Final Exam: January 26, 2018

Duration 3 hours.
One 8.5×11 inch sheet of paper allowed.

Exercise 1.

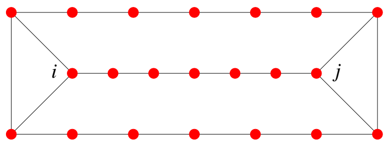

Figure 1: Each edge shown has unit cost. This graph has 21 vertices and 24 edges.

1. What is the cost of a minimum cost traveling salesman tour of the graph shown in Figure 1?
2. What is the value c_{ij} (i.e. the cost of edge (i, j)) in the metric completion of the graph shown in Figure 1?
3. **True** or **False**: The following set of constraints describes an integer program for the traveling salesman problem on the metric completion of a graph.

$$\begin{aligned} \min \sum_{e \in E} c_e \cdot x_e & \\ x(\delta(i)) = 2, &\quad \text{for all } i \in V, \\ x(\delta(S)) \geq 1, &\quad \text{for all } S \subset V \text{ such that } S \neq \emptyset, \\ x_e \geq 0, &\quad \text{for each edge } e. \end{aligned} \tag{P_{TSP}}$$

If your answer is **False**, give correction(s).

(Recall that for a vertex subset, $S \subset V$, $\delta(S)$ denotes the set of edges that have exactly one endpoint in S.)

Exercise 2.

Recall that the *maximum cut problem* is to find a subset $S \subset V$ such that the number of edges crossing the cut $(S, V \setminus S)$ is maximized. Let $G = (V, E)$ be an undirected, unweighted graph. Consider the following relaxation of the maximum cut problem on G.

$$\begin{aligned} \max \sum_{(i,j)\in E} & \frac{1 - v_i \cdot v_j}{2} \\ \text{subject to: } & v_i \cdot v_i = 1 \\ & v_i \in \mathbb{R}^n \end{aligned} \qquad (\mathrm{P}_{cut})$$

Suppose that there exists an optimal solution for (P_{cut}) on G in which for every edge $ij \in E$, it is the case that $v_i \cdot v_j = -\frac{1}{2}$.

a. What is an upper bound on the size of a maximum cut of G in terms of $|E|$? (Give the smallest upper bound you can find.)

b. What is a lower bound on the size of a maximum cut of G in terms of $|E|$? (Give the largest lower bound you can find.)

Exercise 3.

Let $G = (V, E)$ be a simple, bridgeless, cubic graph.[1] Suppose that G has a cycle cover $C \subset E$ such that each cycle in C contains at least eight edges. Moreover, suppose we are given the cycle cover C.

a. Describe how to find a spanning tree of G and a perfect matching in G that intersect in at most $\frac{n}{8}$ edges.

b. Let $J \subseteq V$ denote that set of vertices that have odd degree in the spanning tree T from Part a. Give an upper bound on the weight of a perfect matching of the vertices in J in the metric completion of G.

c. What is the smallest upper bound you can prove on the cost (i.e. cardinality) of a TSP tour in G?

[1] Recall that *bridgeless* means 2-edge-connected and *cubic* means each vertex is adjacent to exactly three edges. A *simple* graph does not contain multi-edges or self-loops.

Exercise 4.

Let $G = (V, E)$ be an undirected, unweighted, 5-edge-connected graph. Our goal is to find subset of edges, $F \subset E$, such that F forms a 3-edge-connected, spanning subgraph of G and $|F|$ is minimum. Consider the following linear programming relaxation for this problem:

$$\begin{aligned} \min \sum_{ij \in E} x_{ij} & \\ x(\delta(S)) \geq 3, &\quad \text{for all } S \subset V \text{ such that } S \neq \emptyset, \\ x_{ij} \geq 0, &\quad \text{for all edges } ij \text{ in } E. \end{aligned} \qquad (P_{3EC})$$

Give a polynomial-time separation oracle for (P_{3EC}).

Exercise 5.

Let $T = (V, A)$ be a tournament and let $n = |V|$. Let $w : V \to \mathbb{R}^+$ denote a nonnegative weight function on the vertices of T. The *minimum weight dominating set problem* is to find a subset of vertices, $S \subset V$, such that:

(i) for each vertex $j \in V$, either j belongs to S or there exists a (directed) edge $(i, j) \in A$ such that $i \in S$, and

(ii) $\sum_{i \in S} w_i$ is minimum.

a. Write an integer program for the minimum weight dominating set problem.

b. Write its linear programming relaxation.

c. Show how to round the relaxation from Part b. to obtain a $\log n$-approximation algorithm for the minimum weight dominating set problem.

Exercise 6.

Let $G = (V, E)$ be an undirected, bipartite graph. Let n denote the number of vertices in G, and let $\mathbf{A}$ denote the adjacency matrix of G. (Recall that $\mathbf{A} = \{a_{ij}\}$ is a symmetric matrix, and $a_{ii} = 0$ for all $i \in V$, and $a_{ij} = 1$ for all $ij \in E$.) Define $\tau(G)$ as follows.

$$\tau(G) \quad = \quad \min_{|\mathbf{x}|=n} \mathbf{x}^\mathsf{T} \mathbf{A} \mathbf{x}$$

True or **False**: $\tau(G) \leq -2 \cdot |E|$.
Justify your answer.

Exercise 7.

Let $G = (V, E)$ be a simple cycle on n vertices. Note that G is an undirected and unweighted graph. Let $S_1 \subset V$ and $S_2 \subset V$ denote two distinct, nonempty subsets of vertices, each corresponding to a minimum cut of G.

True or **False**: Cut S_1 and cut S_2 are equally likely to be output by the routine RANDOM-CONTRACT(G).

Justify your answer. For your convenience, the following subroutines are provided.

RANDOM-CONTRACT(G)

1. Iterate $n - 2$ times:
   (a) Pick edge ij uniformly at random from E.
   (b) $G \leftarrow$CONTRACT(G, ij).

CONTRACT(G, $ij \in E$)

1. Merge vertices i and j to form a single vertex u.
2. Let $V' = \{V \setminus \{i, j\}\} \cup \{u\}$.
3. Let $E' = \{E \setminus \{\delta(i), \delta(j)\}\}$.
4. For each $k \neq i, j$:
   (a) If $ik \in E$, add edge uk to E'.
   (b) If $jk \in E$, add edge uk to E'.
5. Output (multigraph) $G' = (V', E')$.