Homework 1 Solutions

- Exercise 1 - Triangle Transversals

Given an undirected graph $G = (V, E)$, a *triangle transversal* is a subset of edges, $F \subset E$, such that $G' = (V, F')$ contains no triangles. In Lecture 3, we presented a 2-approximation for the problem of finding a minimum cardinality triangle transversal. In class, someone suggested the following algorithm:

REMOVE-EDGE-FROM-TRIANGLE(G)

1. $F \leftarrow \emptyset$.
2. Repeat until G contains no triangles:
   (a) Find any triangle t in G.
   (b) Let e be any edge in t.
      i. Add e to solution set F.
      ii. Remove e from E (i.e., from G).
3. Return F.

Prove or disprove: The algorithm REMOVE-EDGE-FROM-TRIANGLE is a 3-approximation algorithm for the minimum triangle transversal problem.

Solution: The statement is false as shown by the example in Figure 1.

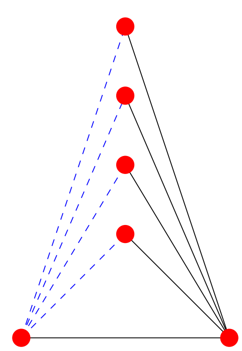

Figure 1: The dotted blue edges could be output by the algorithm. However, there exists a triangle transversal consisting of a single edge.

- Exercise 2 - Tournaments

A *tournament* $T = (V, A)$ is an oriented complete graph: for each pair of distinct vertices i and j in V, there is either an arc ij or ji (but not both). A tournament is *s*trongly connected if for each pair of distinct vertices i and j in V, there is a directed path from i to j and a directed path from j to i.

a. Prove or disprove: A strongly connected tournament on at least three vertices contains a directed triangle. (A directed triangle consists of the arcs ij, jk and ki where i, j and k are three distinct vertices.)

Solution: There are various ways to prove this statement. One way is the following. Consider the directed cycle C formed by an arc $uv \in A$ and a directed path from v to u, which must exist since T is strongly connected. If C is not a triangle, consider any two vertices x and y that are distance at least two apart in both directions. There is an arc connecting x and y which forms a directed cycle of length at most $|C| - 1$. By continuing this process, we can find a directed triangle in T.

Here is another clever solution given by a few students. As shown in lecture, every tournament has a king, k. Since T is strongly connected, there is an arc vk. Since there is no arc kv, there must be a path kwv, since every vertex is at distance at most two from k. Therefore, we have directed triangle kwv.

Here is another solution given by George A. Consider $L = N^-(v)$ and $R = N^+(v)$ for any vertex $v \in V$. Since T is strongly connected, (i) these sets are nonempty, and (ii) there is at least one arc from R to L. This arc forms a triangle with v in T.

b. Prove or disprove: A strongly connected tournament on at least three vertices contains a directed Hamilton cycle. (A Hamilton cycle is a cycle that visits each vertex exactly once.)

Solution: Let C denote a directed cycle in T, which exists by Part a. If C is not a Hamilton cycle, then there exists a vertex $v \notin C$. Suppose there are two distinct vertices $x, y \in C$ such that arcs xv and vy belong to A. Then we claim we can increase the length of C by one. Otherwise, all vertices $v \notin V$ are either *out-vertices* (i.e., for all $x \in V$, we have arc xv) or *in-vertices* (i.e., for all $x \in V$, we have arc vx). Note that there must be at least one out-vertex and at least one in-vertex, since T is strongly connected. Moreover, there must be an out-vertex x and in-vertex y such that arc xy belongs to A. Thus, we can augment C by the directed path u_i, x, y, u_{i+1}, where u_i and u_{i+1} are consecutive vertices in C.

- Exercise 3 - Semi-Kernels in Digraphs

A *semi-kernel* in a digraph $D = (V, A)$ is a subset of vertices, $S \subseteq V$, such that S is a stable set and for every vertex $v \in V$, either $v \in N^+[S]$ or there is some vertex w such that $v \in N^+(w)$ and arc $(u, w) \in A$ for some $u \in S$. In other words, $N^{++}[S] = V$.

Prove or disprove: every digraph has a semi-kernel.

Solution: The following solution is due to Bondy [Bon03]. Consider a maximal induced acyclic subgraph of D (i.e., a subset of vertices $S \subseteq V$ such that adding any vertex in $V \setminus S$ to S would create a directed cycle). Find a kernel K in the acyclic digraph induced on S. Then $N^{++}[K] = V$. A *kernel* is an acyclic dominating set. In an acyclic digraph, a kernel can be found via a straightforward greedy algorithm: add a source (i.e., a vertex with no incoming arcs) to K, remove all dominated vertices, and repeat.

Another proof can by found in [CL72].

- Exercise 4 - Duality

Let $G = (V, E)$ be an undirected graph and let $\mathcal{S}$ be the set of all stable sets. The indicator vector x consists of the entries x_S for each stable set $S \in \mathcal{S}$. In Lecture 4, we gave the following linear programming relaxation for the graph coloring problem.

$$\begin{aligned} &\min \sum_{S \in \mathcal{S}} x_S \\ \text{subject to: } &\sum_{S: v \in S} x_S \geq 1, \quad \text{for all } v \in V, \\ &x_S \geq 0. \end{aligned} \qquad (P_{frac\text{-}color})$$

a. Write the dual linear program for $(P_{frac\text{-}color})$.

Solution: Let y_v denote a variable for each vertex.

$$\max \sum_{v \in V} y_v$$
$$\text{subject to:} \sum_{v \in S} y_v \leq 1, \quad \text{for all } S \in \mathcal{S},$$
$$y_v \geq 0. \qquad (P_{frac\text{-}clique})$$

b. For which discrete optimization problem is the dual a relaxation?

Solution: This is a relaxation of the maximum clique problem.

c. What is the integrality gap of this relaxation?

Many people showed an integrality gap of $\frac{5}{4}$ using a 5-cycle, which has a maximum clique of size 2 and an LP solution with objective value $\frac{5}{2}$, since all stable sets have size at most 2.

A clever solution given by Christian O. is to then amplify this solution k times by replacing vertices with 5-cycles in each of the k rounds to obtain a graph on $N = 5^k$ vertices. The size of the maximum clique remains 2 (since there are no triangles), and the objective value of the LP solution is $\frac{1}{2^k} \cdot N$ (i.e., we can assign value $\frac{1}{2^k}$ to each vertex, since each stable set has size at most 2^k). We have

$$\frac{N}{2^k} = \frac{2^{\log_2 N}}{2^{\log_5 N}} = \frac{2^{\log_2 N}}{2^{\frac{\log_2 N}{\log_2 5}}} = 2^{\log_2 N(1-\frac{1}{\log_2 5})} = N^{1-\epsilon},$$

where $\epsilon \approx .43$. Thus, this gives a *polynomial* integrality gap.

Note that it is NP-hard to approximate the maximum clique of a graph $G = (V, E)$ to within $|V|^{1-\epsilon}$ for any constant $\epsilon > 0$. However, this does not immediately imply the existence of a matching integrality gap for the relaxation ($P_{frac\text{-}clique}$), because this relaxation has exponential size. In fact, since ($P_{frac\text{-}color}$) and ($P_{frac\text{-}clique}$) have the same optimal values (by strong duality), and ($P_{frac\text{-}color}$) is at most $O(\log |V|)$ times the chromatic number of G (as shown in Lecture 4), we can certainly not solve ($P_{frac\text{-}clique}$) efficiently, as this would contradict that fact that approximating the chromatic number to within $|V|^{1-\epsilon}$ is NP-hard for any constant $\epsilon > 0$.

- Exercise 5 - Red and Blue Tournaments

Let $T = (V, A)$ be a tournament (i.e., an oriented complete graph K_n on n vertices). Suppose that each arc is colored either red or blue. For two vertices u and v, we say that *there is a red path from u to v* if there exists a directed path from u to v consisting of only red arcs. We define a *blue path from u to v* analogously. (Note that if $u = v$, then there is trivially a red and a blue path from u to v.

Prove or disprove: There is a vertex u in V such that for each vertex v in V, there is either a red path or a blue path from u to v.

Solution: *I had meant to ask this question with the restriction that the red set of arcs and the blue set of arcs each form a* partial order *(i.e., they contain no cycles). By mistake, I forgot to include this restriction. Many people proved the above statement, which is harder, so it seems to be true. I will post a solution later if I find a nice proof. For now, here is a short proof due to A. Sebő for the restricted statement.*

Let $R \subseteq V$ denote the set of red sources (i.e., vertices that have no incoming red arcs). If R contains only a single vertex, then we are done, since the red arcs form a partial order on V (i.e., every vertex in V is reachable by a red path from some vertex in R). If R contains at least two vertices, x and y with blue arc xy, then remove y from V. Use induction to find a dominating vertex z for the tournament on $V \setminus y$. If $z = x$, then clearly y is also dominated by z on a monochromatic path. If $z \neq x$, then there is a blue path from z to x, and thus a blue path from z to y. Moreover, we can verify the statement for the base case on three vertices.

References

[Bon03] J. A. Bondy. Short proofs of classical theorems. *Journal of Graph Theory*, 44(3):159–165, 2003.

[CL72] V. Chvátal and L. Lovász. Every directed graph has a semi-kernel. In *Hypergraph Seminar, Lecture Notes in Mathematics*, volume 411, page 175. Springer, 1972.