Final Exam: January 26, 2018

Duration 3 hours.
One 8.5×11 inch sheet of paper allowed.

Exercise 1. (3 points.)

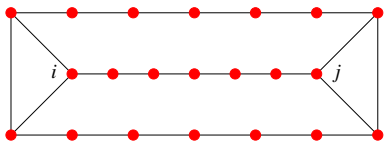

Figure 1: Each edge shown has unit cost. This graph has 21 vertices and 24 edges.

1. What is the cost of a minimum cost traveling salesman tour of the graph shown in Figure 1?

   Solution: 27.

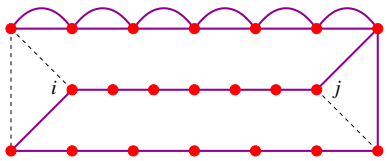

Figure 2: A TSP tour of length 27.

2. What is the value c_{ij} (i.e. the cost of edge (i, j)) in the metric completion of the graph shown in Figure 1?

   Solution: $c_{ij} = 6$.

3. **True** or **False**: The following set of constraints describes an integer program for the traveling salesman problem on the metric completion of a graph.

$$\begin{aligned} \min \sum_{e \in E} c_e \cdot x_e & \\ x(\delta(i)) = 2, & \quad \text{for all } i \in V, \\ x(\delta(S)) \geq 1, & \quad \text{for all } S \subset V \text{ such that } S \neq \emptyset, \\ x_e \geq 0, & \quad \text{for each edge } e. \end{aligned} \tag{P_{TSP}}$$

If your answer is **False**, give the correction(s).

(Recall that for a vertex subset, $S \subset V$, $\delta(S)$ denotes the set of edges that have exactly one endpoint in S.)

Solution: The answer is **False**. The last constraint does not enforce integrality of the solution, so it is not an integer program. Note that since (2) enforces an integer solution to be Eulerian, constraint (3) is sufficient to enforce connectivity. Also, note that in the metric completion, there is an optimal solution that uses each edge at most once.

$$\min \sum_{e \in E} c_e \cdot x_e \tag{1}$$

$$x(\delta(i)) = 2, \quad \text{for all } i \in V, \tag{2}$$

$$x(\delta(S)) \geq 1, \quad \text{for all } S \subset V \text{ such that } S \neq \emptyset, \tag{3}$$

$$x_e \in \{0, 1\}, \quad \text{for each edge } e. \tag{P_{TSP}}$$

Exercise 2. (3 points.)

Recall that the *maximum cut problem* is to find a subset $S \subset V$ such that the number of edges crossing the cut $(S, V \setminus S)$ is maximized. Let $G = (V, E)$ be an undirected, unweighted graph. Consider the following relaxation of the maximum cut problem on G.

$$\begin{aligned} \max \sum_{(i,j)\in E} & \frac{1 - v_i \cdot v_j}{2} \\ \text{subject to: } & v_i \cdot v_i = 1 \\ & v_i \in \mathbb{R}^n \end{aligned} \tag{P$_{cut}$}$$

Suppose that there exists an optimal solution for (P$_{cut}$) on G in which for every edge $ij \in E$, it is the case that $v_i \cdot v_j = -\frac{1}{2}$.

a. What is an upper bound on the size of a maximum cut of G in terms of $|E|$? (Give the smallest upper bound you can find.)

Solution: We are given an *optimal* solution to the above relaxation of the maximum cut problem in which $v_i \cdot v_j = -\frac{1}{2}$ for every edge $ij \in E$. Let $MC(G)$ denote the value of an optimal (integral) maximum cut of G. Then,

$$\begin{aligned} MC(G) &\leq \sum_{ij\in E} \frac{1 - v_i \cdot v_j}{2} \\ &= \sum_{ij\in E} \frac{1 - (-\frac{1}{2})}{2} \\ &= \frac{3|E|}{4}. \end{aligned}$$

b. What is a lower bound on the size of a maximum cut of G in terms of $|E|$? (Give the largest lower bound you can find.)

Solution: Applying the random-hyperplane rounding approach, we have:

$$\begin{aligned} \sum_{ij} \Pr[\text{edge } ij \text{ is cut}] &= \sum_{ij\in E} \frac{\arccos(-\frac{1}{2})}{\pi} \\ &= \sum_{ij\in E} \frac{\frac{2\pi}{3}}{\pi} \\ &= \frac{2|E|}{3}. \end{aligned}$$

Thus, there exists a cut in G consisting of at least this many edges, and $\frac{2|E|}{3}$ is therefore a lower bound on the size of the maximum cut of G.

Exercise 3. (3 points.)

Let $G = (V, E)$ be a simple, bridgeless, cubic graph.[1] Suppose that G has a cycle cover $C \subset E$ such that each cycle in C contains at least eight edges. Moreover, suppose we are given the cycle cover C.

a. Describe how to find a spanning tree of G and a perfect matching in G that intersect in at most $\frac{n}{8}$ edges.

   Solution: Let M denote the perfect matching $E \setminus C$. Find a spanning tree on the components of C (using $\frac{n}{8} - 1$ edges from M) and remove one edge from each cycle in C.

b. Let $J \subseteq V$ denote that set of vertices that have odd degree in the spanning tree T from Part a. Give an upper bound on the weight of a perfect matching of the vertices in J in the metric completion of G.

   Solution: Let $F \subset C$ denote the edges removed from C to form the spanning tree in Part a. Let $M' \subset M$ denote the edges from M belonging to this spanning tree. Note that $M' \cup F$ is a J-join for J since $T \cup M' \cup F$ is Eulerian. We observe that $|M' + F| < \frac{n}{4}$.

c. What is the smallest upper bound you can prove on the cost (i.e. cardinality) of a TSP tour in G?

   Solution: The cycle cover C contains at most $\frac{n}{8}$ cycles. So let $S = C \cup 2M'$ (where $M' = T \cap M$) and note that S is an Eulerian multigraph. Then $|S| \leq n + \frac{n}{4} = \frac{5n}{4}$.

[1] Recall that *bridgeless* means 2-edge-connected and *cubic* means each vertex is adjacent to exactly three edges. A *simple* graph does not contain multi-edges or self-loops.

Exercise 4. (3 points.)

Let $G = (V, E)$ be an undirected, unweighted, 5-edge-connected graph. Our goal is to find subset of edges, $F \subset E$, such that F forms a 3-edge-connected, spanning subgraph of G and $|F|$ is minimum. Consider the following linear programming relaxation for this problem:

$$\begin{aligned} \min \sum_{ij \in E} x_{ij} & \\ x(\delta(S)) \geq 3, & \quad \text{for all } S \subset V \text{ such that } S \neq \emptyset, \\ x_{ij} \geq 0, & \quad \text{for all edges } ij \text{ in } E. \end{aligned} \tag{P_{3EC}}$$

Give a polynomial-time separation oracle for (P_{3EC}).

Solution:

Check if all the values $\{x_{ij}\}$ are nonnegative. If so, find the (global) minimum cut on the graph G using the values $\{x_{ij}\}$ as edge weights. If the minimum cut is at least 3, the values $\{x_{ij}\}$ form a feasible solution to (P_{3EC}). If the minimum cut is strictly less than 3, then this cut is a violated constraint in (P_{3EC}).

Exercise 5. (3 points.)

Let $T = (V, A)$ be a tournament and let $n = |V|$. Let $w : V \to \mathbb{R}^+$ denote a nonnegative weight function on the vertices of T. The *minimum weight dominating set problem* is to find a subset of vertices, $S \subset V$, such that:

(i) for each vertex $j \in V$, either j belongs to S or there exists a (directed) edge $(i, j) \in A$ such that $i \in S$, and

(ii) $\sum_{i \in S} w_i$ is minimum.

a. Write an integer program for the minimum weight dominating set problem.

b. Write its linear programming relaxation.

c. Show how to round the relaxation from Part b. to obtain a $\log n$-approximation algorithm for the minimum weight dominating set problem.

Solution: See notes from Lecture 10.

Exercise 6. (3 points.)

Let $G = (V, E)$ be an undirected, bipartite graph. Let n denote the number of vertices in G, and let $\mathbf{A}$ denote the adjacency matrix of G. (Recall that $\mathbf{A} = \{a_{ij}\}$ is a symmetric matrix, and $a_{ii} = 0$ for all $i \in V$, $a_{ij} = 1$ for all $ij \in E$ and $a_{ij} = 0$ for all $ij \notin E$.) Define $\tau(G)$ as follows.

$$\tau(G) = \min_{|\mathbf{x}|=n} \mathbf{x}^\intercal \mathbf{A}\mathbf{x}$$

True or **False**: $\tau(G) \leq -2 \cdot |E|$.

Justify your answer.

Solution: Let S and $\bar{S}$ denote the two sides of the bipartition of V. For each $i \in S$, set $y_i = 1$, and for each $i \in \bar{S}$, set $y_i = -1$. Since G is bipartite, we can verify that $|\mathbf{y}| = \sum_{i \in V} (y_i)^2 = n$, and

$$\begin{aligned} \mathbf{y}^\intercal \mathbf{A}\mathbf{y} &= \sum_{ij} a_{ij}(y_i \cdot y_j) \\ &= -2|E|. \end{aligned}$$

Thus, the statement is **True**.

Exercise 7. (3 points.)

Let $G = (V, E)$ be a simple cycle on n vertices. Note that G is an undirected and unweighted graph. Let $S_1 \subset V$ and $S_2 \subset V$ denote two distinct, nonempty subsets of vertices, each corresponding to a minimum cut of G.

True or **False**: Cut S_1 and cut S_2 are equally likely to be output by the routine RANDOM-CONTRACT(G).

Justify your answer. For your convenience, the following subroutines are provided.

Solution: The statement is **True**. Every pair of edges in a simple cycle forms a minimum cut. The probability that a particular minimum cut $\{e_1, e_2\}$ is output by the algorithm is:

$$\begin{aligned}\Pr[\text{neither } e_1 \text{ nor } e_2 \text{ is chosen}] &= \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdots \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \\ &= \frac{2}{n(n-1)}.\end{aligned}$$

RANDOM-CONTRACT(G)

1. Iterate $n-2$ times:
   (a) Pick edge ij uniformly at random from E.
   (b) $G \leftarrow$CONTRACT(G, ij).

CONTRACT(G, $ij \in E$)

1. Merge vertices i and j to form a single vertex u.
2. Let $V' = \{V \setminus \{i, j\}\} \cup \{u\}$.
3. Let $E' = \{E \setminus \{\delta(i), \delta(j)\}\}$.
4. For each $k \neq i, j$:
   (a) If $ik \in E$, add edge uk to E'.
   (b) If $jk \in E$, add edge uk to E'.
5. Output (multigraph) $G' = (V', E')$.