \quad \text{for all } v \in V, \\
x_S \geq 0. \\
\tag{P_{frac-color}}
$$

a. Write the dual linear program for $(P_{frac-color})$.

b. For which discrete optimization problem is the dual a relaxation?

c. What is the integrality gap of this relaxation?

- Exercise 5 - Red and Blue Tournaments

Let $T = (V, A)$ be a tournament (i.e., an oriented complete graph $K_n$ on $n$ vertices). Suppose that each arc is colored either red or blue. For two vertices $u$ and $v$, we say that there is a red path from $u$ to $v$ if there exists a directed path from $u$ to $v$ consisting of only red arcs. We define a blue path from $u$ to $v$ analogously. (Note that if $u = v$, then there is trivially a red and a blue path from $u$ to $v$.

Prove or disprove: There is a vertex $u$ in $V$ such that for each vertex $v$ in $V$, there is either a red path or a blue path from $u$ to $v$. 